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SEMICOPULÆ

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Dedicated to Berthold Schweizer on the occasion of his seventy-fifth birthday.

We define the notion of semicopula, a concept that has already appeared in the statistical literature and study the properties of semicopulas and the connexion of this notion with those of copula, quasi-copula, t-norm.

Keywords: semicopula, copula, quasi-copula, aggregation operator, t-norm

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1. INTRODUCTION

The object of the research here reported is the study of the notion of *semicopula*. To the best of our knowledge, this term was used for the first time by Bassan and Spizzichino ([3]) in a statistical context. The concept of semicopula was already known, in a different context, as *conjunctor* (a monotone extension of the Boolean conjunction with neutral element 1) or *t-seminorm* ([23]). However, it has never been studied in its own right at the same level of generality of the present note. As will be seen shortly, this notion is a generalization of that of quasi-copula and, hence, of that of copula. We recall that copulæ were introduced by Sklar ([21, 22]) who proved the theorem that bears his name (for more details, see [16, 20]). The investigations on a class of operations on distribution functions that derive from corresponding operations on random variables defined on the *same* probability space ([2, 18]) lead to the introduction of the concept of quasi-copula ([11, 25]).

Commutative semicopulæ are also a "good" generalization of triangular norms (briefly t-norms), introduced by K. Menger in order to extend the triangle inequality from the setting of metric spaces to probabilistic metric spaces, and successfully used in probability theory, mathematical statistics and fuzzy logic, as generalization of classical logic connectives ([13, 20]).

The paper is organized as follows: semicopulæ are defined in Section 2, where their main properties are given. A compactness question is the subject of Section 3, while Section 4 is devoted to the natural order on semicopulas. The object of Section 5 is the study of a special operation on semicopulas. Multivariate semicopulæ are introduced and briefly studied in Section 6.

2. DEFINITION AND FIRST PROPERTIES

Definition 2.1. A function $S : [0,1]^2 \to [0,1]$ is said to be a *semicopula* if, and only if, it satisfies the two following conditions:

(a) $S(x,1) = S(1,x) = x$ for all x in $[0,1]$;

(b) $S(x,y)$ is increasing in each place.

The class of all semicopulas will be denoted by $\mathcal{S}$.

If S is a semicopula, then for all $x \in [0,1]$

$$0 \le S(x,0) \le S(1,0) = 0,$$

namely $S(x,0) = 0 = S(0,x)$.

The notion of semicopula generalizes other concepts which have received more attention in the literature:
– a semicopula C that is 2-*increasing*, namely which, for all x, x', y, y' in $[0,1]$ with $x \le x'$ and $y \le y'$, satisfies the inequality

$$C(x',y') - C(x,y') - C(x',y) + C(x,y) \ge 0, \tag{2.1}$$

is a copula (see [16, 20]);
– a semicopula Q that satisfies the 1-*Lipschitz property*,

$$|Q(x,y) - Q(x',y')| \le |x - x'| + |y - y'| \tag{2.2}$$

for all x, x', y, y' in $[0,1]$, is a quasi-copula (see [11, 25]);
– a semicopula T that is both commutative

$$T(x,y) = T(y,x), \qquad \text{for all } x \text{ and } y \text{ in } [0,1], \tag{2.3}$$

and associative

$$T(T(x,y),z) = T(x,T(y,z)), \qquad \text{for all } x,\ y,\ z \text{ in } [0,1], \tag{2.4}$$

is a t-norm (see [13, 20]).

Notice also that semicopulæ are binary aggregation operators with neutral element 1 (see, e. g., [4]).

The class $\mathcal{S}$ of semicopulas strictly includes the class $\mathcal{Q}$ of quasi-copulas, which, in its turn, strictly includes the class $\mathcal{C}$ of copulas, $\mathcal{C} \subset \mathcal{Q} \subset \mathcal{S}$. Moreover, $\mathcal{S}_E$ will denote the set of commutative semicopulas; these correspond to exchangeable random variables. Of course, $\mathcal{S}_E$ is a proper subset of $\mathcal{S}$ and it strictly includes the set $\mathcal{T}$ of t-norms. Finally, the family of continuous semicopulas will be denoted by $\mathcal{S}_C$.

Example 2.1. The following function Z is a semicopula, but it is not a quasi-copula:

$$Z(x,y) = \begin{cases} 0, & (x,y) \in [0,1[^2, \\ \min\{x,y\}, & \text{elsewhere.} \end{cases}$$

Example 2.2. The following function S is a semicopula, but, because it is not associative, it is not a t-norm:

$$S(x,y) = xy \max\{x,y\}.$$

Example 2.3. The following function S is an associative semicopula, but it is not commutative

$$S(x,y) = \begin{cases} 0, & (x,y) \in [0,1/2] \times [0,1[, \\ \min\{x,y\}, & \text{elsewhere.} \end{cases}$$

Example 2.4. The function S_θ ($\theta > 1$) defined by

$$S_\theta(x,y) := \begin{cases} xy^\theta, & x \le y, \\ x^\theta y, & \text{elsewhere,} \end{cases}$$

is a continuous semicopula, but not a quasi-copula, because if, for instance, $\theta = 2$, one has

$$S_2(8/10,9/10) - S_2(8/10,8/10) = 136/1000 > 1/10;$$

thus S_2 is not 1-Lipschitz.

Proposition 2.1. If $S : [0,1]^2 \to [0,1]$ is a semicopula, then for all x and y in $[0,1]$,

$$Z(x,y) \le S(x,y) \le \min\{x,y\} = M(x,y). \tag{2.5}$$

P r o o f. If S is a semicopula, then, for all $x, y \in [0,1[$, one has

$$0 = S(x,0) \le S(x,y) \le \min\{x,y\},$$

and, if $x = 1$ (analogously $y = 1$), then $S(1,y) = y = \min\{y,1\}$. □

In other words, S is a semicopula if, and only if, S is a binary aggregation operator satisfying (5).

By introducing a condition stronger than 1-Lipschitz on semicopulas, we obtain

Proposition 2.2. Let S be a semicopula. Then S satisfies the kernel property, viz.

$$\forall x, x', y, y' \in [0,1] \qquad |S(x,y) - S(x',y')| \le \max\{|x-x'|, |y-y'|\}, \tag{2.6}$$

if, and only if, $S = M$.

Proof. It is known that condition (6) on a semicopula S is equivalent to its sub-shift invariance (see, e. g., [5]), i. e., for all $x, y, a \in [0,1]$ such that $x + a$ and $y + a$ are in $[0,1]$

$$S(x+a, y+a) \leq a + S(x,y).$$

In particular, if $x \geq y$, $S(x + (1-x), y + (1-x)) \leq 1 - x + S(x,y)$, so that $S(x,y) \geq y = x \wedge y$, and analogously if $x \leq y$. In view of Proposition 2.1 the proof is complete. □

A related concept is that of *co-semicopula*, which we introduce here in analogy with what is done in the case of t-conorms (see [13, 20]).

Definition 2.2. A function $S^* : [0,1]^2 \to [0,1]$ is called a *co-semicopula* if it is increasing in each place and satisfies the boundary condition

$$S^*(x,0) = S^*(0,x) = x \qquad \text{for all } x \in [0,1].$$

The co-semicopula is the dual operation of a semicopula, according to the following result, which can be easily proved.

Proposition 2.3. A function S^* is a co-semicopula if, and only if, there exists a semicopula S such that, for all x and y in $[0,1]$,

$$S^*(x,y) = 1 - S(1-x, 1-y). \tag{2.7}$$

This latter proposition allows to study only the properties of semicopulas and to obtain the corresponding ones for co-semicopulas by (7).

It must be noticed that no assumption on the (left- or right-) continuity of a semicopula has hitherto been made; but, in this case, the next result can be useful (the proof is the same as that of Lemma 3.1 in the authors' paper [8]).

Proposition 2.4. For a semicopula S, the following statements are equivalent:

(a) S is (left-)continuous in each place;

(b) S is jointly (left-)continuous.

Definition 2.3. A semicopula S is said to be *convex* if, for all x, y, u and v in $[0,1]$ one has, for every $\alpha \in [0,1]$,

$$S\left(\alpha\, x + (1-\alpha)\, u, \alpha\, y + (1-\alpha)\, v\right) \leq \alpha\, S(x,y) + (1-\alpha)\, S(u,v).$$

It is said to be *concave* if $-S$ is convex.

The following proposition can be proved as in [1, Corollary 1].

Proposition 2.5. The (semi-)copula M is the only concave semicopula.

We recall that the Fréchet-Hoeffding lower bound ([15]) for both copulas and quasi-copulas is the copula W defined by

$$W(x,y) := \max\{0, x+y-1\};$$

this is called Lukasiewicz copula in [13].

Proposition 2.6. If a semicopula S is convex and symmetric, then $S \leq W$.

Proof. Since S is convex and symmetric, it is Schur-convex; therefore, if x and y are in $[0,1]$ with $x+y \leq 1$, then $S(x,y) \leq S(x+y,0) = 0 = W(x,y)$, while, if $x+y \geq 1$, then $S(x,y) \leq S(1, x+y-1) = x+y-1 = W(x,y)$. □

By using Definition 2.1, one can easily prove that the functions of the following four examples are semicopulæ.

Example 2.5. (Weighted arithmetic mean) If S_0 and S_1 are semicopulæ, then for all $\theta \in [0,1]$ both the weighted arithmetic mean $(1-\theta)S_0 + \theta S_1$ and the weighted geometric mean $S_0^\theta \, S_1^{1-\theta}$ are semicopulæ. In other words, the set $\mathcal{S}$ of semicopulas is convex and log-convex.

Example 2.6. (Ordinal sum) Let $\{J_i\}_{i\in I}$ denote a family (possibly infinite) of nonempty, pairwise disjoint open subintervals $J_i :=]a_i, b_i[$ of $[0,1]$. Let $\{S_i\}$ be a collection of semicopulas with the same index set $\{J_i\}$. The ordinal sum of $\{S_i\}$ with respect to $\{J_i\}$ is the function S defined by

$$S(x,y) := \begin{cases} a_i + (b_i - a_i)\; S_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right), & (x,y) \in J_i^2; \\ M(x,y), & \text{elsewhere.} \end{cases}$$

It is easily shown that an ordinal sum of semicopulas is a semicopula, which will be denoted by $S = (\langle a_i, b_i, S_i \rangle)_{i\in I}$.

Example 2.7. (Transformed semicopulæ) Let S be a semicopula and let φ be an increasing bijection of $[0,1]$. The function S_φ, defined, for all x and y in $[0,1]$, by

$$S_\varphi(x,y) = \varphi^{-1}\left(S\left(\varphi(x), \varphi(y)\right)\right)$$

is also a semicopula, called the *transform of* S.

Example 2.8. (Frame semicopulæ) Let the points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

partition the unit interval $[0, 1]$; the *frame semicopula* S_f corresponding to the above partition is defined by

$$S_f(x, y) := \begin{cases} t_{i-1}, & \text{if } (x, y) \in [t_{i-1}, 1[^2 \setminus [t_i, 1[^2, \\ x \wedge y, & \text{if } x \vee y = 1, \\ 0, & \text{if } x \wedge y = 0. \end{cases}$$

Notice that a frame semicopula can always be modified by changing its value in any of the "frames" $[t_{i-1}, 1[^2 \setminus [t_i, 1[^2$ from t_{i-1} to any $\bar{t}_{i-1} \in]t_{i-2}, t_{i-1}[$. Moreover, if continuity questions arise, one may choose as the value taken on the side of each frame one of the values taken on the two adjoining frames.

Definition 2.4. Let S be a semicopula. The *horizontal section of S at $b \in [0, 1]$* is the function $h_b : [0, 1] \to [0, 1]$ defined by $h_b(t) := S(t, b)$; the *vertical section of S at $a \in [0, 1]$* is the function $v_a : [0, 1] \to [0, 1]$ defined by $v_a(t) := S(a, t)$; the *diagonal section of S* is the function $\delta_S : [0, 1] \to [0, 1]$ defined by $\delta_S(t) := S(t, t)$.

Proposition 2.7. Let S be a semicopula and δ its diagonal. Then

(a) $\delta(0) = 0$ and $\delta(1) = 1$;

(b) $\delta(t) \leq t$ for all $t \in [0, 1]$;

(c) δ is increasing;

(d) if $\delta(t) = t$ for all $t \in [0, 1]$, then $S = M$;

(e) if $\delta(t) = 0$ for all $t \in [0, 1[$, then $S = Z$.

Proof. The statements (a), (b) and (c) are direct consequences of Definition 2.1. Now, suppose that $\delta(t) = t$ for all t in $[0, 1]$. For all $x, y \in [0, 1]$, if $x \geq y$, then

$$S(y, y) = y \leq S(x, y) \leq S(1, y) = y;$$

whereas if $x < y$, then

$$S(x, x) = x \leq S(x, y) \leq S(x, 1) = x;$$

that is $S(x, y) = \min\{x, y\}$. The proof of statement (e) is analogous. □

As in the case of copulas (see [9, 10, 17]), given a function δ satisfying properties (a), (b) and (c), it is always possible to construct a semicopula whose diagonal section is δ; for instance:

$$S_\delta(x, y) := \begin{cases} \delta(x) \wedge \delta(y), & \text{if } x, y \in [0, 1[, \\ x \wedge y, & \text{elsewhere.} \end{cases}$$

S_δ is a *diagonal semicopula* associated with δ.

Example 2.9. Consider the function $\delta : [0,1] \to [0,1]$, given for all $t \in [0,1]$ by

$$\delta(t) = \begin{cases} 0, & t \in [0,1/2[; \\ 1/2, & t \in [1/2,1[; \\ 1, & t = 1. \end{cases}$$

The diagonal semicopula associated to δ is the semicopula S_δ given by

$$S_\delta(x,y) = \begin{cases} x \wedge y, & x \vee y = 1, \\ 1/2, & (x,y) \in [1/2,1[^2; \\ 0, & \text{elsewhere.} \end{cases}$$

Notice, however that a semicopula is not uniquely determined by its diagonal. For example, if $\delta(t) = t^2$ for all $t \in [0,1]$, there are two semicopulæ, $\Pi(x,y) = xy$ and $S_\delta(x,y) = x^2 \wedge y^2$, for $(x,y) \in [0,1[^2$, with diagonal section equal to δ.

3. COMPACTNESS

Let X denote the set of all functions from $[0,1]^2$ to $[0,1]$ equipped with the product topology (which corresponds to pointwise convergence).

Theorem 3.1. The class $\mathcal{S}$ of semicopulas is a compact subset of X (under the topology of pointwise convergence).

Proof. Since X is a product of compact spaces, it is well known from Tychonoff Theorem (see, e. g., [12]) that X is compact. The proof is completed by showing that $\mathcal{S}$ is a closed subset of X, namely, that, given a sequence $\{S_n\}_{n\in\mathbf{N}}$ in $\mathcal{S}$, if S_n converges pointwise to S, then S belongs to $\mathcal{S}$. In fact, for all $x, x', y \in [0,1]$ and $n \in \mathbf{N}$, one has

$$S_n(x,1) = x \underset{n\to+\infty}{\longrightarrow} x = S(x,1) = S(1,x),$$

and, if $x \le x'$, $S_n(x,y) \le S_n(x',y)$ implies $S(x,y) \le S(x',y)$, which is the desired conclusion. □

A sequence $\{S_n : n \in \mathbf{N}\}$ of semicopulas is a Cauchy sequence with respect to pointwise convergence if, for every $\epsilon > 0$ and for every point (x,y) in $[0,1]^2$, there exists a natural number $n_0 = n_0(\epsilon, x, y)$ such that

$$|S_n(x,y) - S_m(x,y)| < \epsilon,$$

whenever $n, m \ge n_0$. As an immediate consequence, each Cauchy sequence of semicopulas converges pointwise to some semicopula; in other words $\mathcal{S}$ is complete. We note that there are Cauchy sequences of (continuous) t-norms whose pointwise limit is not a t-norm (see [13]); therefore $\mathcal{T}$ is neither a complete nor a compact subset of $\mathcal{S}$.

By connecting Example 2.5 and Theorem 3.1, it follows that $\mathcal{S}$ is a compact and convex subset of X; therefore, in view of the Krein–Millman Theorem (see, e. g., [7]), one has

Corollary 3.1. The class $\mathcal{S}$ of semicopulas is the convex hull of the set formed by extremal points of $\mathcal{S}$, where a semicopula A is said to be *extremal* if, for all B and C in $\mathcal{S}$, and for all $\lambda \in]0,1[$, $A = \lambda B + (1-\lambda) C$ implies $A = B = C$.

Next we show that the semicopulæ Z and M are extremal.

Given the semicopula Z, suppose that there exist B and C in $\mathcal{S}$ and $\lambda \in]0,1[$ such that $Z(x,y) = \lambda B(x,y) + (1-\lambda) C(x,y)$ on $[0,1]^2$. For all $x, y \in [0,1[$, the equality

$$Z(x,y) = 0 = \lambda B(x,y) + (1-\lambda) C(x,y)$$

implies

$$B(x,y) = 0 = C(x,y),$$

so that one has $B = Z = C$ on $[0,1]^2$.

Using the same notations, we consider the semicopula M and suppose

$$M(x,y) = \lambda B(x,y) + (1-\lambda) C(x,y)$$

on $[0,1]^2$. In particular, for every $x \in [0,1]$; then the equality

$$M(x,x) = x = \lambda B(x,x) + (1-\lambda) C(x,x)$$

implies

$$\delta_B(x) = \delta_C(x) = x,$$

which, in view of Proposition 2.7, yields $B = C = M$.

4. ORDER

Proposition 2.1 suggests a partial order on the set of semicopulas.

Definition 4.1. If S_1 and S_2 are semicopulæ, S_1 is said to be *smaller than* S_2, and one writes $S_1 \prec S_2$, if $S_1(x,y) \leq S_2(x,y)$ for all x, y in $[0,1]$.

This is a partial ordering, because not every pair of semicopulas is comparable: it is sufficient to consider the copulas of Example 2.18 in [16] or the following example.

Example 4.1. Let S_1 and S_2 be, respectively, the two ordinal sums given by

$$S_1(x,y) = (\langle 0, 1/2, Z \rangle) = \begin{cases} 0, & (x,y) \in [0,1/2[^2, \\ \min\{x,y\}, & \text{elsewhere;} \end{cases}$$

and by

$$S_2(x,y) = (\langle 1/2, 1, Z \rangle) = \begin{cases} 1/2, & (x,y) \in [1/2,1[^2, \\ \min\{x,y\}, & \text{elsewhere.} \end{cases}$$

Then $S_1(1/4,1/4) \leq S_2(1/4,1/4)$, but $S_1(3/4,3/4) \geq S_2(3/4,3/4)$.

Let $\mathcal{A}$ be a nonempty subset of $\mathcal{S}$. We denote by $\vee\mathcal{A}$ and $\wedge\mathcal{A}$, respectively, the pointwise supremum and infimum of $\mathcal{A}$, that is, for each $(x,y) \in [0,1]^2$,

$$\vee\mathcal{A}(x,y) := \sup\{S(x,y), S \in \mathcal{A}\}, \qquad \wedge\mathcal{A}(x,y) := \inf\{S(x,y), S \in \mathcal{A}\}.$$

Proposition 4.1. $\mathcal{S}$ is a complete lattice, that is, for every $\mathcal{A} \subset \mathcal{S}$, $\mathcal{A} \neq \emptyset$, $\vee\mathcal{A}$ and $\wedge\mathcal{A}$ are in $\mathcal{S}$.

Proof. Let $\mathcal{A}$ be a nonempty subset of $\mathcal{S}$. For all $x, x', y \in [0,1]$ such that $x \leq x'$, one has

$$\vee\mathcal{A}(x,1) = \sup\{S(x,1), S \in \mathcal{A}\} = \sup\{x, S \in \mathcal{A}\} = x,$$

that is $\vee\mathcal{A}$ satisfies the condition (i) of Definition 2.1; moreover,

$$\vee\mathcal{A}(x,y) = \sup\{S(x,y), S \in \mathcal{A}\} \leq \sup\{S(x',y), S \in \mathcal{A}\} = \vee\mathcal{A}(x',y),$$

that is $\vee\mathcal{A}$ satisfies the condition (ii) of Definition 2.1, and hence $\vee\mathcal{A}$ is a semicopula. Analogously one can prove that $\wedge\mathcal{A}$ is a semicopula. □

In particular, the minimum (and the maximum) of two semicopulas is a semicopula. This result holds for quasi-copulas, but neither for copulas ([19]) nor for t-norms, as the following example shows.

Example 4.2. Consider the two t-norms Π and T, defined for all $x, y \in [0,1]$ by $\Pi(x,y) = xy$ and

$$T(x,y) = \begin{cases} 0, & (x,y) \in [0,1/2] \times [0,1/2]\,; \\ x, & (x,y) \in [0,1/2] \times]1/2,1]\,; \\ y, & (x,y) \in]1/2,1] \times [0,1/2]\,; \\ 2xy - x - y + 1, & (x,y) \in]1/2,1] \times]1/2,1]\,. \end{cases}$$

Let S be the pointwise minimum of Π and T. Then

$$S\,(S(5/10, 6/10), 8/10) = 24/100,$$

while

$$S\,(5/10, S(6/10, 8/10)) = 0,$$

i. e. S is not associative, and hence it is not a t-norm.

In [19], it was proved that the class $\mathcal{Q}$ of quasi-copulas is the Dedekind–MacNeille extension (the DM-extension, for short) of the set $\mathcal{C}$ of copulas, that is $\mathcal{Q}$ contains lower and upper bounds (i. e. pointwise infima and suprema) of all subsets of $\mathcal{C}$, in the same way as the real numbers are the extension of the set of rationals by Dedekind's cuts (for more details on lattice theory, see, e. g., [24]). In view of Proposition 4.1, $\mathcal{S}$ is a extension of class $\mathcal{T}$ of t-norms and $\mathcal{S}$ strictly includes the DM-extension of $\mathcal{T}$, because the supremum (infimum) of a subset of t-norms is commutative. This suggests the following

Conjecture. The DM-extension of $\mathcal{T}$ is the set of commutative semicopulas, or, equivalently, S is a commutative semicopula if, and only if, S is the supremum of a subset of t-norms.

5. POINTWISE INDUCED SEMICOPULÆ

Let A and B be semicopulæ and let φ be a mapping from $[0,1]^2$ into $[0,1]$. Then a new mapping $\psi(A,B) : [0,1]^2 \to [0,1]$ is also defined via

$$\psi(A,B)(x,y) := \varphi\left(A(x,y), B(x,y)\right). \tag{5.1}$$

We shall investigate under which conditions the mapping $\psi(A,B)$ just introduced is a semicopula, for *every* choice of A and B in $\mathcal{S}$; in other words, when does (8) induce a map $\psi : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$? When this occurs, we shall say that φ *induces pointwise the binary operation* ψ on $\mathcal{S}$. The following Lemma will be needed.

Lemma 5.1. Let s_1, s_2 and t be points in $[0,1[$ with $s_1 \le s_2$. Then there exist two semicopulæ A and B and two points (x_1,y_1) and (x_2,y_2) in $[0,1]^2$, with $x_1 \le x_2$ and $y_1 \le y_2$ such that

$$A(x_1,y_1) = s_1 \quad \text{and} \quad A(x_2,y_2) = s_2,$$
$$B(x_1,y_1) = t = B(x_2,y_2).$$

Proof. Three cases will be considered.

Case 1: $t \le s_1 \le s_2$. Let A be the ordinal sum given by

$$A = (\langle s_i, s_{i+1}, Z\rangle)_{i\in I},$$

with $I = \{0,1,2,3\}$ and $s_0 = 0$, $s_3 = 1$, so that

$$A(x,y) = \begin{cases} 0, & (x,y) \in [0,s_1[^2, \\ s_1, & (x,y) \in [s_1,s_2[^2, \\ s_2, & (x,y) \in [s_2,1[^2, \\ x \wedge y, & \text{elsewhere,} \end{cases}$$

and let B be the ordinal sum given by

$$B = (\langle t_i, t_{i+1}, Z\rangle)_{i\in I},$$

with $t_0 = 0$, $t_1 = t$, $t_2 = 1$, so that

$$B(x,y) = \begin{cases} 0, & (x,y) \in [0,t[^2, \\ t, & (x,y) \in [t,1[^2, \\ x \wedge y, & \text{elsewhere.} \end{cases}$$

Then

$$A(s_1,s_1) = s_1, \qquad A(s_2,s_2) = s_2, \qquad B(s_1,s_1) = t = B(s_2,s_2).$$

Case 2: $s_1 \le t \le s_2$. Choose B as in the previous case and let A be the frame semicopula defined by

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [s_1,1[^2, \\ s_1, & (x,y) \in [s_1,1[^2 \setminus]t,1[^2, \\ t, & (x,y) \in]t,1[^2 \setminus [s_2,1[^2, \\ s_2, & (x,y) \in [s_2,1[^2, \\ x \wedge y, & x \vee y = 1. \end{cases}$$

Then

$$A(t,t) = s_1, \qquad A(s_2,s_2) = s_2 \qquad \text{and} \qquad B(t,t) = B(s_2,s_2) = t.$$

Case 3: $s_1 \le s_2 \le t$. Choose B as in two previous cases and define A to be the frame semicopula

$$A(x,y) := \begin{cases} 0, & (x,y) \in [0,1[^2 \setminus [t,1[^2, \\ s_1, & (x,y) \in [t,1[^2 \setminus [x_1,1[^2, \\ s_2, & (x,y) \in [x_1,1[^2, \\ x \wedge y, & x \vee y = 1. \end{cases}$$

where we have chosen the point x_1 subject to the only condition $t < x_1 < 1$. Then

$$A(t,t) = s_1, \qquad A(x_1,x_1) = s_2, \qquad B(x_1,x_1) = B(t,t) = t,$$

which proves the assertion. □

Theorem 5.1. The following statements are equivalent:

(a) for all semicopulas A and B, $\psi(A,B)$ is a semicopula;

(b) for every $s \in [0,1[$ the functions $t \mapsto \varphi(t,s)$ and $t \mapsto \varphi(s,t)$ are increasing and $\varphi(x,x) = x$ for every $x \in [0,1]$.

Proof. (a) $\Longrightarrow$ (b) If $\psi(A,B)$ is a semicopula, then

$$x = \psi(A,B)(x,1) = \varphi\left(A(x,1), B(x,1)\right) = \varphi(x,x).$$

Let s_1, s_2 and t be in $[0,1[$ with $s_1 \le s_2$. Then, because of Lemma 5.1, there are two points (x_1,y_1) and (x_2,y_2) in $[0,1]^2$ with $x_1 \le x_2$ and $y_1 \le y_2$ such that $A(x_1,y_1) = s_1$, $A(x_2,y_2) = s_2$ and $B(x_1,y_1) = B(x_2,y_2) = t$. Therefore

$$\begin{aligned} \varphi(s_1,t) &= \varphi\left(A(x_1,y_1), B(x_1,y_1)\right) = \psi(A,B)(x_1,y_1) \le \psi(A,B)(x_2,y_2) \\ &= \varphi\left(A(x_2,y_2), B(x_2,y_2)\right) = \varphi(s_2,t). \end{aligned}$$

In an analogous manner, one proves that, for all $s \in [0,1[$, the function $t \mapsto \varphi(s,t)$ is increasing.

The converse implication, (b) $\Longrightarrow$ (a), is just a matter of a straightforward verification. □

As a consequence of the preceding theorem, every idempotent aggregation operator induces pointwise a binary operation on $\mathcal{S}$. Examples of idempotent aggregation operators are given, for instance, by the geometric and harmonic means and, in general, the quasi-arithmetic mean, defined by

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$$

for every stricly increasing bijection f of $[0,1]$.

It is known from [14] that the *kernel property* completely characterizes the functions that induce pointwise a binary operation on the class $\mathcal{Q}$ of quasi-copulas. In the case of copulas, instead, this problem is still open.

6. MULTIVARIATE SEMICOPULÆ

The notion of semicopula can be extended in a natural way to the n-dimensional case ($n \geq 3$).

Definition 6.1. A function $S : [0,1]^n \to [0,1]$ is said to be an *n-semicopula* if it satisfies the two following conditions

(a) $S(x_1, x_2, \ldots, x_n) = x_i$ for x_i in $[0,1]$ and $x_j = 1$ for all $j \neq i$;

(b) $S(x_1, x_2, \ldots, x_n)$ is increasing in each place.

As a convention, the identity on $[0,1]$, $id_{[0,1]}$, is the only 1-semicopula.

Given a family of n-semicopulas $\{S_n\}_{n\in\mathbf{N}}$, the corresponding aggregation operator $A : \cup_{n\in\mathbf{N}} [0,1]^n \to [0,1]$, where $A_n = S_n$ for all $n \in \mathbf{N}$, has neutral element 1 and annihilator element 0.

Higher dimensional semicopulæ are easily constructed from lower dimensional ones, in view of the following results, the easy proof of which will not be reproduced here.

Proposition 6.1. Let H be a 2-semicopula and let S_m and S_n be, respectively, an m-semicopula and an n-semicopula ($m, n \in \mathbf{N}$); then the function $S : [0,1]^{m+n} \to [0,1]$ defined by

$$S(x_1, \ldots, x_{m+n}) := H\left(S_m(x_1, \ldots, x_m), S_n(x_{m+1}, \ldots, x_{m+n})\right) \qquad (6.1)$$

is an $(m+n)$-semicopula.

Aggregation operators of type (9) are called *double aggregation operators*; they allow to combine two input lists of information coming from different sources into a single output (see [6] for more details).

Proposition 6.2. Let $S_1, S_2, \ldots, S_n$ be bivariate semicopulæ; then the function $S : [0,1]^{n+1} \longrightarrow [0,1]$ defined by

$$S(x_1, x_2, \ldots, x_{n+1}) := S_n\left(S_{n-1}\left(S_{n-2}\left(\ldots S_3\left(S_2\left(S_1(x_1, x_2), x_3\right), x_4\right), \ldots, x_n\right), x_{n+1}\right)\right)$$

is an $(n+1)$-semicopula.

In the opposite direction we can construct lower dimensional semicopulas from higher dimensional ones.

Proposition 6.3. Any m-marginal, $m \geq 2$, of an n-semicopula S_n, $m < n$ is an m-semicopula, viz., if S_n is an n-semicopula, then the function $S_m : [0,1]^m \longrightarrow [0,1]$ defined by

$$S_m(x_1, x_2, \ldots, x_m) = S_n(x_1, x_2, \ldots, x_m, 1, 1, \ldots, 1)$$

is an m-semicopula, and so is any function obtained from it by permuting its arguments.

Propositions 6.1, 6.2 and 6.3 can be analogously proved also in the case of quasi-copulas.

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