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RESIDUAL IMPLICATIONS AND CO-IMPLICATIONS FROM IDEMPOTENT UNINORMS

DANIEL RUIZ AND JOAN TORRENS

This paper is devoted to the study of implication (and co-implication) functions defined from idempotent uninorms. The expression of these implications, a list of their properties, as well as some particular cases are studied. It is also characterized when these implications satisfy some additional properties specially interesting in the framework of implication functions, like contrapositive symmetry and the exchange principle.

Keywords: t-norm, t-conorm, idempotent uninorm, aggregation, implication function
AMS Subject Classification: 03B52, 06F05, 94D05

1. INTRODUCTION

Introduced in the field of aggregation functions in [21] and [11], uninorms have proved to be useful not only in this field, but also in many others like expert systems, neural networks, fuzzy system modelling, fuzzy logic, etc. There are three different known classes of uninorms, stated in [4], the $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ class, representable uninorms and idempotent uninorms. The first two classes are studied in [11] whereas the third one is studied in [5]. From these studies many other papers on uninorms have appeared, even some generalizations of these operators like in [16]. Moreover, implication operators derived from t-norms are extensively studied, as in [1] and [12], but also those derived from uninorms. Implication functions derived from representable uninorms, as well as from uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, have been studied in [8] and [7], respectively. There are also some works involving idempotent uninorms, like [18] and [19] but, dealing with implication functions, only some few results can be found in [9] and only with respect to left-continuous and right-continuous idempotent uninorms.

Uninorms are a kind of aggregation functions that have proven to be useful in many fields. One of them, where residual implications play an important role, is fuzzy mathematical morphology, see [10] and [13]. Fuzzy morphological operators are defined precisely from idempotent conjunctive uninorms in [13], and the properties of the residual implications of such uninorms are essential to obtain good morphological properties.

The main goal of this paper (which is an extended version with proofs of [20]) is to study those implication functions defined from idempotent uninorms in general. We specially study the case of implications obtained from residuation, that is,

$$I(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$$

for all $x, y \in [0,1]$. In this case we give first the general expression of such implications as well as a list of the properties that they satisfy. It is derived from their expression that all idempotent uninorms with the same associated function g have the same residual implication. It is also proved that some other properties, including contrapositive symmetry, are satisfied only in particular cases: when the associated function of the idempotent uninorm is a strong negation. Another way to define implication functions from disjunctive idempotent uninorms is the one given by

$$I(x,y) = U(N(x),y) \qquad \text{for all} \quad x,y \in [0,1]$$

where N is a strong negation. In the special case when the associated function of U coincide with N, both kinds of implications become extremely close. Moreover, they coincide when U is right-continuous as it was already proved in [9]. The study of the exchange principle is also done and it brings us examples of non left-continuous conjunctive uninorms such that their derived implications satisfy this important property. Finally, the last section of this paper gives a similar study for co-implications.

2. PRELIMINARIES

We assume the reader to be familiar with some basic notions concerning t-norms and t-conorms which can be found for instance in [14]. Also some results on uninorms in general, that will be used in the paper without further mention, can be found in [11] and [14].

Definition 1. (See [11].) A *uninorm* is a two-place function $U : [0,1] \times [0,1] \longrightarrow [0,1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0,1]$, called the *neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.

It is clear that the function U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0,1\}$ and a uninorm U is said conjunctive when $U(1,0) = 0$ and disjunctive when $U(1,0) = 1$.

Definition 2. Let U be a uninorm. If there is a t-norm T and a t-conorm S such that U is given by

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } 0 \leq x,y \leq e \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } e \leq x,y \leq 1 \\ \min(x,y) & \text{if } \min(x,y) < e \leq \max(x,y) \end{cases}$$

then U is said to be in $\mathcal{U}_{\min}$, and if U is given by

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } 0 \leq x, y \leq e \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } e \leq x, y \leq 1 \\ \max(x,y) & \text{if } \min(x,y) \leq e < \max(x,y) \end{cases}$$

then U is said to be in $\mathcal{U}_{\max}$.

Definition 3. A uninorm U with neutral element $e \in (0,1)$ is representable if and only if there is a strictly increasing, continuous function $h : [0,1] \to [-\infty, +\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by

$$U(a,b) = h^{-1}(h(a) + h(b))$$

for all $(a,b) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) \in \{0,1\}$.

Definition 4. A binary operator $U : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be idempotent whenever $U(x,x) = x$ for all $x \in [0,1]$.

In [2], Czogala–Drewniak give the general form of idempotent, associative and increasing binary operators with a neutral element (see also Theorem 3). Particular cases of operators with these properties are of course, idempotent uninorms. A detailed characterization for the cases of left-continuous and right-continuous idempotent uninorms is given in the following theorems.

Theorem 1. (De Baets [5].) A binary operator U is a left-continuous idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with fix point e, satisfying $g(g(x)) \geq x$ for all $x \leq g(0)$ and $g(x) = 0$ for all $x > g(0)$ such that, for all $x, y \in [0,1]$, U is given by

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y \leq g(x) \text{ and } x \leq g(0) \\ \max(x,y) & \text{elsewhere.} \end{cases}$$

Theorem 2. (De Baets [5].) A binary operator U is a right-continuous idempotent uninorm with neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with fix point e, satisfying $g(g(x)) \leq x$ for all $x \geq g(1)$ and $g(x) = 1$ for all $x < g(1)$ such that, for all $x, y \in [0,1]$, U is given by

$$U(x,y) = \begin{cases} \max(x,y) & \text{if } y \geq g(x) \text{ and } x \geq g(1) \\ \min(x,y) & \text{elsewhere.} \end{cases}$$

A complete characterization of Czogala–Drewniak's operators can be found in [15], as follows.

Theorem 3. (Martín–Mayor–Torrens [15].) Let F be a binary operator on $[0,1]$. F is associative, increasing, idempotent and has a neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with $g(e) = e$, $g(x) = 0$ for all $x > g(0)$, $g(x) = 1$ for all $x < g(1)$, satisfying

$$\inf\{y \mid g(y) = g(x)\} \le g(g(x)) \le \sup\{y \mid g(y) = g(x)\} \tag{1}$$

for all $x \in [0,1]$, such that

$$F(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } y = g(x) \text{ and } x < g(g(x)) \\ \max(x,y) & \text{if } y > g(x) \text{ or } y = g(x) \text{ and } x > g(g(x)) \\ \min(x,y) \text{ or } \max(x,y) & \text{if } y = g(x) \text{ and } x = g(g(x)). \end{cases}$$

Moreover, in this case F must be commutative except perhaps on the set of points (x,y) such that $y = g(x)$ with $x = g(g((x))$.

Remark 1. Let $g : [0,1] \to [0,1]$ be a decreasing function with $g(e) = e$. Note that condition (1) becomes $g(g(x)) = x$ for all $x \in [0,1]$ where g is strictly decreasing. On the other hand, when g is constant in an interval (a,b) then $g(g(x))$ must be such that $a \le g(g(x)) \le b$.

Let us point out also that the theorem above gives a characterization of all idempotent uninorms, requiring only commutativity in points (x,y) such that $y = g(x)$ and $x = g(g(x))$. In particular, this characterization includes those given in Theorems 1 and 2 for left-continuous and right-continuous idempotent uninorms. In fact, if the function F is left-continuous it must be equal to the minimum for all points (x,y) such that $y = g(x)$ and thus the function g must satisfy $g(g(x)) \ge x$ for all $x \in [0,1]$ and similarly for right-continuity.

3. IMPLICATION FUNCTIONS DEFINED FROM IDEMPOTENT UNINORMS

In view of the theorems above any idempotent uninorm U (continuous on one side or not) is determined by a decreasing function g. In what follows we will refer to this function g as the *associated function* of U. Moreover, from now on, any idempotent uninorm U with neutral element e and associated function g will be denoted by $U = (e,g)$. Note however that for some functions g, there are a lot of idempotent uninorms with the same neutral element e and the same associated function g, and of all these uninorms at most one can be left-continuous and at most one right-continuous.

Definition 5. A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication function or simply an implication if it satisfies:

- I is non increasing in the first place and non decreasing in the second one.

- I satisfies:

$$I(0,0) = I(1,1) = 1 \quad \text{and} \quad I(1,0) = 0.$$

From the definition it follows that $I(x,1) = 1$ and $I(0,x) = 1$ for all $x \in [0,1]$ and so the restriction of I to $\{0,1\}^2$ coincides with the classical implication.

Definition 6. Let U be a uninorm. We will denote by I_U the binary operator given by:

$$I_U = \sup\{z \mid z \in [0,1], U(x,z) \leq y\}.$$

When I_U is an implication function, we will say that I_U is the residual implication of U.

The fact of being the operator I_U an implication function, and the properties that satisfies, becomes important in several contexts like:

- Fuzzy relational equations, where the residual implicators (as well as the residual co-implicators, see next section) are the key for solving fuzzy relational equations of the form $R \circ X = A$, where R is a fuzzy relation and A is a fuzzy set (see for instance [3]).
- Fuzzy mathematical morphology, where residual implicators play an essential role in order to define the erosion and the dilation operators. In this context, properties of the implicators like the modus ponens, contrapositive symmetry, or the exchange principle directly derive in good morphological properties of the mentioned morphological operators (see [13] and [17]).

In this way, the study of when the operator above is an implication function is given in [7] and [8] for representable uninorms, as well as for uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, respectively. For idempotent uninorms only some results for left and right-continuous cases are given in [9]. In the general case we have the following

Proposition 1. Let $U = (e,g)$ be any idempotent uninorm. I_U is an implication function if and only if $g(0) = 1$.

Proof. Non-increasingness in the first place and non-decreasingness in the second one are trivial from the definition. On the other hand, it is clear from the definition of I_U that $I_U(1,1) = 1$ and $I_U(1,0) = 0$, but in order to have $I_U(0,0) = 1$, we need that $U(x,0) = 0$ for all $x < 1$ and this occurs if and only if $g(0) = 1$. □

The following theorem includes Theorem 8 in [9] as a particular case.

Theorem 4. Consider $U = (e,g)$ any idempotent uninorm with $g(0) = 1$. The residual implication I_U is given by:

$$I_U(x,y) = \begin{cases} \min(g(x),y) & \text{if } y < x \\ \max(g(x),y) & \text{if } y \geq x. \end{cases} \tag{2}$$

Proof. We divide the proof in some cases.

- When $y < x$ and $y < g(x)$. In this case, we have $U(x,y) = \min(x,y) = y$. If we take z satisfying $y < z$, $U(x,z) \in \{x,z\} > y$, and then

$$I_U(x,y) = \sup\{z \mid z \in [0,1], U(x,z) \leq y\} = y.$$

- If $y < x$ and $y \geq g(x)$. If we take z satisfying $z < g(x) \leq y$, $U(x,z) = \min(x,z) = z < y$; but if z satisfies $g(x) < z$, then $U(x,z) = \max(x,z) \geq x > y$, and we can conclude that

$$I_U(x,y) = \sup\{z \mid z \in [0,1], U(x,z) \leq y\} = g(x).$$

- If $y \geq x$ and $y > g(x)$. Now, $U(x,y) = \max(x,y) = y$. But if we take z satisfying $g(x) < y < z$, $U(x,z) = \max(x,z) = z > y$, and then

$$I_U(x,y) = \sup\{z \mid z \in [0,1], U(x,z) \leq y\} = y.$$

- When $y \geq x$ and $y \leq g(x)$. In this case, if we take z satisfying $z < g(x)$ then $U(x,z) = \min(x,z) = x \leq y$, but if z satisfies $y < g(x) < z$, then $U(x,z) = \max(x,z) = z > y$, and we can conclude that

$$I_U(x,y) = \sup\{z \mid z \in [0,1], U(x,z) \leq y\} = g(x).$$

Now, from the steps above expression (2) follows easily. □

As a corollary of the theorem above we obtain the following

Corollary 1. All idempotent uninorms with the same neutral element e and the same associated function g with $g(0) = 1$, have the same residual implication, given by expression (2).

Remark 2. Note that the corollary above gives no contradiction with Theorem 8 in [9] since there, expression in case ii) is actually the same that the one given in cases i) and iii), which also coincides with expression (2).

Example 1. Now we can give the expression of I_U when U is an idempotent uninorm and member of $\mathcal{U}_{\min}$. In that case, the associated function of U is

$$g(x) = \begin{cases} 1 & \text{if } x < e \\ e & \text{if } x \geq e \end{cases}$$

and I_U is:

$$I_U(x,y) = \begin{cases} y & \text{if } y < x \text{ and } y \leq e \\ e & \text{if } y < x \text{ and } y > e \\ y & \text{if } y \geq x \text{ and } x \geq e \\ 1 & \text{if } y \geq x \text{ and } x < e \end{cases}$$

that can be viewed in Figure 1.

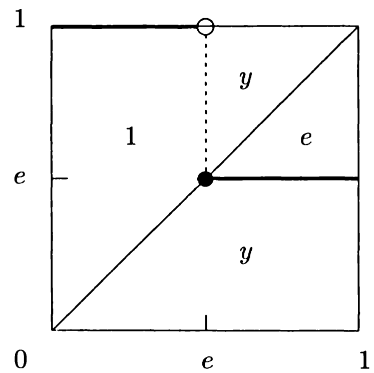

Fig. 1. I_U when U is an idempotent uninorm in $\mathcal{U}_{\min}$.

The following proposition is derived from results in [7] and [8] and it can also be trivially deduced from Theorem 4.

Proposition 2. Let $U = (e, g)$ be an idempotent uninorm with $g(0) = 1$, and I_U its residual implication. Then

i) $I_U(e, y) = y$ for all $y \in [0, 1]$.

ii) $I_U(x, y) \geq e$ if $x \leq y$.

iii) (Generalized Modus Ponens) $U(x, I_U(x, y)) \leq y$ *for all* $(x, y) \in [0, 1]^2$ if and only if U is left-continuous, and in that case U is conjunctive.

Proposition 3. Let $U = (e, g)$ be an idempotent uninorm with $g(0) = 1$, and I_U its residual implication. Then

i) $I_U(x, .)$ is right-continuous whereas $I_U(., y)$ is left-continuous if and only if so is g.

ii) $I_U(x, x) = \max(x, g(x))$.

iii) $I_U(x, y) \geq y$ if and only if $y \geq x$ or $(y < x$ and $y \leq g(x))$.

iv) $I_U(x, g(x)) = g(x)$.

v) $I_U(x, e) = g(x)$.

Proof. All the statements are straightforward. □

One special case, that will be characterized in several ways in next propositions, is when the associated function g is a strong negation N. It is specially interesting, mainly because in this case we have a lot of nice properties.

Proposition 4. Let $U = (e, g)$ be an idempotent uninorm with $g(0) = 1$, I_U its residual implication and $N : [0,1] \to [0,1]$ a strong negation. Then

$$I_U(x, e) = N(x) \qquad \text{for all} \quad x \in [0,1]$$

if and only if $g = N$. Moreover, in this case we have

$$I_U(x, N(x)) = N(x) \qquad \text{for all} \quad x \in [0,1]. \tag{3}$$

Proof. It follows from points iv) and v) in the previous proposition. □

Remark 3. Property described by expression (3) has been recently studied in [1] for residual implications from t-norms, due to its applicability in the framework of inclusion grade indicators constructed from implications.

Another important property, also satisfied when g is a strong negation, is contrapositive symmetry, that is

$$I_U(x, y) = I_U(N(y), N(x))) \qquad \text{for all} \quad x \in [0,1].$$

This property has been studied in [9] for left and right-continuous cases. For the general case we have the following proposition.

Proposition 5. Let $U = (e, g)$ be an idempotent uninorm with $g(0) = 1$, I_U its residual implication and N a strong negation. Then I_U has contrapositive symmetry with respect to N if and only if $g = N$.

Proof. When $g = N$, by one side we have:

$$I_U(x, y) = \begin{cases} \min(N(x), y) & \text{if } \ y < x \\ \max(N(x), y) & \text{if } \ y \geq x \end{cases}$$

and by the other

$$\begin{aligned} I_U(N(y), N(x)) &= \begin{cases} \min(N(N(y)), N(x)) & \text{if } \ N(x) < N(y) \\ \max(N(N(y)), N(y)) & \text{if } \ N(x) \geq N(y) \end{cases} \\ &= \begin{cases} \min(y, N(x)) & \text{if } \ y < x \\ \max(y, N(x)) & \text{if } \ y \geq x \end{cases} \end{aligned}$$

and this proves that I_U has contrapositive symmetry with respect to N.

Conversely, let us first show that $N(e) = e$. We have, using that I_U has contrapositive symmetry with respect to N,

$$e = I_U(e, e) = I_U(N(e), N(e)) = \max\{g(N(e)), N(e)\}.$$

Now, if $N(e) \geq g(N(e))$, then $e = N(e)$. If $N(e) < g(N(e))$, then $g(N(e)) = e$ and $N(e) < e$. Consequently, for all $x \in (N(e), e)$ we have $g(x) = e$ and also $N(x) \in (N(e), e)$ and we can write that

$$x = I_U(e, x) = I_U(N(x), N(e)) = \min(g(N(x)), N(e)) = N(e)$$

which is a contradiction.
Then, using that $N(e) = e$, we have for all $x \in [0, 1]$

$$g(x) = I_U(x, e) = I_U(N(e), N(x)) = I_U(e, N(x)) = N(x).$$ □

Example 2. Consider the strong negation $N(x) = 1 - x$ and the right continuous idempotent uninorm $U = (1/2, N)$. In this case we have

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < 1-x \\ \max(x,y) & \text{if } y \geq 1-x \end{cases}$$

and its residual implication

$$I_U(x,y) = \begin{cases} \min(1-x,y) & \text{if } y < x \\ \max(1-x,y) & \text{if } y \geq x \end{cases}$$

satisfies contrapositive symmetry with respect to N by previous proposition. This residual implication can be viewed in Figure 2.

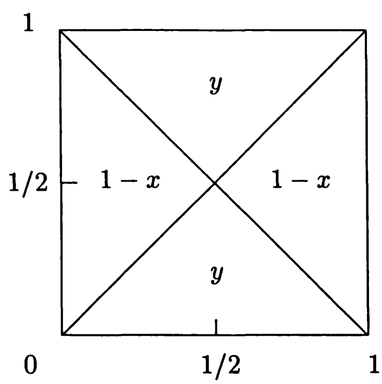

Fig. 2. I_U with $U = (1/2, N)$ and $N(x) = 1 - x$.

In [7] it was defined for any uninorm U and strong negation N the binary operator

$$I_{U,N} = U(N(x), y)$$

that is obviously an implication if and only if U is disjunctive. The case of representable uninorms was studied in [8] whereas, concerning left and right-continuous idempotent uninorms, it was proved in [9] the following

Proposition 6. (De Baets–Fodor [9].) Let N be a strong negation and U_r (U_l) the right (left) continuous idempotent uninorm with N as associated function. Then the following equalities hold:

$$I_{U_r,N} = I_{U_r} = I_{U_l}.$$

From Corollary 1, it is clear that the result above can be generalized to any idempotent uninorm $U = (e, N)$ as follows:

Proposition 7. Let N be a strong negation and $U = (e, N)$ any idempotent uninorm. Then the following equality holds:

$$I_{U_r,N} = I_U.$$

Moreover, it can be proved an if and only if version of this result.

Proposition 8. Let N be a strong negation and $U = (e, g)$ any idempotent uninorm. Then $I_{U,N} = I_U$ if and only if $g = N$ and U is right-continuous.

Proof. If $g = N$ and U is right-continuous, we have $U = U_r$ and the proposition above proves $I_{U,N} = I_U$. Conversely, if $I_{U,N} = I_U$ we have by one side

$$I_{U,N}(x, e) = U(N(x), e) = N(x)$$

and by the other, using proposition 3 v),

$$I_U(x, e) = g(x).$$

Thus $g(x) = N(x)$ for all $x \in [0, 1]$. Moreover, applying $I_{U,N}(x, x) = I_U(x, x)$ for all x we obtain, using Proposition 3 ii),

$$U(N(x), x) = \max(N(x), x)$$

following the right-continuity of U. □

To finish this section, let us study the exchange principle. Given an implication I, it verifies the exchange principle if

$$I(x, I(y, z)) = I(y, I(x, z)) \tag{4}$$

for all x, y, z in $[0, 1]$.

Proposition 9. Let $U = (e, N)$ be any idempotent uninorm with N a strong negation. Then I_U verifies the exchange principle.

Proof. As it is said in corollary 1, if we take two uninorms with the same generator function, they have the same residual implicator. Then, if we take $U_r = (e, N)$:

$$U_r(x, y) = \begin{cases} \min(x, y) & \text{if } y < N(x) \\ \max(x, y) & \text{if } y \geq N(x). \end{cases}$$

We know that $I_U = I_{U_r}$ and, by the previous proposition, that $I_{U_r,N} = I_{U_r}$. Then $I_U(x, y) = I_{U_r,N}(x, y) = U_r(N(x), y)$. Now, using that U_r is associative and commutative, we have that

$$\begin{aligned} I_U(x, I_U(y, z)) &= U_r(N(x), U_r(N(y), z)) = U_r(U_r(N(x), N(y)), z)) \\ &= U_r(U_r(N(y), N(x)), z)) = U_r(N(y), U_r(N(x), z)) \\ &= I_U(y, I_U(x, z)), \end{aligned}$$

for all x, y, z in $[0, 1]$. □

All idempotent uninorms such that their residual implications satisfy this important property, including consequently those given in the proposition above, are characterized in next theorem.

Theorem 5. Let $U = (e, g)$ be any idempotent uninorm with $g(0) = 1$. Then I_U satisfies the exchange principle if and only if the following property is satisfied:

$$\text{if } g(g(x)) < x \text{ for some } x \in [0, 1], \text{ then } x > e \text{ and } g(x) = e. \tag{5}$$

Proof. First, suppose that I_U satisfies the exchange principle, and $a \in [0, 1]$ which $g(g(a)) < a$. Now we divide the proof in several cases.

- First note that $a \neq e$ because $g(g(e)) = e$.
- If $a < g(a)$, then we have $g(g(a)) < a < g(a)$ and by one side

$$\begin{aligned} I_U(a, I_U(g(a), a)) &= I_U(a, \min(g(g(a)), a)) = I_U(a, g(g(a))) \\ &= \min(g(g(a)), g(a)) = g(g(a)), \end{aligned}$$

and by the other

$$I_U(g(a), I_U(a,a)) = I_U(g(a), \max(g(a),a)) = I_U(g(a), g(a))$$

$$= \max(g(g(a)), g(a)) = g(a).$$

And then $g(a) = g(g(a))$, but this lead us to a contradiction.

- If $a > g(a)$, then $g(a) \leq g(g(a))$, and we have by one side

$$I_U(a, I_U(g(a), g(g(a)))) = I_U(a, g(g(a))) = \min(g(g(a)), g(a)) = g(a),$$

and by the other

$$I_U(g(a), I_U(a, g(g(a)))) = I_U(g(a), \min(g(a), g(g(a)))) = I_U(g(a), g(a))$$

$$= \max(g(g(a)), g(a)) = g(g(a)).$$

And then $g(a) = g(g(a))$, that means that $g(a) = e$ and $a > e$, because e is the only fixpoint of g.

In any case, if I_U satisfies the exchange principle, it satisfies (5).
Conversely, suppose that g satisfies (5). Since $I_U(x,y) \in \{g(x), y\}$, we divide the proof in several cases depending on the values of $I_U(x,z)$ and $I_U(y,z)$.

1) If $I_U(x,z) = z$ and $I_U(y,z) = z$. We have:

$$I_U(x, I_U(y,z)) = I_U(x,z) = z = I_U(y,z) = I_U(y, I_U(x,z))$$

and then the exchange principle is satisfied.

2) $I_U(y,z) = z$ and $I_U(x,z) = g(x)$. Then, by one side we have

$$I_U(x, I_U(y,z)) = I_U(x,z) = g(x)$$

and by the other

$$I_U(y, I_U(x,z)) = I_U(y, g(x)).$$

Now we study the value of $I_U(y, g(x))$.

- If $z < y$ then $I_U(y,z) = \min(g(y), z) = z$ and consequently $z \leq g(y)$.
  * If $z < x$ then $I_U(x,z) = \min(g(x), z) = g(x)$ and therefore $z \geq g(x)$. Using that $g(x) \leq z < y$ and $g(x) \leq z \leq g(y)$ we can compute the value of $I_U(y, g(x))$:

$$I_U(y, g(x)) = \min(g(y), g(x)) = g(x).$$

  * If $z \geq x$ then we have that $x \leq z \leq g(y)$ and $x \leq z < y$ that implies that $g(y) \leq g(x)$.

· If $x \neq g(y)$ then $x < g(y)$. By definition of idempotent uninorm, this means that $y \leq g(x)$ and then

$$I_U(y, g(x)) = \max(g(y), g(x)) = g(x).$$

· If $x = g(y)$ then $g(y) \leq g(x) = g(g(y))$ and we have

$$\begin{aligned} I_U(y, g(x)) &= I_U(y, g(g(y))) \\ &= \begin{cases} \min(g(y), g(g(y))) & \text{if } g(g(y)) < y \\ \max(g(y), g(g(y))) & \text{if } g(g(y)) \geq y \end{cases} \\ &= \begin{cases} g(y)(= e) & \text{if } g(g(y)) < y \\ g(g(y)) & \text{if } g(g(y)) \geq y. \end{cases} \end{aligned}$$

Now, using that g satisfies (5), we obtain:

$$I_U(y, g(x)) = g(g(y)) = g(x).$$

– If $z \geq y$ the proof is similar to the previous case.

And in any case $I_U(y, g(x)) = g(x)$ and the exchange principle is satisfied.

3) If $I_U(y, z) = g(y)$ and $I_U(x, z) = z$. This case is similar to the previous one because x and y play a symmetric role in the equation (4).

4) If $I_U(y, z) = g(y)$ and $I_U(x, z) = g(x)$. We have by one side

$$I_U(x, I_U(y, z)) = I_U(x, g(y)),$$

and by the other

$$I_U(y, I_U(x, z)) = I_U(y, g(x)).$$

– If $x \neq g(y)$ and $y \neq g(x)$, by definition, if $x > g(y)$ then $y \geq g(x)$ but $y \neq g(x)$, and therefore if $x > g(y)$ then $y > g(x)$. Similarly we have that if $y > g(x)$, then $x > g(y)$. That is, $y > g(x)$ if and only if $x > g(y)$. Then,

$$I_U(x, g(y)) = \begin{cases} \min(g(x), g(y)) & \text{if } x > g(y) \\ \max(g(x), g(y)) & \text{if } x < g(y) \end{cases}$$

and

$$I_U(y, g(x)) = \begin{cases} \min(g(x), g(y)) & \text{if } y > g(x) \\ \max(g(x), g(y)) & \text{if } y < g(x) \end{cases}$$

are the same, and the exchange principle is satisfied.

– If $y = g(x)$ we have that:

$$I_U(y, g(x)) = I_U(g(x), g(x)) = \max(g(g(x)), g(x))$$

and

$$
\begin{aligned}
I_U(x,g(y)) &= I_U(x,g(g(x))) \\
&= \begin{cases} \max(g(x),g(g(x))) & \text{if } g(g(x)) \geq x \\ \min(g(x),g(g(x))) & \text{if } g(g(x)) < x \end{cases} \\
&= \max(g(x),g(g(x)))
\end{aligned}
$$

because g satisfies (5). Therefore $I_U(x,g(y)) = I_U(y,g(x))$.

– If $x = g(y)$, the case is similar to the previous one.

Consequently if g satisfies (5) then I_U satisfies the exchange principle. □

Remark 4. Note that in the previous theorem we have found non left-continuous uninorms such that their residual implications I_U satisfy the exchange principle. In particular, idempotent uninorms in $\mathcal{U}_{\min}$ (that are right-continuous) satisfy the condition in the theorem above and, consequently, their residual implications satisfy the exchange principle.

4. CO–IMPLICATION FUNCTIONS AND DUALITY

Similarly to the previous section, we define

Definition 7. A binary operator $J : [0,1] \times [0,1] \to [0,1]$ is said to be a co-implication function or simply an co-implication if it satisfies:

- J is non increasing in the first place and non decreasing in the second one.
- $J(0,0) = J(1,1) = 0$ and $J(0,1) = 1$.

While residual implicators can be viewed as a fuzzy generalization of the classical implication ($p \Longrightarrow q$), residual co-implicators generalize the classical co-implication ($q \not\Longleftrightarrow p$).

Definition 8. Let U be a uninorm. We will denote by J_U the binary operator given by:

$$J_U = \inf\{z \mid z \in [0,1], U(x,z) \geq y\}.$$

We will say that J_U is the residual co-implication of U if J_U is a co-implication function.

Similarly to the case of implication functions, the following results can be proved.

Proposition 10. Let $U = (e,g)$ be any idempotent uninorm. J_U is a co-implication function if and only if $g(1) = 0$.

Although, in fuzzy mathematical morphology, the morphological operators are usually defined throughout residual implications, co-implication functions and their properties are also essential to obtain certain good morphological properties, like for instance the idempotence of fuzzy opening and fuzzy closing (see [6]).

Theorem 6. Consider $U = (e, g)$ any idempotent uninorm with $g(1) = 0$. The residual co-implication J_U is given by:

$$J_U(x, y) = \begin{cases} \min(g(x), y) & \text{if } y \leq x \\ \max(g(x), y) & \text{if } y > x. \end{cases} \tag{6}$$

Remark 5. Comparing (2) and (6) we can see that, given any uninorm with g as associated function satisfying $g(0) = 1$ and $g(1) = 0$, both I_U and J_U coincide except on the set of points (x, x).

Recall that, given any idempotent uninorm $U = (e, g)$ and a strong negation N, we can construct the dual operator

$$\widetilde{U}(x, y) = N(U(N(x), N(y)))$$

that is also an idempotent uninorm. Its neutral element is $\widetilde{e} = N(e)$ and its associated function is $\widetilde{g}(x) = N(g(N(x)))$. For example, if $U \in \mathcal{U}_{\min}$, then $\widetilde{U} \in \mathcal{U}_{\max}$.

Now, let J be any co-implication, then the dual operator

$$\widetilde{J}(x, y) = N(J(N(x), N(y)))$$

is an implication. Moreover, given an idempotent uninorm $U = (e, g)$ with $g(1) = 0$, we have that the following equalities hold, for any strong negation N:

$$\widetilde{J_U} = I_{\widetilde{U}} \qquad \text{and} \qquad \widetilde{I_U} = J_{\widetilde{U}}.$$

That is, for any idempotent uninorm $U = (e, g)$ with $g(1) = 0$, the dual operator of the residual co-implication J_U of U, is the residual implication of $\widetilde{U}$.

Remark 6. Note that in the special case of $g = N$, U and $\widetilde{U}$ have the same associated function, N, and consequently $\widetilde{J_U} = I_U$.

From this duality it is easy to see that each result for implications proved in the section above has its corresponding result for co-implications. We state here the result corresponding to the exchange principle and we leave the others to the reader.

Theorem 7. Let $U = (e, g)$ be any idempotent uninorm with $g(1) = 0$. The following items are equivalent:

i) J_U satisfies the exchange principle.

ii) $I_{\widetilde{U}}$ satisfies the exchange principle.

iii) If $x < g(g(x))$ for some $x \in [0,1]$, then $x < e$ and $g(x) = e$.

Proof. For all x, y, z in $[0,1]$ we have that if J_U satisfies the exchange principle

$$\begin{aligned}
I_{\widetilde{U}}(x, I_{\widetilde{U}}(y,z)) &= \widetilde{J_U}(x, \widetilde{J_U}(y,z)) = N(J_U(N(x), N(\widetilde{J_U}(y,z)))) \\
&= N(J_U(N(x), N(N(J_U(N(y), N(z)))))) \\
&= N(J_U(N(x), J_U(N(y), N(z)))) \\
&= N(J_U(N(y), J_U(N(x), N(z)))) \\
&= N(J_U(N(y), N(N(J_U(N(x), N(z)))))) = \widetilde{J_U}(y, \widetilde{J_U}(x,z)) \\
&= I_{\widetilde{U}}(y, I_{\widetilde{U}}(x,z)),
\end{aligned}$$

then $I_{\widetilde{U}}$ satisfies the exchange principle. Conversely, a similar proof shows that if $I_{\widetilde{U}}$ satisfies the exchange principle, J_U does, and we have equivalence between i) and ii).

Now, by applying Theorem 5, we know that $I_{\widetilde{U}}$ satisfies the exchange principle if and only if the following equivalent statements hold

$$\begin{gathered}
\text{If } x > \widetilde{g}(\widetilde{g}(x)) \text{ for some } x \in [0,1], \text{ then } x > \widetilde{e} \text{ and } \widetilde{g}(x) = \widetilde{e} \\
\Updownarrow \\
\text{If } x > N(g(N(N(g(N(x)))))) \text{ for some } x \in [0,1], \text{ then } x > N(e) \text{ and} \\
N(g(N(x))) = N(e), \\
\Updownarrow \\
\text{If } x > N(g(g(N(x)))) \text{ for some } x \in [0,1], \text{ then } x > N(e) \text{ and } g(N(x)) = e, \\
\Updownarrow \\
\text{If } N(x) < g(g(N(x))) \text{ for some } x \in [0,1], \text{ then } N(x) < e \text{ and } g(N(x)) = e, \\
\Updownarrow \\
\text{If } x < g(g(x)) \text{ for some } x \in [0,1], \text{ then } x < e \text{ and } g(x) = e,
\end{gathered}$$

and consequently, ii) is equivalent to iii). □

Remark 7. Now we have that given any idempotent uninorm in $\mathcal{U}_{\max}$ (left-continuous and disjunctive uninorm), its residual co-implication J_U satisfies the exchange principle.

Example 3. Consider $N(x) = \sqrt{1-x^2}$, and U the right-continuous idempotent uninorm $U = (\sqrt{2}/2, N)$ given by the expression:

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } \; y < \sqrt{1-x^2} \\ \max(x,y) & \text{if } \; y \geq \sqrt{1-x^2} \end{cases}$$

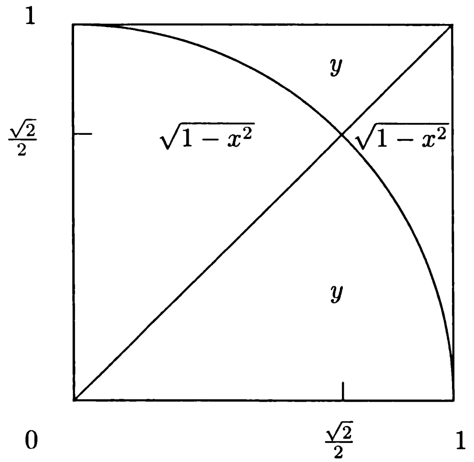

Fig. 3. J_U with $U = (\sqrt{2}/2, N)$ and $N(x) = \sqrt{1-x^2}$.

its residual implication

$$I_U(x,y) = \begin{cases} \min(\sqrt{1-x^2}, y) & \text{if } y < x \\ \max(\sqrt{1-x^2}, y) & \text{if } y \geq x \end{cases}$$

and its residual co-implication

$$J_U(x,y) = \begin{cases} \min(\sqrt{1-x^2}, y) & \text{if } y \leq x \\ \max(\sqrt{1-x^2}, y) & \text{if } y < x \end{cases}$$

that can be viewed in Figure 3. Note that the only difference between I_U and J_U is in the set of points $\{(x,x)/x \in [0,1]\}$. In this case, I_U and J_U satisfy the exchange principle and both satisfy contrapositive symmetry with respect to N.

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