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WEAK STRUCTURE AT INFINITY AND ROW-BY-ROW DECOUPLING FOR LINEAR DELAY SYSTEMS

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We consider the row-by-row decoupling problem for linear delay systems and show some close connections between the design of a decoupling controller and some particular structures of delay systems, namely the so-called weak structure at infinity. The realization by static state feedback of decoupling precompensators is studied, in particular, generalized state feedback laws which may incorporate derivatives of the delayed new reference.

Keywords: structure at infinity, row-by-row decoupling, delay systems
AMS Subject Classification: 93B10, 93C23

1. INTRODUCTION

For linear finite dimensional systems, the structure at infinity or the Smith–McMillan form at infinity are well known tools for the characterization of the solvability of some control problems like model matching [7], disturbance rejection (see for example [8]), and row-by-row decoupling [2]. Connections with Silverman's structure algorithm have also been established [17]. For linear infinite dimensional systems and in the particular case of bounded operators, the structure at infinity was introduced by Hautus [4] and later described in several equivalent ways and used for the characterization of solvability conditions for some control problems in [9]. A particular attention was then paid to the class of linear delay systems, with a first contribution by the present authors [10]. However, the structure at infinity defined there is too weak to prevent the potential compensators from being anticipative, as it was pointed out in [16]. Later, in [12] has been introduced the concept of *strong* structure at infinity (which can only be defined for some classes of infinite dimensional systems) for which non-anticipative solutions to control problems can be designed and realized by static state feedback. In the present paper, we use the *weak* structure at infinity in order to design a broader class of precompensators achieving row-by-row decoupling. This may be compared to [14] where disturbance rejection was considered. These precompensators are decomposed into two parts: a strong proper precompensator which may be realized by static state feedback and a weak proper precompensator which can be realized by generalized static state feedback, namely feedback which contains the derivative of the new control. The results given here are in a general form at least for systems with commensurate delays. If the new control is not smooth enough, then the decoupling problem cannot be solved by generalized static state feedback.

The paper is organized as follows. In Section 2 we describe the delay system considered in the paper and the problem of decoupling. In Section 3 we give basic notions and recall classical results concerning linear systems without delays, then we recall some notions and results for systems with delays in Section 4. In Section 5 we solve the row-by-row decoupling problem for delay systems in a general framework.

2. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

2.1. System description

We consider linear time-invariant systems with delays described by:

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-1) + B_0u(t) \\ y(t) = C_0x(t) \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ is the control input, $y(t) \in \mathcal{Y} \approx \mathbb{R}^m$ is the output to be controlled. Without loss of generality, we can assume that B_0 is of full column rank. In order to simplify the notation and some computations, we limit ourselves to systems with a single delay in the state. All results and considerations given here remain valid for systems with several commensurate delays in the state, in the control and in the output.

The transfer function matrix of system (1) is

$$T(s,e^{-s}) = C_0(sI - A_0 - A_1e^{-s})^{-1}B_0$$

and may be expanded in two different ways, namely as a power series expansion, either in the variable e^{-s} (with coefficients function of s) or in the variable s (with coefficients function of e^{-s}). The first expansion is

$$T(s,e^{-s}) = \sum_{j=0}^{\infty} T_j(s)e^{-js}, \tag{2}$$

where $T_j(s) = C_0(sI - A_0)^{-1}\left[A_1(sI - A_0)^{-1}\right]^j B_0$. Each matrix $T_j(s)$ may be decomposed using the following constant matrices introduced by Kirillova and Churakova and compared with other tools in [19]:

$$\begin{aligned} Q_i(j) &= A_0Q_{i-1}(j) + A_1Q_{i-1}(j-1), \\ Q_0(0) &= I,\ Q_i(j) = 0,\ i < 0 \quad \text{or} \quad j < 0. \end{aligned} \tag{3}$$

We have

$$T_j(s) = \sum_{i=0}^{\infty} C_0Q_i(j)B_0s^{-(i+1)}.$$

The other expression, which will be used in this paper, is the following one

$$T(s, e^{-s}) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} C_0 Q_i(j) B_0 e^{-js} \right) s^{-(i+1)}. \tag{4}$$

We can remark that

$$\sum_{j=0}^{i} C_0 Q_i(j) B_0 e^{-js} = C_0 (A_0 + A_1 e^{-s})^i B_0. \tag{5}$$

These expressions may be obtained by a simple calculation using relations (3), see [16, 19].

2.2. Problem formulation

Our objective concerns decoupling of systems like (1).

The "open-loop" definition of decoupling is the following: Find a precompensator $K(s, e^{-s})$ and non identically zero scalar transfer functions $h_i(s, e^{-s})$, $i = 1, \ldots, m$, such that

$$T(s, e^{-s}) K(s, e^{-s}) = \operatorname{diag}\{h_1(s, e^{-s}), \ldots, h_m(s, e^{-s})\}.$$

We are interested in feedback implementations of such decoupling precompensators, when they exist, and we want to connect the properties of $K(s, e^{-s})$ which make it realizable in a feedback form and the type of, more or less restricted, feedback laws which may be used.

We have previously shown [12] that any decoupling solution $K(s, e^{-s})$ belonging to some particular class of precompensators, called *strong biproper* (see Section 4), is equivalent to a *static state feedback control* law of the type: $u(s) = F(e^{-s})x(s) + G(e^{-s})v(s)$, where $F(e^{-s})$ and $G(e^{-s})$ are rational transfer function matrices with respect to the variable e^{-s}, $F(e^{-s})$ being strong proper and $G(e^{-s})$ strong biproper.

The aim of the present paper is to consider a broader class of decoupling precompensators, called *weak biproper* (see Section 4) and to show their equivalence with *generalized static state feedback control* laws of the type: $u(s) = F(e^{-s})x(s) + G(s, e^{-s})v(s)$, where $F(e^{-s})$ is strong proper and $G(s, e^{-s})$ weak biproper. This amounts to accepting in the control law some delayed derivatives of the new reference input $v(t)$ and looking for more general solutions, assuming that the reference input $v(t)$ is smooth enough for all its involved derivatives to exist and to be bounded.

Remark 1. $F(e^{-s})$ and $G(s, e^{-s})$ are proper transfer function matrices (in the usual sense) with respect to the variable $z = e^s$ and with constant coefficients for $F(e^{-s})$ and rational (including polynomials in s) for $G(s, e^{-s})$. The following illustrative example shows how this may occur. Let us consider the system

$$T(s, e^{-s}) = \begin{bmatrix} s^{-3} & (s^{-4} + s^{-2})e^{-s} \\ 0 & s^{-1} \end{bmatrix}.$$

It can be easily checked that there is no static state feedback law which decouples this system. However, $K(s,e^{-s}) = \begin{bmatrix} 1 & -(s^{-1}+s)e^{-s} \\ 0 & 1 \end{bmatrix}$ is a decoupling precompensator realizable by the generalized static feedback with $F(e^{-s}) = -\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{-s}$ and $G(s,e^{-s}) = \begin{bmatrix} 1 & -se^{-s} \\ 0 & 1 \end{bmatrix}$. In the time domain the control law is given by: $u_1(t) = -x_4(t-1) + v_1(t) - \dot{v}_2(t-1)$ and $u_2(t) = v_2(t)$. More details will be given in Section 5.

3. FINITE DIMENSIONAL SYSTEMS

The basic notions used in this paper are notions of properness. Let us recall in this section the case of a classical finite dimensional linear system given by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{6}$$

where $x(t) \in \mathcal{X} \approx \mathbb{R}^n$ is the state, $u(t) \in \mathcal{U} \approx \mathbb{R}^m$ is the control input, $y(t) \in \mathcal{Y} \approx \mathbb{R}^m$ is the output to be controlled. The matrix B is of full column rank. The transfer function matrix of the system is

$$T(s) = C(sI - A)^{-1}B.$$

The matrix $T(s)$ is rational and strictly proper, the properness being defined by the following.

Definition 2. A complex valued rational function $f(s)$ is called proper if $\lim f(s)$ is finite when $|s| \to \infty$. It is called strictly proper if this limit is 0. It is called biproper if this limit is invertible.

As for linear systems in finite dimensional spaces one considers in fact only rational functions, properness means that the degree of the numerator is less than or equal to the degree of the denominator and strict properness means that the equality cannot hold. A fundamental result is the existence of a canonical form at infinity (Smith–McMillan form at infinity) for strictly proper matrices (but also for general rational matrices, see for instance [6]).

Theorem 3. Given a strictly proper system, with transfer $T(s)$, there exist (non unique) biproper matrices $B_1(s)$ and $B_2(s)$ such that

$$B_1(s)\,T(s)\,B_2(s) = \begin{bmatrix} \Delta(s) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Delta(s) = \operatorname{diag}\{s^{-n_1}, \dots, s^{-n_r}\}$. The strictly positive integers n_i are called the orders of the zero at infinity and the list of integers $\{n_1, \dots, n_r\}$ is the *structure at infinity* and $\Delta(s)$ is denoted by $\Sigma_\infty(C, A, B)$ or $\Sigma_\infty T(s)$. The integer r is the rank of the system.

Because of the basic properties of the Laplace transform, the structure at infinity of $T(s)$ allows to describe the behavior of the system (6) at time $t = 0$.

Another important tool which is useful to characterize several properties of linear systems is the maximal (A, B)-invariant subspace contained in $\operatorname{Ker} C$, see [20]. It will be denoted by $\mathcal{V}_*(\operatorname{Ker} C, A, B)$. We shall also use the alternative expression of this subspace given by Hautus [5]:

$$\begin{aligned} &\mathcal{V}_*(\operatorname{Ker} C, A, B) = \{x \in \operatorname{Ker} C : x = (sI - A)\xi(s) - B\omega(s)\}, \\ &\xi(s),\ \omega(s) \quad \text{strictly proper}, \quad \xi(s) \in \operatorname{Ker} C,\ |s| > s_0. \end{aligned} \tag{7}$$

This is called a $(\xi - \omega)$-representation for $\mathcal{V}_*(\operatorname{Ker} C, A, B)$. The following result is well known and has been established by several authors.

Theorem 4. The following propositions are equivalent:

1. There exists a biproper precompensator $K(s)$ such that
$$T(s)K(s) = \operatorname{diag}\{h_1(s), \ldots, h_m(s)\}.$$
2. The global and the row by row structures at infinity are equal:
$$\Sigma_\infty(C, A, B) \quad = \quad \begin{bmatrix} \Sigma_\infty(c_1, A, B) \\ \vdots \\ \Sigma_\infty(c_m, A, B) \end{bmatrix},$$
where Σ_∞ denotes the canonical form at infinity for the given system, c_i, $i = 1, \ldots, m$ being the rows of the matrix C.
3. The so-called Falb–Wolovich matrix
$$D = \begin{bmatrix} c_1 A^{n_1 - 1} B \\ \vdots \\ c_m A^{n_m - 1} B \end{bmatrix},$$
is invertible. The integer n_i, $i = 1, \ldots, m$ is the order of the zero at infinity of each row subsystem: $c_i A^{n_i - 1} B \neq 0$ and $c_i A^j B = 0$ for $j < n_i - 1$.
4. There exists a feedback law $u(t) = Fx(t) + Gv(t)$, such that
$$C(sI - A - BF)^{-1} BG = \operatorname{diag}\{h_1(s), \ldots, h_m(s)\}.$$
5. $\operatorname{Im} B = \sum_{i=1}^m \operatorname{Im} B \cap \mathcal{V}_*(C_i, A, B)$, $\operatorname{Im} B \cap \mathcal{V}_*(C_i, A, B) \neq \{0\}$, where $C_i = \bigcap_{j \neq i}^m \operatorname{Ker} c_j$.

The relation between the precompensator $K(s)$ and the feedback law (F, G) is given by
$$K(s) = \left(I - F(sI - A)^{-1} B\right)^{-1} G. \tag{8}$$

Proof. For the proofs of the equivalence of the statements $1-4$ see for example [1, 2, 3, 20] and references given there. Statement 5 being less "classical". let us give a proof of the equivalence between 1 and 5, based on the $(\xi-\omega)$-representation (7) of $\mathcal{V}_*(\operatorname{Ker} C, A, B)$.

If the system is decouplable by precompensator (statement 1), there exists a biproper precompensator $K(s)$ such that

$$C(sI-A)^{-1}BK(s) = \operatorname{diag}\{h_1(s), \ldots, h_m(s)\}.$$

The matrix $K(s)$ may be written as $K(s) = V + W(s)$, where V is a non singular constant matrix (because $K(s)$ is biproper) and $W(s)$ is strictly proper. Let v_i and $\omega_i(s)$ be the ith columns of the matrices V and $W(s)$ respectively. Then $\{v_i, i = 1, \ldots, m\}$ forms a basis in $\mathbb{R}^m$. If we take

$$\xi_i(s) = (sI-A)^{-1}B(v_i + \omega_i(s)),$$

then $\xi_i(s) \in \mathcal{C}_i$. On the other hand $\xi_i(s)$ and $\omega_i(s)$ are strictly proper. Hence, for all $i = 1, \ldots, m$ one has

$$Bv_i \in \mathcal{V}_*(\mathcal{C}_i, A, B)$$

As $\{v_i, i = 1, \ldots, m\}$ forms a basis of $\mathcal{U}$ and B is assumed to be of full column rank, then $\{Bv_i, i = 1, \ldots, m\}$ is a basis of $\operatorname{Im} B$. Hence, statement 5 holds.

Conversely assume that condition 5 is satisfied. Then for $\{v_i, i = 1, \ldots, m\}$ linearly independent, one has

$$Bv_i = (sI-A)\xi_i(s) - B\omega_i(s)$$

with $\xi_i(s), \omega_i(s)$ strictly proper and $\xi_i(s) \in \mathcal{C}_i$, i.e. $C\xi_i(s) = c_i\xi_i(s)$. For $V = [v_1 \ \ldots \ v_m]$ and $W(s) = [\omega_1(s) \ \ldots \ \omega_m(s)]$, if we define $K(s) = V + W(s)$, then $K(s)$ is biproper and

$$C(sI-A)^{-1}BK(s) = \operatorname{diag}\{c_1\xi_1(s), \ldots, c_m\xi_m(s)\}.$$

This means that the system is row-by-row decoupled by the precompensator $K(s)$ and we have $h_i(s) = c_i\xi_i(s)$. □

4. STRUCTURAL NOTIONS FOR DELAY SYSTEMS

The transfer function matrix of a delay system is not rational in s. Moreover, it is not analytical at infinity. Therefore the notions of properness must be precised.

Definition 5. A complex valued function $f(s)$ is called weak proper if $\lim f(s)$ is finite when $s \in \mathbb{R}$ tends to ∞. It is called strictly weak proper if this limit is 0. A matrix $B(s)$ is weak biproper if it is weak proper and if this limit is invertible. Weak proper is replaced by strong proper if the same occurs when $s \in \mathbb{C}$ and $\Re\mathrm{e}(s) \to \infty$.

This is a general definition, but in this paper we use this notion for functions $f(s, e^{-s})$ rational with respect to variables s and e^s.

Let us precise that a matrix $B(s)$ is weak (respectively strong) biproper iff $B(s) = B_0 + W(s)$, where B_0 is constant and invertible and $W(s)$ is strictly weak (resp. strong) proper.

It is obvious that strong properness implies weak properness. If the function is analytical at infinity both notions coincide, because the limits at infinity are the same. The strong properness was used in [4] and [9] in the description of the structure at infinity for infinite dimensional systems. In [10, 16, 14] the weak notion was used in order to define the structure at infinity of delay systems and to solve some control problems, even if in [10, 16] this notion was not yet clearly precised and separated from the strong one.

Let us recall the following result using weak properness and introduced in [12].

Theorem 6. Given a system like (1), with transfer $T(s, e^{-s})$, there exist (non unique) weak biproper rational matrices $B_1(s, e^{-s})$ and $B_2(s, e^{-s})$ such that

$$B_1(s,e^{-s})T(s,e^{-s})B_2(s,e^{-s}) = \begin{bmatrix} \Delta_0(s) & 0 & \cdots & 0 & 0 \\ 0 & \Delta_1(s)e^{-s} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \Delta_k(s)e^{-ks} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\Delta_i(s) = \mathrm{diag}\{s^{-n_{i,1}}, \ldots, s^{-n_{i,j_i}}\}$ and $n_{i,j} \leq n_{i,j+1}$, $i = 1, \ldots, k$. The list of integers

$$\{n_{i,j},\ i = 1, \ldots, k; j = j_1, \ldots, j_i\}$$

is called the *weak structure at infinity* of the system (1) and is noted $\Sigma_\infty^w T(s, e^{-s})$.

Some additional assumptions may insure that the weak structure at infinity also gives a strong structure at infinity: in that case the matrices $B_i(s, e^{-s})$ are strong biproper (see [12]).

5. THE ROW-BY-ROW DECOUPLING PROBLEM FOR DELAY SYSTEMS

Our purpose is to give, for a linear time delay system, a more general solution for the row-by-row decoupling problem.

The given problem was studied by several authors [15, 18, 16] but only partial solutions were given. In [13] an abstract geometric approach was developed using Hautus' definition of (A, B)-invariant subspaces. The result given there is limited to the strong definition of properness. At the end of the present paper, we shall extend this result to the weak proper case. Note however that it is difficult to compute the corresponding subspaces. Our approach developed first in [14] for the disturbance rejection problem is extended here to the row-by-row decoupling problem. The weak structure at infinity given in the previous section allows to give the following general formulation and solution for this control problem by generalized static state feedback.

Theorem 7. The following propositions are equivalent:

1. The row-by-row decoupling problem for the delay system (1) is solvable by a *weak biproper* precompensator:

$$T(s,e^{-s})K(s,e^{-s}) = \operatorname{diag}\left\{h_1(s,e^{-s}),\ldots,h_m(s,e^{-s})\right\}.$$

2. The global and the row by row *weak structures at infinity* are equal:

$$\Sigma_\infty^w(C_0,A_0,A_1,B_0) = \begin{bmatrix} \Sigma_\infty^w(c_1,A_0,A_1,B_0) \\ \vdots \\ \Sigma_\infty^w(c_m,A_0,A_1,B_0) \end{bmatrix},$$

where c_i's are the rows of the matrix C_0.

3. The generalized Falb-Wolovich matrix:

$$D_0 = \begin{bmatrix} c_1 Q_{n_1-1}(k_1)B_0 \\ \vdots \\ c_m Q_{n_m-1}(k_m)B_0 \end{bmatrix},$$

is invertible, where for each row i the integers n_i and k_i are such that: $c_i Q_{n_i-1}(k_i)B_0 \neq 0$ and $c_i Q_l(j)B_0 = 0$ for $l < n_i - 1$ and $j < k_i$.

4. The decoupling problem is solvable by *generalized static state feedback*

$$u = F(e^{-s})x + G(s,e^{-s})v,$$

where $F(e^{-s})$ is strong proper and $G(s,e^{-s})$ weak biproper.

Proof.
$1 \Rightarrow 2$. Let $A(e^{-s}) = A_0 + A_1 e^{-s}$. Assume that condition 1 is satisfied. This gives, for each i:

$$c_i(sI - A(e^{-s}))^{-1}B_0 K(s,e^{-s}) = [0 \ \cdots \ h_i(s,e^{-s}) \ \cdots \ 0]$$

and then each row i of the system has the structure at infinity of

$$[0 \ \cdots \ h_i(s,e^{-s}) \ \cdots \ 0].$$

On the other hand the (weak) global structure at infinity of $T(s,e^{-s})$ is invariant under the multiplication by (weak) biproper $K(s,e^{-s})$, namely is that of the matrix

$$\operatorname{diag}\left\{h_1(s,e^{-s}),\ldots,h_m(s,e^{-s})\right\},$$

which means that 2 holds.

$2 \Rightarrow 3$. Suppose that condition 2 is verified. The integers n_i and k_i for $i = 1, \ldots, m$ describe the weak structure at infinity of each row i. Then

$$T(s, e^{-s}) = \operatorname{diag}\left\{s^{-n_1}e^{-k_1 s}, \ldots, s^{-n_m}e^{-k_m s}\right\}(D_0 + W(s, e^{-s})) \tag{9}$$

and $W(s, e^{-s})$ is strictly weak proper. If D_0 is not invertible, then by elementary operations one can reduce some row or column of D_0 and then the global structure at infinity would not coincide with $\operatorname{diag}\{s^{-n_1}e^{-k_1 s}, \ldots, s^{-n_m}e^{-k_m s}\}$, which is not possible by hypothesis. Then, 3 holds.

$3 \Rightarrow 4$. Suppose that condition 3 is verified. Then, from factorization (9) (which is always true), $D_0 + W(s, e^{-s})$ is weak biproper, because

$$\lim_{\mathbb{R} \ni s \to \infty} (D_0 + W(s, e^{-s})) = D_0,$$

and D_0 is invertible. Let us then denote $K(s, e^{-s}) \stackrel{\text{def}}{=} (D_0 + W(s, e^{-s}))^{-1}$. This compensator $K(s, e^{-s})$ is also weak biproper and achieves decoupling:

$$T(s, e^{-s})K(s, e^{-s}) = \operatorname{diag}\left\{s^{-n_1}e^{-k_1 s}, \ldots, s^{-n_m}e^{-k_m s}\right\},$$

which means that (1) holds. We shall now show that there exist strong proper $F(e^{-s})$ and weak proper $G(s, e^{-s})$ which realize this particular $K(s, e^{-s})$, namely:

$$K(s, e^{-s}) = \left[I - F(e^{-s})(sI - A(e^{-s}))^{-1}B_0\right]^{-1} G(s, e^{-s}). \tag{10}$$

Let us now analyze the matrix $W(s, e^{-s})$. Each entry of this matrix is rational in the first argument s, namely if we denote each entry by $w(s, z)$, with $z := e^{-s}$ (in order to distinguish both arguments), one can decompose it as follows:

$$w(s, z) = w_0(z) + w_1(s, z) + w_2(s, z),$$

where $w_0(z)$ is strictly proper (in the classical sense) with respect to z, $w_1(s, z)$ is strictly proper with respect to s, and $w_2(s, z)$ is polynomial in s (not strictly proper). This gives a decomposition of the matrix $W(s, z)$:

$$W(s, z) = W_0(z) + W_1(s, z) + W_2(s, z),$$

with the same properties. This implies that $W_1(s, e^{-s})$ is strictly strong proper and $W_2(s, e^{-s})$ is strictly weak proper. This allows to write $D_0 + W(s, e^{-s})$ in the form

$$D_0 + W(s, e^{-s}) = D(e^{-s}) + W_1(s, e^{-s}) + W_2(s, e^{-s}),$$

where $D(e^{-s}) = D_0 + W_0(e^{-s})$.

If c_p denotes the pth row of the matrix C_0, then, according to the decomposition of the transfer function matrix (4), one can see that each row p of the matrix $D(e^{-s})$ can be written as

$$\sum_{j=0}^{\infty} c_p Q_{n_p - 1}(k_p + j) B_0 e^{-js}, \quad p = 1, \ldots, m.$$

In the same way, each row of $W_1(s,e^{-s})$ may be decomposed as

$$\sum_{i=1}^{\infty}\left(\sum_{j=0}^{n_p+i-1} c_p Q_{n_p+i-1}(k_p+j)B_0 e^{-js}\right)s^{-i}, \quad p=1,\dots,m.$$

Let us denote

$$K_1(s,e^{-s}) \stackrel{\text{def}}{=} (D(e^{-s})+W_1(s,e^{-s}))^{-1}$$

and

$$K_2(s,e^{-s}) \stackrel{\text{def}}{=} K(s,e^{-s})-K_1(s,e^{-s}).$$

A simple calculation gives

$$\begin{aligned} K_2(s,e^{-s}) &= -(D(e^{-s})+W_1(s,e^{-s}))^{-1}W_2(s,e^{-s})K(s,e^{-s}) \\ &= -K_1(s,e^{-s})W_2(s,e^{-s})K(s,e^{-s}). \end{aligned}$$

Hence, $K_1(s,e^{-s})$ is strong biproper and $K_2(s,e^{-s})$ is strictly weak proper because $W_2(s,e^{-s})$ is strictly weak proper and $K(s,e^{-s})$ is weak biproper.

We first give a feedback realization of $K_1(s,e^{-s})$ and then show that $K(s,e^{-s})$ is realizable by generalized static state feedback.

Let us define

$$V_1(s,e^{-s}) \stackrel{\text{def}}{=} W_1(s,e^{-s})B_g^{-1}$$

where B_g^{-1} is the left inverse of the (full column rank) matrix B_0. This means that

$$W_1(s,e^{-s}) = V_1(s,e^{-s})B_0.$$

More precisely, from (5), each row of $V_1(s,e^{-s})$ is given by

$$\sum_{i=1}^{\infty} c_p A^{n_p-1+i}(e^{-s})s^{-i}, \quad p=1,\dots,m.$$

Let us denote

$$F_1(s,e^{-s}) \stackrel{\text{def}}{=} -D^{-1}(e^{-s})V_1(s,e^{-s})(sI-A(e^{-s})).$$

Let us show, by a simple calculation, that $F_1(s,e^{-s})$ does not depend on the first argument. In order to do that, let us consider each row $v_p(s,e^{-s}), p=1,\dots,m$, of the matrix $V_1(s,e^{-s})(sI-A(e^{-s}))$. It may be written as

$$v_p(s,e^{-s}) = \sum_{i=1}^{\infty} c_p A^{n_p-1+i}(e^{-s})s^{-i+1} - \sum_{i=1}^{\infty} c_p A^{n_p+i}(e^{-s})s^{-i}$$

after a change of indices in the first sum, we get

$$v_p(s,e^{-s}) = \sum_{i=0}^{\infty} c_p A^{n_p+i}(e^{-s})s^{-i} - \sum_{i=1}^{\infty} c_p A^{n_p+i}(e^{-s})s^{-i}$$

and then

$$v_p(s, e^{-s}) = c_p A^{n_p}(e^{-s}),$$

for all $p = 1, \ldots, m$ which means that $V_1(s, e^{-s})(sI - A(e^{-s}))$ does not depend on the first argument. This implies that $F_1(s, e^{-s})$ also does not depend on the first argument.

Let us then denote:

$$F(e^{-s}) \stackrel{\text{def}}{=} F_1(s, e^{-s}), \quad G_1(e^{-s}) \stackrel{\text{def}}{=} D^{-1}(e^{-s}).$$

Then

$$-G_1^{-1}(e^{-s})F(e^{-s})(sI - A(e^{-s}))^{-1}B_0 = V_1(s, e^{-s})B_0 = W_1(s, e^{-s})$$

and

$$G_1^{-1}(e^{-s}) - G_1^{-1}(e^{-s})F(e^{-s})(sI - A(e^{-s}))^{-1}B_0 = D(e^{-s}) + W_1(s, e^{-s}).$$

This obviously gives

$$\begin{array}{rcl} K_1(s, e^{-s}) & \stackrel{\text{def}}{=} & (D(e^{-s}) + W_1(s, e^{-s}))^{-1} \\ & = & \left[I - F(e^{-s})(sI - A(e^{-s}))^{-1}B_0\right]^{-1} G_1(e^{-s}) \end{array} \tag{11}$$

which means that $F(e^{-s})$ and $G_1(e^{-s})$ realize $K_1(s, e^{-s})$, in the sense that:

$$C_0\left[sI - A(e^{-s}) - B_0F(e^{-s})\right]^{-1} B_0G_1(e^{-s}) = C_0[sI - A(e^{-s})]^{-1}B_0K_1(s, e^{-s}).$$

Let us then denote

$$G_2(s, e^{-s}) \stackrel{\text{def}}{=} \left[I - F(e^{-s})(sI - A(e^{-s}))^{-1}B_0\right] K_2(s, e^{-s}),$$

which gives

$$K_2(s, e^{-s}) = \left[I - F(e^{-s}()sI - A(e^{-s}))^{-1}B_0\right]^{-1} G_2(s, e^{-s}). \tag{12}$$

It is essential to note that $G_2(s, e^{-s})$ is strictly weak proper because $K_2(s, e^{-s})$ has the same property and the other factor is strong biproper. Then, by construction, terms in s^k, with positive integer k, are multiplied by strictly positive powers of the delay operators e^{-s}.

From (11) and (12) we get

$$K(s, e^{-s}) = K_1(s, e^{-s}) + K_2(s, e^{-s}) = \left[I - F(e^{-s})(sI - A(e^{-s}))^{-1}B_0\right]^{-1} G(s, e^{-s}),$$

where $G(s, e^{-s}) = G_1(e^{-s}) + G_2(s, e^{-s})$. This means that $F(e^{-s})$ and $G(s, e^{-s})$ realize $K(s, e^{-s})$:

$$C_0(sI - A(e^{-s}) - B_0F(e^{-s}))^{-1}B_0G(s, e^{-s}) = C_0(sI - A(e^{-s}))^{-1}B_0K(s, e^{-s}),$$

and then

$$C_0(sI - A(e^{-s}) - B_0F(e^{-s}))^{-1}B_0G(s, e^{-s}) = \text{diag}\left\{s^{-n_1}e^{-k_1 s}, \ldots, s^{-n_m}e^{-k_m s}\right\}.$$

Note that, as $G_2(s,e^{-s})$ may contain weak proper terms (namely terms like $s^k e^{-ps}$), in the decomposition of $G(s,e^{-s})$, such generalized terms appear also.
$4 \Rightarrow 1$. Suppose now that condition 4 holds, i.e. the decoupling problem is solvable by generalized static state feedback. Then

$$\begin{aligned} T_{F,G}(s,e^{-s}) &\stackrel{\text{def}}{=} C_0(sI - A(e^{-s}) - B_0F(e^{-s}))^{-1}B_0G(s,e^{-s}) \\ &= \operatorname{diag}\left\{h_1(s,e^{-s}),\ldots,h_m(s,e^{-s})\right\}. \end{aligned}$$

where $h_i(s,e^{-s}) \neq 0$ for each i. Then $T_{F,G}(s,e^{-s}) = T(s,e^{-s})K(s,e^{-s})$ where $K(s,e^{-s})$ given by

$$K(s,e^{-s}) = \left[I - F(e^{-s})(sI - A(e^{-s}))^{-1}B_0\right]^{-1} G(s,e^{-s})$$

is obviously weak biproper (due to the form of $F(e^{-s})$ and $G(s,e^{-s})$). Then

$$T(s,e^{-s})K(s,e^{-s}) = \operatorname{diag}\left\{h_1(s,e^{-s}),\ldots,h_m(s,e^{-s})\right\},$$

i.e. condition 1 is satisfied, which ends the proof of Theorem 4. □

Let us illustrate these generalized static state feedback solutions on the example given in Remark 1. The corresponding state space representation is given by the matrices:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The structural condition may be easily verified ($n_1 = 3, k_1 = 0, n_2 = 1, k_2 = 0$). The generalized Falb–Wolovich matrix is $D_0 = I$. We have:

$$D(e^{-s}) = D_0, \quad W_1(s,e^{-s}) = \begin{bmatrix} 0 & s^{-1}e^{-s} \\ 0 & 0 \end{bmatrix}, \quad W_2(s,e^{-s}) = \begin{bmatrix} 0 & se^{-s} \\ 0 & 0 \end{bmatrix}.$$

A decoupling precompensator is $K(s,e^{-s}) = \begin{bmatrix} 1 & -(s^{-1}+s)e^{-s} \\ 0 & 1 \end{bmatrix} = K_1(s,e^{-s}) + K_2(s,e^{-s})$, with

$$K_1(s,e^{-s}) = \begin{bmatrix} 1 & -s^{-1}e^{-s} \\ 0 & 1 \end{bmatrix}, \quad K_2(s,e^{-s}) = \begin{bmatrix} 0 & -se^{-s} \\ 0 & 0 \end{bmatrix}.$$

Our calculation gives

$$F(e^{-s}) = -\begin{bmatrix} (1 \;\; 0 \;\; 0 \;\; 0)\, A^3(e^{-s})B_0 \\ (0 \;\; 0 \;\; 0 \;\; 1)\, A(e^{-s})B_0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{-s},$$

and

$$G(s,e^{-s}) = \begin{bmatrix} 1 & -se^{-s} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -se^{-s} \\ 0 & 0 \end{bmatrix}.$$

The decoupled system has the transfer function matrix

$$T(s,e^{-s})K(s,e^{-s}) = \operatorname{diag}\{s^{-3}, s^{-1}\}$$

Corollary 8. If, in Theorem 7, weak (structure or properness) is replaced by strong, then the feedback contains only static terms, and no derivative of the reference is needed.

Proof. The assumptions of the corollary imply that the weak structure at infinity is also the strong structure at infinity [12], this gives $K(s,e^{-s}) = K_1(s,e^{-s})$, and then $G(s,e^{-s}) = G_1(e^{-s})$. The precompensator is realizable by static state feedback. No derivative of the delayed reference is needed. □

In Theorem 7, the geometric formulation was omitted (statement 5 in Theorem 4). The following theorem gives an analogous result. In order to formulate this result, let us introduce the Hautus like definition of (A,B)-invariant subspace for delay systems (see [10] for the introduction of this tool and application to disturbance decoupling for delay systems and [13] for the same statements in terms of strong properness and for the design of strong decoupling precompensator).

For $\mathcal{C}_i = \bigcap_{j\neq i}^m \operatorname{Ker} c_j$, let $\mathcal{V}_\Sigma(\mathcal{C}_i, A(e^{-s}), B_0)$, $i = 1,\dots,m$ be the subspaces

$$\mathcal{V}_\Sigma(\mathcal{C}_i, A(e^{-s}), B_0) = \left\{x \in \mathcal{C}_i : x = (sI - A(e^{-s}))\xi(s,e^{-s}) - B_0\omega(s,e^{-s})\right\},$$

with strictly weak proper $\xi(s,e^{-s})$ and $\omega(s,e^{-s})$ such that $\xi(s,e^{-s}) \in \mathcal{C}_i$ for $s \in \mathbb{R}$ and $s > s_0$.

Theorem 9. The system (1) is decouplable by weak biproper precompensation iff $\operatorname{Im} B_0 = \sum_{i=1}^m \operatorname{Im} B_0 \cap \mathcal{V}_\Sigma(\mathcal{C}_i, A(e^{-s}), B_0)$, with $\operatorname{Im} B_0 \cap \mathcal{V}_\Sigma(\mathcal{C}_i, A(e^{-s}), B_0) \neq \{0\}$.

Proof. The proof is the same as the proof of the equivalence of statements 1 and 5 for Theorem 4. We only need to replace $\mathcal{V}_*$ by $\mathcal{V}_\Sigma$, A by $A(e^{-s})$, B by B_0 and proper by weak proper. □

We have here limited our presentation to systems with a single delay in the state. Note that the statements and some details of the proofs may rather easily be reformulated for systems having also delays in the controls and the outputs.

6. CONCLUSION

In order to solve in a general form and without prediction the row-by-row decoupling problem for delay systems, we use the weak structure at infinity which is well defined for linear time delay systems. The general solution is of feedback type. However, we need some smoothness of the new reference v. This is the counterpart of the generality. For practical use, this means that we can use only some classes of references or, if the reference is not smooth enough, we need in fact very high gain in approximation. The results given here may be also considered, with some slight modification, for more general delay systems: with several delays in the state or of neutral type.

The problem of stability is not investigated here. It is obvious that, without stability, the realization of such generalized decoupling techniques would be of no practical interest. For systems without delays, the existence of stable solutions is independent on the fact that derivatives of the reference are or are not used: it just depends on conditions relating unstable invariant zeros. For systems with delays, this problem of stability needs further investigations, taking also into account that stabilizing control laws may often require distributed delays!... This problem is now under consideration, as well as, for effective implementation, the numerical aspects linked with the realization of such generalized feedback laws.

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