Kybernetika

Zdena Riečanová

Archimedean atomic lattice effect algebras in which all sharp elements are central

Kybernetika, Vol. 42 (2006), No. 2, 143--150

Persistent URL: http://dml.cz/dmlcz/135705

KYBERNETIKA — VOLUME *42* (2006), NUMBER 2, PAGES 143 – 150

ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRAS IN WHICH ALL SHARP ELEMENTS ARE CENTRAL

Zdenka Riečanová

We prove that every Archimedean atomic lattice effect algebra the center of which coincides with the set of all sharp elements is isomorphic to a subdirect product of horizontal sums of finite chains, and conversely. We show that every such effect algebra can be densely embedded into a complete effect algebra (its MacNeille completion) and that there exists an order continuous state on it.

Keywords: lattice effect algebra, sharp and central element, block, state, subdirect decomposition, MacNeille completion

AMS Subject Classification: 06D35, 03G12, 03G25, 81P10

1. INTRODUCTION

Lattice effect algebras generalize orthomodular lattices (including Boolean algebras) and MV-algebras [1]. Effect algebras, introduced in 1994 by Foulis and Bennet [3], or equivalent in some sense, algebraic structure D-poset introduced in 1994 by Kôpka and Chovanec [7] may be carriers of probability measures, where elements of these structures represent properties, questions or events with fuzziness, uncertainty or unsharpness. In spite of these facts there are (even finite lattice) effect algebras admitting no states and no probabilities (see [4] and [12]).

The aim of this paper is to give a full description of all Archimedean atomic lattice effect algebras in which all sharp elements are central. An important subfamily of Archimedean effect algebras (studied by us in this paper) is a family of all Archimedean atomic MV-effect algebras (MV-algebras), or more general family of all Archimedean atomic distributive effect algebras, detailed description of which has been given in [15]. Full description of all complete atomic modular effect algebras has been given in [17].

In fact, an effect algebra with the partial order derived from the effect algebraic $\oplus$-operation is a bounded poset. It is well-known that there are effect algebras (even orthomodular lattices) the effect algebraic $\oplus$-operation of which cannot be extended onto their MacNeille completions (completions by cuts, [8]). We show that each effect algebra from the studied family of Archimedean atomic effect algebras can be embedded into a complete effect algebra, which is its MacNeille completion.

As a consequence we obtain the statement about existence of (o)-continuous state (completely additive probability) on these effect algebras.

2. BASIC DEFINITIONS AND FACTS

A model for an effect algebra is the standard effect algebra of positive selfadjoint operators dominated by the identity on a Hilbert space. In a general form, an effect algebra was introduced by Foulis and Bennett [3].

Definition 2.1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect-algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfies the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in P$ there exists a unique $b \in P$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. Moreover, if we write $a \oplus b = c$ for $a, b, c \in E$, then we mean both that $a \oplus b$ is defined and $a \oplus b = c$. In every effect algebra E we can define the partial operation $\ominus$ and the partial order $\leq$ by putting

$$a \leq b \quad \text{and} \quad b \ominus a = c \quad \text{iff} \quad a \oplus c \quad \text{is defined and} \quad a \oplus c = b\,.$$

Since $a \oplus c = a \oplus d$ implies $c = d$, the operation $\ominus$ and the relation $\leq$ are well defined. If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (a *complete effect algebra*). If, moreover, E is a modular or distributive lattice then E is called *modular* or *distributive* effect algebra

Recall that a set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if

i) $1 \in Q$,

ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

Assume that $(E_1; \oplus_1, 0_1, 1_1)$ and $(E_2; \oplus_2, 0_2, 1_2)$ are effect algebras. An injection $\varphi : E_1 \to E_2$ is called an *embedding* if $\varphi(1_1) = 1_2$ and for $a, b \in E_1$ we have $a \leq b'$ iff $\varphi(a) \leq (\varphi(b))'$ in which case $\varphi(a \oplus_1 b) = \varphi(a) \oplus_2 \varphi(b)$. We can easily see that then $\varphi(E_1)$ is a sub-effect algebra of E_2 and we say that E_1 and $\varphi(E_1)$ are *isomorphic*, or that E_1 *is up to isomorphism a sub-effect algebra of* E_2. Clearly, if E_1 and E_2 are lattice effect algebras then $\varphi(E_1)$ is a sublattice of E_2. We usually identify E_1 with $\varphi(E_1)$.

Recall that a direct product $\prod\{E_\kappa \mid \kappa \in H\}$ of effect algebras E_κ is a cartesian product with $\oplus$, 0, 1 defined "coordinatewise". An element $z \in E$ is called *central* if the intervals $[0, z]$ and $[0, z']$ with the inherited $\oplus$-operation are effect algebras in their own right and $E \cong [0, z] \times [0, z']$, see [5]. The set $C(E) = \{z \in E \mid z \text{ is central}\}$ is called a *center* of E. If $C(E) = \{0, 1\}$ then E is called *irreducible*.

Definition 2.2. A *subdirect product* of a family $\{E_\kappa \mid \kappa \in H\}$ of lattice effect algebras is a sublattice-effect algebra Q (i. e., Q is simultaneously a sub-effect algebra and a sublattice) of the direct product $\prod\{E_\kappa \mid \kappa \in H\}$ such that each restriction of the natural projection pr_{κ_i} to Q is onto E_{κ_i}. Moreover, Q is a *sub-direct product decomposition* of a lattice effect algebra E if there exists an isomorphism $\varphi : E \to Q$ (of E onto Q).

A *horizontal sum* of finite chains $C_1 = \{0, a, 2a, \dots, 1 = n_a a\}$ and $C_2 = \{0, b, 2b, \dots, 1 = n_b b\}$ is a lattice effect algebra $E = C_1 \cup C_2$ with identified elements 0 and 1 and such that $C_1 \cap C_2 = \{0, 1\}$ and ka and ℓb for $k \neq n_a$ and $\ell \neq n_b$ are noncomparable, i. e., $ka \vee \ell b = 1$ and $ka \wedge \ell b = 0$. In the same manner we define a horizontal sum of any family of finite chains.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-*orthogonal* if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus_{k=1}^{n-1} a_k$ exists and $\bigoplus_{k=1}^{n-1} a_k \leq a_n'$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K | K \subseteq G,\ K \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K | K \subseteq G,\ K \text{ is finite}\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa \in H_1}$). We refer the reader to [11].

An effect algebra $(E; \oplus, 0, 1)$ is called *Archimedean* if for no nonzero element $e \in E$ the elements $ne = \underbrace{e \oplus e \cdots \oplus e}_{n\text{-times}}$ exist for all $n \in N$. An Archimedean effect algebra is called *separable* if every $\oplus$-orthogonal system of elements of E is at most countable. We can show that *every complete effect algebra is Archimedean* [10].

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if nx exists for every $n \in N$. We write $\mathrm{ord}(x) = n_x \in N$ if n_x *is the greatest positive integer such that* $n_x x$ *exists in* E. Clearly, in an Archimedean effect algebra $n_x < \infty$ for every $x \in E$.

Recall that elements x and y of a lattice effect algebra are called *compatible* (written $a \leftrightarrow b$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$. For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements of E are compatible then E is called an *MV-effect algebra*.

An element a of an effect algebra E is called an *atom* if $0 \leq b < a$ implies $b = 0$ and E is called *atomic* if for every $x \in E$, $x \neq 0$ there is an atom $a \in E$ with $a \leq x$. Clearly every finite effect algebra is atomic.

For more details we refer the reader to [2] and the references given therein. We review only a few properties.

Statements of the following Lemma are well-known and follows from the facts that in every lattice effect algebra E for $x, y \in E$ we have

$$x \oplus y = (x \vee y) \oplus (x \wedge y) \quad \text{iff} \quad x \leq y',$$

moreover

$$(x \vee z) \oplus y = (x \oplus y) \vee (y \oplus z) \quad \text{iff} \quad x, z \leq y'\,.$$

Lemma 2.3. Let E be an Archimedean atomic lattice effect algebra. Then for any atoms $a, b \in E$, $a \neq b$:

(i) $a \leftrightarrow b$ iff $a \leq b'$ iff $a \oplus b = a \vee b$ exists iff $ka \oplus \ell b = ka \vee \ell b$ exists for all $k \in \{1, \dots, n_a\}$, $\ell \in \{1, \dots, n_b\}$ (see [14]).

(ii) $n_a a \leq n_b b \Rightarrow a \not\leftrightarrow b$.

(iii) $ka \leq lb$, $k \neq n_a$, $l \neq n_b$ implies $a = b$ and $k = l$ (see [14]).

Lemma 2.4.

i) [6] If E is a lattice and $Y \subseteq E$ with $\bigvee Y$ existing in E then $x \leftrightarrow Y \Rightarrow z \wedge (\bigvee Y) = \bigvee \{z \wedge y \mid y \in Y\}$ and $z \leftrightarrow \bigvee Y$.

ii) If E is an Archimedean lattice effect algebra and $x \in E$ then $x \in S(E)$ iff $y \leq x \Rightarrow n_y y \leq x$.

3. BLOCKS, SHARP AND CENTRAL ELEMENTS

In [9] it has been shown that every maximal subset M of pairwise compatible elements of a lattice effect algebra E is a sub-effect algebra and a sublattice of E, called a *block* of E. Moreover, E is a union of its blocks. Clearly, blocks of E are MV-effect algebras.

In every lattice effect algebra E the set $S(E) = \{x \in E \mid x \wedge x' = 0\}$ is an orthomodular lattice (see [6]) and $B(E) = \{x \in E \mid x \leftrightarrow E\}$ is an MV-effect algebra such that $S(E)$ and $B(E)$ are sub-lattices and sub-effect algebras of E. Moreover, we have shown that $z \in C(E)$ iff $x = (x \wedge z) \vee (x \wedge z')$ for all $x \in E$ which gives $C(E) = B(E) \cap S(E)$ for every lattice effect algebra E. Further, $S(E)$, $B(E)$ and $C(E)$ are closed with respect to all existing infima and suprema in E [11]. In general, $C(E) = \{0, 1\}$ does not imply $C(S(E)) = \{0, 1\}$.

In this section we give a full description of all Archimedean atomic lattice effect algebras with $C(E) = S(E)$. Recall that for an effect algebra E and $p \in C(E)$ the interval $[0, p]$ is an effect algebra with inherited $\oplus$-operation and the unit p. In this case, for every $x \in [0, p]$ we have $x \oplus (x' \wedge p) = p$.

Theorem 3.1. Let E be an Archimedean atomic lattice effect algebra. The following conditions are equivalent:

i) $C(E) = S(E)$.

ii) E is isomorphic to a subdirect product of horizontal sums of finite chains.

In this case $C(E) = S(E)$ is atomic and every block of E is atomic.

(iii) $C(E)$ is atomic and for every atom p of $C(E)$ the interval $[0, p]$ is a horizontal sum of finite chains.

P r o o f. (i) $\Rightarrow$ (ii) Since $S(E) = C(E) \subseteq B(E) = \bigcap \{M \subseteq E \mid M \text{ block of } E\}$, we see that $S(E) \subseteq M$ for every block M of E.

Let $a \neq b$ be any two atoms of E. Assume that $n_b b < n_a a$. Let M_a is a block of E such that $a \in M_a$. Then $\{a, n_b b, n_a a\} \subseteq M_a$ because $\{n_b b, n_a a\} \subseteq S(E) \subseteq M_a$. Since $n_b b < n_a a = a \oplus \cdots \oplus a$ (n-times), we obtain by Riesz-Decomposition-Property that $a \leq n_b b$. It follows by Lemma 2.3 that $n_a a \leq n_b b$, a contradiction. This proves that $S(E)$ is atomic and $\{n_a a \mid a \in E \text{ atom of } E\}$ is the set of all atoms of $S(E) = C(E)$. Note, only, that in [14] it has been shown that $x \in S(E)$ iff there is a set $\{a_\kappa \mid \kappa \in H\}$ of atoms of E such that $x = \bigoplus\{n_{a_\kappa} a_\kappa \mid \kappa \in H\}$. Now, by [16, Theorem 3.1], E is isomorphic to a subdirect product of $\prod\{[0,p] \mid p \in E$ an atom of $S(E)\}$. Further, $p \in S(E)$ is an atom of $S(E)$ iff there exists an atom a of E with $n_a a = p$. Let $A_p = \{a_\kappa \in E \mid a_\kappa \text{ atom of } E \text{ with } n_{a_\kappa} a_\kappa = p\}$. Then the interval $[0,p]$ in E is a horizontal sum of chains $\{0, a_\kappa, \ldots, n_{a_\kappa} a_\kappa = p\}$, because for any $a_{\kappa_1} \neq a_{\kappa_2}$ with $n_{a_{\kappa_1}} a_{\kappa_1} = n_{a_{\kappa_2}} a_{\kappa_2}$ we have $k a_{\kappa_1} = \ell a_{\kappa_2}$ iff $k = n_{a_{\kappa_1}}$ and $\ell = n_{a_{\kappa_2}}$ (see [14]) and $a_{\kappa_1} \not\leftrightarrow a_{\kappa_2}$ by Lemma 2.3. Moreover, by Lemma 2.3, for atom b of E with $b \leq p$ we have $n_b b \leq p$ and hence $n_b b = p$ because p is an atom of $S(E)$.

(ii) $\Rightarrow$ (i) Assume now that E is isomorphic to a subdirect product Q of $\prod\{E_\kappa \mid \kappa \in H\}$, where E_κ are horizontal sums of finite chains. It follows that $C(E_\kappa) = S(E_\kappa) = \{0_\kappa, 1_\kappa\}$, $\kappa \in H$. By [16, Theorem 3.3] we obtain that $S(E) = \varphi^{-1}(\prod\{S(E_\kappa) \mid \kappa \in H\})$ and $C(E) = \varphi^{-1}(\prod\{C(E_\kappa) \mid \kappa \in H\})$, where $\varphi : E \to \prod\{E_\kappa \mid \kappa \in H\}$ is the embedding. We see that $C(E) = S(E)$ is an atomic Boolean algebra. Further, for every $\kappa \in H$, blocks of E_κ are finite chains, hence they are atomic. Moreover, by [16, Theorem 3.3] $M \subseteq E$ is a block of E iff there are blocks $M_\kappa \subseteq E_\kappa$, $\kappa \in H$ such that $M = \varphi^{-1}\Big(\prod\{M_\kappa \mid \kappa \in H\}\Big)$. This proves that all blocks of E are atomic.

iii) $\Leftrightarrow$ (ii) This follows by [16, Theorem 3.1]. □

Note that every Archimedean MV-effect algebra E satisfies conditions from Theorem 3.1. In this case $E = B(E)$ and for every atom p of $C(E)$ the interval $[0,p]$ is a finite chain. On the other hand, if the interval $[0,p]$ is a horizontal sum of at least two finite chains then $C([0,p]) = S([0,p]) = B([0,p]) = \{0,p\}$. It follows that E is a Boolean algebra iff $C(E) = S(E) = B(E) = E$. Further, we obtain the following

Corollary of Theorem 3.1: If E is an Archimedean atomic lattice effect algebra then the following conditions are equivalent:

i) $C(E) = S(E) = B(E) \neq E$.

ii) E is isomorphic to a subdirect product of horizontal sums of finite chains and at least one factor of this product is a horizontal sum of at least two chains.

It is well known that every poset $(P; \leq)$ has the *MacNeille completion* (completion by cuts). By [18] the MacNeille completion of a poset P is any complete lattice $\widehat{P}$ into which the poset P can be supremum and infimum densely embedded, i. e., for each $x \in \widehat{P}$ there are $Q, M \subseteq P$ such that $x = \bigvee \varphi(M) = \bigwedge \varphi(Q)$, where $\varphi : P \to \widehat{P}$ is the embedding. We usually identify P with $\varphi(P)$.

A complete effect algebra $(\widehat{E}; \widehat{\oplus}, \widehat{0}, \widehat{1})$ is called a *MacNeille completion of an effect algebra* $(E; \oplus, 0, 1)$ if, up to isomorphisms, E is a sub-effect algebra of $\widehat{E}$ and, as poset,

$\widehat{E}$ is a MacNeille completion of E (see [8]). It is known that there are (even finite) effect algebras the MacNeille completions of which are not again effect algebras (see [8]).

Recall that a complete effect algebra E is (o)-continuous (order-continuous) if for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of E and every $y \in E$ we have $x_\alpha \uparrow x \Rightarrow x_\alpha \wedge y \uparrow x \wedge y$. In this case lattice operations $\vee$ and $\wedge$ in E are (o)-continuous.

Theorem 3.2. Every Archimedean atomic lattice effect algebra with $C(E) = S(E)$ has a MacNeille completion isomorphic to a complete effect algebra $\widehat{E} \cong \prod\{[0,p] \subseteq E \mid p \in E \text{ atom of } C(E)\}$. Moreover, $\widehat{E}$ is an (o)-continuous lattice.

Proof. Clearly, $\widehat{E}$ is complete, since every $[0,p]$ is a horizontal sum of finite chains and hence it is a complete lattice. Further, E is a sub-direct product (up to isomorphism) of $\widehat{E}$ and hence E and $\widehat{E}$ have the same set of all atoms (we identify here E with $\varphi(E)$, where $\varphi : E \to \widehat{E}$ is the embedding). Thus E is dense in $\widehat{E}$, since for every $x \in \widehat{E}$ there are atoms a_α, $\alpha \in \mathcal{E}$, of E such that $x = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$ (see [14, Theorem 3.1]). This proves that $\widehat{E}$ is a MacNeille completion of E. For every atom p of $C(E)$ the interval $[0,p]$ is chain-finite, and therefore (o)-continuous. This proves the (o)-continuity of $\widehat{E}$. □

Remark 3.3 Note that if a lattice effect algebra E is isomorphic to a subdirect product of the direct product $\prod\{E_\kappa \mid \kappa \in H\}$, where E_κ are horizontal sums of finite chains, then

(i) E is an Archimedean atomic MV-effect algebra iff for every $\kappa \in H$, E_κ is a finite chain,

(ii) E is an Archimedean atomic distributive effect algebra iff for every $\kappa \in H$, E_κ is either a finite chain or a horizontal sum of two chains $\{0_\kappa, b^i_\kappa, 1_\kappa = 2b^i_\kappa\}$, $i = 1, 2$ (see [15]).

(iii) E is an Archimedean atomic modular effect algebra iff for every $\kappa \in H$, E_κ is either a finite chain or a horizontal sum of arbitrary many chains $\{0_\kappa, b^i_\kappa, 1_\kappa = 2b^i_\kappa\}$, $i \in \mathcal{I}$ (see [17]).

Remark 3.4 There are Archimedean modular effect algebras for which $S(E) \neq C(E)$. For example, the horizontal sum of Boolean algebra $\{0, a, a', 1 = a \oplus a'\}$ and a chain $\{0, b, 1 = 2b\}$. In this case $S(E) = \{0, a, a', 1\}$ and $C(E) = \{0, 1\}$, since $a \not\leftrightarrow b$. All complete atomic modular effect algebras are completely described in [17]. On the other hand conditions (i) and (ii) of Remark 3.3 are full characterisations of all Archimedean atomic MV-effect, respectively all distributive effect algebras, since they all satisfy $C(E) = S(E)$.

Recall that a map $\omega : E \to [0,1]$ is called a *state* on an effect algebra $(E; \oplus, 0, 1)$ if $\omega(1) = 1$ and $x \leq y' \Rightarrow \omega(x \oplus y) = \omega(x) + \omega(y)$ for $x, y \in E$. A state ω is called

(o)-continuous if for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of E such that $x_\alpha \downarrow x$ (i.e., $x = \bigwedge\{x_\alpha \mid \alpha \in \mathcal{E}\}$ and $\alpha_1 \le \alpha_2 \Rightarrow x_{\alpha_1} \ge x_{\alpha_2}$) we have $\omega(x_\alpha) \downarrow 0$. The question about existence of states are important, since there exist (even finite) effect algebras admitting no states (see [4, 12]).

Theorem 3.5. On every Archimedean atomic lattice effect algebra E with $C(E) = S(E)$ there exists an (o)-continuous state.

Proof. Let the complete effect algebra $\widehat{E}$ be a MacNeille completion of E from Theorem 3.2. Then $C(\widehat{E}) = S(\widehat{E})$ is a complete atomic Boolean algebra with the set $\{n_a a \mid a \in E \text{ atom of } E\}$ of all atoms of $C(\widehat{E})$ and $C(E)$ as well. By "Smearing Theorem of states" from [13] there exists an (o)-continuous state on $\widehat{E}$, since $\widehat{E}$ is (o)-continuous. The restriction $\omega|E$ is an (o)-continuous state on E. □

Remark 3.6 Finally note that on the effect algebra E in Theorem 3.5 there exists a *faithful* (o)-continuous state ω on E (i.e., $\omega(x) = 0 \Rightarrow x = 0$) iff the set $\{n_a a \mid a \in E$ atom of $E\}$ is at most countable (clearly every interval $[0, n_a a]$ may include an arbitrary set of atoms of E). In this case if $\{p_1, p_2, \dots\}$ is the set of all atoms of $C(E)$ and positive real number c_k are such that $\sum_{k=1}^{\infty} c_k = 1$ then there exists a unique (o)-continuous state ω on E such that $\omega(p_k) = c_k$ for $k = 1, 2, \dots$. This is because if for an atom a of E we have $\omega(n_a a) = c_k$ then $\omega(a) = \frac{c_k}{n_a}$. Moreover, this state ω is subadditive (i.e., $\omega(x \vee y) \le \omega(x) + \omega(y)$ for any $x, y \in E$) iff E is modular (see [17]).

Theorem 3.7. Let E be an Archimedean atomic effect algebra with $C(E) = S(E)$. The following conditions are equivalent:

(i) E is separable.

(ii) The set of all atoms of $C(E)$ is at most countable.

(iii) There exists an (o)-continuous faithful state on E.

ACKNOWLEDGEMENT

This work was supported by grants G-1/3025/06 and G-1/3115/06 of the Ministry of Education of the Slovak Republic and by Science and Technology Assistance Agency under the contract No. APVT-51-032002.

(Received January 25, 2006.)

REFERENCES

[1] C. C. Chang: Algebraic analysis of many-valued logics. Trans. Amer. Math. Soc. *88* (1958), 467–490.

[2] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structures. Kluwer Academic Publishers, Dordrecht – Boston – London and Ister Science, Bratislava 2000.

[3] D. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. *24* (1994), 1331–1352.
[4] R. J. Greechie: Orthomodular lattices admitting no states. J. Combin. Theory Ser. A *10* (1971), 119–132.
[5] R. J. Greechie, D. Foulis, and S. Pulmannová: The center of an effect algebra. Order *12* (1995), 91–106.
[6] G. Jenča and Z. Riečanová: On sharp elements in lattice effect algebras. BUSEFAL *80* (1999), 24–29.
[7] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.
[8] Z. Riečanová: MacNeille completions of D-posets and effect algebras. Internat. J. Theor. Phys. *39* (2000), 859–869.
[9] Z. Riečanová: Generalization of blocks for D-lattices and lattice ordered effect algebras. Internat. J. Theor. Phys. *39* (2000), 231–237.
[10] Z. Riečanová: Archimedean and block-finite lattice effect algebras. Demonstratio Math. *33* (2000), 443–452.
[11] Z. Riečanová: Orthogonal sets in effect algebras. Demonstratio Math. *34* (2001), 3, 525–532.
[12] Z. Riečanová: Proper effect algebras admitting no states. Internat. J. Theor. Phys. *40* 40 (2001), 1683–1691.
[13] Z. Riečanová: Smearings of states defined on sharp elements onto effect algebras. Internat. J. Theor. Phys. *41* (2002), 1511–1524.
[14] Z. Riečanová: Continuous effect algebra admitting order-continuous states. Fuzzy Sets and Systems *136* (2003), 41–54.
[15] Z. Riečanová: Distributive atomic effect algebras. Demonstratio Math. *36* (2003), 247–259.
[16] Z. Riečanová: Subdirect decompositions of lattice effect algebras. Internat. J. Theor. Phys. *42* (2003), 1425–1433.
[17] Z. Riečanová: Modular atomic effect algebras and the existence of subadditive states. Kybernetika *40* (2004), 459–468.
[18] J. Schmidt: Zur Kennzeichnung der Dedekind–MacNeilleschen Hulle einer geordneten Menge. Arch. Math. *17* (1956), 241–249.

Zdenka Riečanová, Department of Mathematics, Faculty of Electrical Engineering and Information Technology STU, Ilkovičova 3, 812 19 Bratislava 1. Slovak Republic.
e-mail: zdenka.riecanova@stuba.sk