Sudhansu S. Maiti; Atanu Biswas

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A NEW CHARACTERIZATION OF GEOMETRIC DISTRIBUTION

Sudhansu S. Maiti and Atanu Biswas

A characterization of geometric distribution is given, which is based on the ratio of the real and imaginary part of the characteristic function.

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1. INTRODUCTION

The geometric distribution, given by the cumulative distribution function (cdf)

$$F(x) = \Pr(X \le x) = \begin{cases} 1 - \theta^{x+1}, & if x = 0, 1, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

and the probability mass function (pmf)

$$f(x) = (1-\theta)\theta^x,\ x = 0, 1, \ldots, \tag{1}$$

for $0 < \theta < 1$, is the discrete analog of exponential distribution. Clearly, $F(x)$ is right continuous. If X follows an exponential distribution, $[X]$, the integer part of X, has a geometric distribution (see Kalbfleish and Prentice, [12], Chapt. 3). The exponential distribution is widely referenced probability law used in reliability and life testing for continuous data as the simplest choice. Exponential distribution has several nice properties by which the statistical analyses become simpler. When the lives of some equipment and components are being measured by the number of completed cycles of operations or strokes, or in case of periodic monitoring of continuous data, the geometric distribution is a natural choice. It possesses most of the nice properties of the exponential distribution, of course in the discrete set up.

Geometric distribution is characterized by the discrete version of the *lack of memory* and the constant hazard rate properties, which is also satisfied by the exponential distribution. Xekalaki [19], Hitha and Nair [11] and Roy and Gupta [17] have examined some characterization results for discrete models. Some characterizations of the geometric distribution are given by Rogers [16], Clawford [3], Srivastava [18], Galambos [9], El-Neweihi and Govindarajulu [4], Rao and Sreehari [15], Arnold [1], Ferguson [5, 6, 7] and Fosam and Shanbhag [8], among others. Almost all theses characterizations are on the results of the order statistics and the lack of memory property. In the present paper, we give an alternative characterization in a completely different view point.

We consider the form

$$\phi(t) = C(t) + \mathrm{i}\, S(t),$$

the natural expression of any characteristic function (cf) $\phi(t)$, with $S(t) = \mathrm{E}(\sin tX)$ and $C(t) = \mathrm{E}(\cos tX)$ are the imaginary and the real parts of $\phi(t)$. Meintanis and Iliopoulos [13] illustrated that $S(t)/C(t)$ is linear in t for exponential distribution. Here we are interested to see whether a similar result in the discrete set up characterizes the geometric distribution, the discrete analog of the exponential distribution. Then, it might be straightforward to use the geometric distribution in the discrete life testing problems. In this short note, we present a characterization of the geometric distribution based on the ratio $S(t)/C(t)$. The result is given in Section 2. Section 3 concludes.

2. THE CHARACTERIZATION

Note that, for the geometric distribution (1), the cf is given by

$$\begin{aligned} \phi(t) = \mathrm{E}(\exp(\mathrm{i}\, tX)) &= (1-\theta)(1-\theta\exp(\mathrm{i}\, t))^{-1} \\ &= (1-\theta)\left[\sum_{j=0}^{\infty}\theta^j\cos jt + \mathrm{i}\sum_{j=0}^{\infty}\theta^j \sin jt\right] \\ &= C(t) + \mathrm{i}\, S(t). \end{aligned}$$

We first state the following Theorem from Rainville ([14], Chapt. 8, pp. 129–130).

Theorem 1. If

$$f_1(x) = \sum_{j=0}^{\infty} a_j x^j \quad \text{in } |x| < R_1,$$

and

$$f_2(x) = \sum_{j=0}^{\infty} b_j x^j \quad \text{in } |x| < R_2,$$

and if $b_0 \neq 0$, then

$$\frac{f_1(x)}{f_2(x)} = \sum_{j=0}^{\infty} q_j x^j \quad \text{in } |x| < R,$$

where $R = \min\{R_1, R_2, |z|\}$, with z being the zero of $f_2(x)$ nearest to $x = 0$.

The q_j's are determined as follows:

$$q_0 = a_0/b_0,$$

and for $j \geq 1$,

$$b_0 q_j = a_j - \sum_{u=1}^{\infty} b_u q_{j-u}.$$

Now we state a special case of Cantor's Theorem from Bary ([2], Chapt. II, p. 193).

Lemma 1. Let $g(x)$ be a function defined at non-negative integers. Then

$$\sum_{j=0}^{\infty} \sin(jt) g(j) = 0 \quad \text{for all } t,$$

implies $g(j) = 0$ for $j = 1, 2, \ldots$.

Now we present the characterization theorem as follows.

Theorem 2. Among all distributions of nonnegative integer valued random variables, the geometric distribution is the only one for which

$$S(t) = \frac{\theta \sin t}{1 - \theta \cos t} C(t) \quad \text{for all } t.$$

P r o o f. In Theorem 1, we put $f_1 = S(t)$, $f_2 = C(t)$, $a_j = \sin jt$, $b_j = \cos jt$, $b_0 = 1$ and $R_1 = R_2 = 1$. Under this set up z comes out to be unity. Consequently, we have

$$\frac{S(t)}{C(t)} = \sum_{j=0}^{\infty} q_j \theta^j,$$

with

$$q_0 = a_0 / b_0 = 0$$

and for $j \geq 1$,

$$q_j = \sin jt - \sum_{k=1}^{n-1} q_{j-k} \cos kt.$$

Clearly,

$$q_1 = \sin t,$$

$$q_2 = \sin 2t - \sin t \cos t = \sin t \cos t.$$

Suppose, for $j = 1, \ldots, n$, $q_j = \sin t \cos^{j-1} t$, which holds for $j = 1, 2$.

Hence,

$$\begin{aligned} q_{n+1} &= \sin(n+1)t - \sin t \sum_{k=1}^{n} \cos kt \cos^{n-k} t \\ &= \cos t \left\{ \sin nt - \sin t \sum_{k=1}^{n-1} \cos kt \cos^{n-1-k} t \right\} \\ &= \cdots\cdots\cdots \\ &= \sin t \cos^n t. \end{aligned}$$

Thus, $q_j = \sin t \cos^{j-1} t$ for all j.

Consequently, we get

$$\frac{S(t)}{C(t)} = \sum_{j=1}^{\infty} \theta^j \sin t \cos^{j-1} t = \frac{\theta \sin t}{1 - \theta \cos t},$$

as $|\theta \cos t| < 1$.

Now, we prove the reverse part. If

$$\frac{S(t)}{C(t)} = \frac{\theta \sin t}{1 - \theta \cos t},$$

we immediately have

$$(1 - \theta \cos t)\mathrm{E}(\sin tX) = (\theta \sin t)\, \mathrm{E}(\cos tX),$$

which gives

$$\begin{aligned} 0 &= \mathrm{E}(\sin tX) - \theta \mathrm{E}(\sin(X+1)t) \\ &= \sum_{j=0}^{\infty} \sin jt[f(j) - \theta f(j-1)]. \end{aligned}$$

Since this is true for all t, from special case of Cantor's Theorem stated earlier, we immediately get $f(j) - \theta f(j-1) = 0$ for $j = 1, 2, \ldots$, and hence

$$f(j) = \theta f(j-1) = \theta^j f(0),$$

which, together with $\sum_{j=0}^{\infty} f(j) = 1$, gives geometric pmf (1) for $f(j)$. Hence the result follows. □

3. CONCLUDING REMARK

In this short note, a different characterization of geometric distribution, through the ratio of imaginary and real parts of the cf, is provided. Note that, for an exponential distribution with mean $1/\theta$, we have

$$S(t)/C(t) = \theta t \quad \text{for all } t.$$

(See Meintanis and Iliopoulos [13].) It is interesting to note that, unlike the case of exponential, the characterization of geometric distribution is not linear in t for $S(t)/C(t)$. It is a periodic function with period 2π. The characterization of exponential and its discrete analog (geometric) are quite different.

The application of this result will be based on the empirical cf $\phi_n(t) = n^{-1} \sum_{j=1}^{n} \exp(itX_j)$, where $X_1, \dots, X_n$ are random samples. A goodness-of-fit test of the empirical cf, as in the line of Henze and Meintanis [10] is under study. We hope to pursue this in a future communication.

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Sudhansu S. Maiti, Department of Statistics, Visva-Bharati University, Santiniketan – 731 235. India.
e-mail: dssm1@rediffmail.com

Atanu Biswas, Applied Statistics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata – 700 108. India.
e-mail: atanu@isical.ac.in