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STATES ON UNITAL PARTIALLY-ORDERED GROUPS[1]

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We study states on unital po-groups which are not necessarily commutative as normalized positive real-valued group homomorphisms. We show that in contrast to the commutative case, there are examples of unital po-groups having no state. We introduce the state interpolation property holding in any Abelian unital po-group, and we show that it holds in any normal-valued unital ℓ-group. We present a connection among states and ideals of po-groups, and we describe extremal states on the state space of unital po-groups.

1. INTRODUCTION

Nowadays there appears a whole family of non-commutative generalizations of MV-algebras: pseudo MV-algebras of G. Georgescu and A. Iorgulescu [14], or equivalently, generalized MV-algebras of J. Rachůnek [17], pseudo BL-algebras [4]. For them the author [5] proved that any pseudo MV-algebra is always an interval in a unital ℓ-group (G, u) with a strong unit u. In addition, pseudo-effect algebras were introduced in [9, 10]. Such algebras are sometimes also intervals in unital po-groups.

A non-commutative reasoning can be met in the every-day life very often. Many human processes are depending on the order of variables, as well as in quantum mechanics there exist experiments, using polarizing filters to a beam of particles, where the result depends on the order of used filters, that is, it can happen that we detect particles or not depending on the order of used oriented filters.

States on algebraic structures are generalizations of probability measures on Boolean algebras. States on MV-algebras were introduced by F. Chovanec [3] and by D. Mundici [16], by the author on pseudo MV-algebras, and by G. Georgescu [13] on pseudo BL-algebras. States have the intent of capturing the notion of "average degree of truth" of a proposition. Since every pseudo MV-algebra is an interval in a unital ℓ-group, [5], in [6] there was shown that there is a one-to-one correspondence between states on pseudo MV-algebras and states on unital ℓ-groups (= normalized positive real-valued homomorphisms on unital ℓ-groups).

States on pseudo-effect algebras were studied in [11], and since sometimes they are also intervals in unital po-groups, we concentrate our interest to states (= normalized positive real-valued homomorphisms) on unital po-groups. We recall that theory of

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states on commutative unital po-groups is very well developed in the monograph [15] by K. R. Goodearl. We were inspired by this book in our research, but the principal problem is that in contrast to commutative unital po-groups, it is not clear whether every unital po-group possesses at least one state. In [6], it was shown that there is a unital ℓ-group which has no states. In addition, there was shown that every normal-valued (hence any representable or commutative) unital ℓ-group possesses at least one state. Some properties of extremal states on pseudo MV-algebras are exhibited in [7].

Therefore, we have introduced the state interpolation property which holds in any Abelian unital po-group. This notion enables us to study some properties of states important for commutative unital po-groups.

2. STATES ON PSEUDO-EFFECT ALGEBRAS

According to [9, 10], a partial algebra $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* ([9, 10]) if, for all $a,b,c \in E$, the following holds

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) for any $a \in E$, there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a \le b$ iff there exists an element $c \in E$ such that $a+c=b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. If E is a lattice under $\le$, we say that E is a *lattice pseudo-effect algebra.* If $+$ is commutative, i. e., if $a+b=b+a$, E is said to be an *effect algebra.*

Let E be a pseudo-effect algebra. Let $/$, $\backslash$ be two partial binary operations on E such that, for $a,b \in E$, a / b is defined iff $b \backslash a$ is defined iff $a \le b$, and such that in this case we have

$$(b \backslash a) + a = a + (a / b) = b.$$

The basic properties of pseudo-effect algebras can be found in [9] – [11].

Let $E = (E;+,0,1)$ be a pseudo-effect algebra. We define $a^- := 1 \backslash a$ and $a^\sim := a / 1$ for any $a \in E$.

For example if (G,u) is a unital (not necessary Abelian) po-group with a strong unit u (sometimes it is sufficient to assume only $u > 0$), and

$$\Gamma(G,u) := [0,u] = \{g \in G : \ 0 \le g \le u\},$$

then $(\Gamma(G,u);+,0,u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G,u)$. In [10], there are conditions showing when a pseudo-effect algebra can be represented in this way.

We recall that a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;[2]

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

In [5] it was shown that every pseudo MV-algebra is isomorphic to $\Gamma(G, u)$, where (G, u) is a unital ℓ-group with a strong unit u, where $a \oplus b := (a + b) \wedge u$, $a \odot b = (a - u + b) \vee 0$ and $a^- = u - a$ and $a^\sim = -a + u$.

If M is a pseudo MV-algebra, then the partial operation $a+b$ is defined iff $a \leq b^-$, and then $a + b := a \oplus b$, and $(M; +, 0, 1)$ is a pseudo-effect algebra.

A *state* on a pseudo-effect algebra E is a mapping $m : E \to [0, 1]$ such that (i) $m(1) = 1$, and (ii) $m(a + b) = m(a) + m(b)$ whenever $a + b$ is defined in E.

Let G be a po-group. A mapping $f : G \to \mathbb{R}$ such that (i) $f(g_1 + g_2) = f(g_1) + f(g_2)$, $g_1, g_2 \in G$, (ii) $f(g) \geq 0$ for any $g \in G^+$ is said to a *positive homomorphism*. If (G, u) $(u \neq 0)$ is a unital po-group with strong unit, then any positive homomorphism s on G with $s(u) = 1$ is said to be a *state* on (G, u).

If s is a state on a unital po-group (G, u), then $m := s|\Gamma(G, u)$ is a state on the pseudo-effect algebra $E = \Gamma(G, u)$. Conversely, if m is a state on $\Gamma(G, u)$ then sometimes it can be uniquely extended to a state s on (G, u). Such a situation can happen when E is a pseudo MV-algebra [6, Thm. 3.9] as well as for pseudo-effect algebras having a special form of the Riesz decomposition property [11]. Therefore, in what follows, we concentrate our interest to study of states on unital po-groups.

3. STATES ON PO–GROUPS

In the present section, we study states on unital po-groups and introduce some important notions known from theory of states on commutative unital po-groups inspired by the monograph [15].

[2] As usually, $\odot$ has a higher priority than $\oplus$.

Lemma 3.1. Let G be a po-group, let H be a subgroup of G, and let $x \in G$. Let $f: H \to \mathbb{R}$ be a positive homomorphism, and set

$$\begin{aligned} r &= \inf\{f(z)/n : z \in H, n \geq 1, nx \leq z\}, \\ p &= \sup\{f(y)/m : y \in H, m \geq 1, y \leq mx\}. \end{aligned}$$

(a) $-\infty \leq p \leq r \leq +\infty$.

(b) If $x + h = h + x$ for all $h \in H$ and if there exists a positive homomorphism $g: H + \mathbb{Z}x \to \mathbb{R}$ extending f, then $p \leq g(x) \leq r$.

(c) If $x+h = h+x$ for all $h \in H$ and if q is a real number such that $p \leq q \leq r$, then f extends to a positive homomorphism $g : H + \mathbb{Z}x \to \mathbb{R}$ such that $g(x) = q$.

Proof. (a) If no element mx ($m \in \mathbb{N}$) lies above any element of H, then $p = -\infty$, while if no element nx ($n \in \mathbb{N}$) lies below any element of H, then $r = +\infty$. In either of these cases, $p \leq r$. If there exist $y, z \in H$ and $m, n \geq 1$ such that $y \leq mx$ and $nx \leq z$, then $ny \leq nmx \leq mz$ and so $f(y)/m \leq f(z)/n$. Thus $p \leq r$ in all cases.

(b) If $y \in H$ and $m \geq 1$ with $y \leq mx$, then $f(y) = g(y) \leq mg(x)$, whence $f(y)/m \leq g(x)$. Thus $p \leq g(x)$. If $z \in H$ and $n \geq 1$ with $nx \leq z$, then $ng(x) \leq g(z) = f(z)$, whence $g(x) \leq f(z)/n$. Thus $g(x) \leq r$.

(c) We claim that if $h \in H$ and $k \in \mathbb{Z}$ such that $h + kx \geq 0$, then $f(h) + kq \geq 0$. If $k = 0$, then $h \geq 0$ and so $f(h) + kq = f(h) \geq 0$, because f is positive. If $k > 0$, then as $-h \leq kx$, we have $f(-h)/k \leq p \leq q$ whence $f(h) + kq \geq 0$. If $k < 0$, then as $(-k)x \leq h$ we have $q \leq r \leq f(h)/(-k)$, and again $f(h) + kq \geq 0$.

Now if $h \in H$ and $k \in \mathbb{Z}$ with $h + kx = 0$, then $f(h) + kq = 0$. Indeed, since $h + kx \geq 0$ and $-h - kx \geq 0$, we obtain $f(h) + kq \geq 0$ and $f(-h) - kq \geq 0$, so that $f(h) + kq = 0$. Therefore f extends to a well-defined positive homomorphism $g: H + \mathbb{Z}x \to \mathbb{R}$ such that $g(x) = q$. □

Proposition 3.2. Let (G, u) be a nonzero po-group with strong unit. For any $x \in G$, we set

$$\begin{aligned} f^*(x) &= \inf\{l/n : l \in \mathbb{Z}, n \geq 1, nx \leq lu\}, \\ f_*(x) &= \sup\{k/i : k \in \mathbb{Z}, i \geq 1, ku \leq ix\}. \end{aligned}$$

(a) $-\infty < f_*(x) \leq f^*(x) < +\infty$.

(b) If s is a state on (G, u), then $f_*(x) \leq s(x) \leq f^*(x)$.

(c) $f_*(0) = 0 = f^*(0)$ and $f_*(u) = 1 = f^*(u)$.

(d) $f^*(x) = -f_*(-x)$, $f^*(-x) = -f_*(x)$.

(e) $f^*(jx) = jf^*(x)$ and $f_*(jx) = jf_*(x)$ for any $j \geq 1$.

(f) If $x + y = y + x$, then $f^*(x+y) \leq f^*(x) + f^*(y)$ and $f_*(x+y) \geq f_*(x) + f_*(y)$.

(g) If $x \leq y$, then $f_*(x) \leq f_*(y)$ and $f^*(x) \leq f^*(y)$.

Proof. Since G is nonzero, $ju < 0$ for all integers $j < 0$.

(a) There exist positive integers k, l such that $-x \le ku$ and $x \le lu$, whence $f_*(x) \ge -k > -\infty$ and $f^*(x) \le l < \infty$. Given integers $k, l \in \mathbb{Z}$ and $i, n \ge 1$ such that $ku \le ix$ and $nx \le lu$, we have $knu \le inx \le liu$, and so $(li - kn)u \ge 0$, consequently $li - kn \ge 0$, i.e., $l/n \ge k/i$. Thus $f_*(x) \le f^*(x)$.

(b) This is evident.

(c) This is clear.

(d) Let $nx \le lu$, $l \in \mathbb{Z}$, then $-lu \le n(-x)$ which gives $f_*(x) \ge -l/n$ so that $f_*(-x) \ge -f^*(x)$.

On the other hand, let $k'u \le i(-x)$ for $k' \in \mathbb{Z}$, $i \ge 1$. Then $ix \le -k'u$ which gives $f^*(x) \le -k/l$, i.e., $f^*(x) \le -f_*(-x)$. Hence, $f^*(x) \le -f_*(-x) \le f^*(x)$, i.e., $f^*(x) = -f_*(-x)$.

(e) Let $n_1 \le l_1 u$ and $j \ge 1$. Then $n_1(jx) \le l_1 ju$ which proves $f^*(jx) \le l_1 j/n_1 = jf^*(x)$. On the other hand, let $n(jx) \le lu$. Then $f^*(x) \le l/(nj)$, so that $jf^*(x) \le l/n$ which gives $jf^*(x) \le f^*(jx)$.

(f) Let $x, y \in G$ and $x + y = y + x$. There exist integers $n_1, n_2 \ge 1$ and $l_1, l_2 \in \mathbb{Z}$ such that $n_1 x \le l_1 u$, $n_2 y \le l_2 u$, which gives $n_1 n_2 x + n_1 n_2 y = n_1 n_2 (x + y) \le (n_2 l_1 + n_1 l_2)u$ so that $f^*(x + y) \le f^*(x) + f^*(y)$.

In a similar way we can prove $f_*(x + y) \ge f_*(x) + f_*(y)$.

(g) It is evident. □

It is worthy to recall that if (G, u) is a non-zero Abelian unital po-group with strong unit then it possesses at least one state [15, Cor. 4.4.]. On the other hand, there exists an example of a unital ℓ-group having no state [6, Cor. 7.4].

Proposition 3.3. Let (G, u) be a nonzero po-group with strong unit, let $x \in G^+$, and define

$$\begin{aligned} f'^*(x) &= \inf\{l/n : \ l, n \ge 1,\ nx \le lu\}, \\ f'_*(x) &= \sup\{k/i : \ k \ge 0, i \ge 1,\ ku \le ix\}. \end{aligned}$$

Then

$$0 \le f'_*(x) = f_*(x) \le f^*(x) = f'^*(x) < \infty,$$

where $f_*(x)$ and $f^*(x)$ are defined in Proposition 3.2.

Proof. Since $0u \le x$, we have $f'_*(x) \ge 0$. We show that $f'_*(x), f'^*(x)$ coincide with $f_*(x)$ and $f^*(x)$, respectively.

Obviously $f^*(x) \le f'^*(x)$. Now consider any $l \in \mathbb{Z}$ and $n \ge 1$ such that $nx \le lu$. If $l > 0$, then $f'^*(x) \le l/n$ by definition of $f'^*(x)$. If $l < 0$, then $x \le nx \le lu < 0$ which is impossible. If $l = 0$, then $x \le nx \le lu = 0$ whence $x = 0$. In this case $n'x = 0 < u$ for all $n' \ge 1$, so that $f'^*(x) \le 1/n'$ for all $n' \ge 1$, and hence $f'^*(x) \le 0 = l/n$. Therefore $f'^*(x) \le l/u$ in all cases which proves $f'^*(x) \le f^*(x)$.

On the other hand, $f'_*(x) \le f_*(x)$. Now consider any $k \in \mathbb{Z}$ and $i \ge 1$ such that $ku \le ix$. If $k \ge 0$, then $k/i \le f'_*(x)$ by definition of $f'_*(x)$, while if $k < 0$, then $k/i < 0 \le f'_*(x)$, so that $f_*(x) \le f'_*(x)$. □

Proposition 3.4. Let (G,u) be a nonzero po-group with strong unit, let $x \in \Gamma(G,u)$, and define

$$f^{''*}(x) = \inf\{l/n : \ l,n \geq 1, \ nx \leq lu\},$$
$$f''_*(x) = \sup\{k/i : \ k \geq 0, i \geq 1, \ ku \leq ix\}.$$

Then

$$0 \leq f''_*(x) = f'_*(x) = f_*(x) \leq f^*(x) = f^{'*}(x) = f^{''*}(x) \leq 1,$$

where $f_*(x), f'_*(x)$ and $f^*(x), f^{'*}(x)$ are defined in Propositions 3.2–3.3.

Proof. This follows the same ideas as the proof of Proposition 3.3. □

Proposition 3.5. Every linearly ordered unital po-group (G,u) possesses a unique state. Moreover, $f^*(x) = f_*(x)$ for any $x \in G$.

Proof. Let C_1 and C_2 be two proper convex subgroups of G. Then either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. If not, then there exist two elements x, y such that $x \in C_1 \setminus C_2$ and $y \in C_2 \setminus C_1$. Since $x \leq y$ or $y \leq x$, we can derive a contradiction.

Let now C_0 be the set-theoretical union of all proper convex subgroups of G. Then C_0 is a maximal convex proper directed subgroup of G. We note that, for $x \in G^+$, $x \in C_0$ iff $nx < u$ for any $n \geq 1$. We claim that C_0 is a normal subgroup of G. Let $x \in G$ and $h \in C_0$. If $x \in C_0$, then $x + h = (x + h - x) + x$. If $x \notin C_0$, then for any $h \in C_0$, $nh < u$ for every $n \geq 1$. Therefore, $x + h = (x + h - x) + x$. Put $h_1 = x + h - x$. Then $nh_1 = x + nh - x = x - (x - nh)$. Since $0 < x - nh \leq x < u$, we have $nh_1 \in C_0$, hence $h_1 \in C_0$.

Similarly we prove that, for all $x \in G$ and $h \in C_0$, there exists $h_2 \in C_0$ such that $h + x = x + h_2$.

Then $(G/C_0, u/C_0)$ is a linearly ordered unital group with strong unit which contains no nonzero proper convex subgroup. Consequently, the group $(G/C_0, u/C_0)$ is Archimedean and hence commutative, and by Hölder's theorem [2], it is isomorphic to some subgroup of $(\mathbb{R}, 1)$. This means that $(G/C_0, u/C_0)$ has a unique state μ. Then $s(x) := \mu(x/C_0)$, $x \in G$, is a state on (G,u).

Define $f_*(x)$ and $f^*(x)$ by Proposition 3.2. Then $f_*(x) \leq f^*(x)$. We show that $f_*(x) = f^*(x)$ for any $x \in G$. Suppose that $f_*(x) < f^*(x)$ for some $x \in G$. Hence we may choose two integers k and $n \geq 1$ such that $f_*(x) < k/n < f^*(x)$. Since $k/n < f^*(x)$, we have $nx \not\leq ku$, and consequently $ku \leq nx$. But then $k/n \leq f_*(x)$, a contradiction. This proves that s is a unique state on (G,u). □

Proposition 3.6. Let (G,u) be a nonzero unital po-group with strong unit. The function f_* is a state on (G,u) if and only if so is f^*. In this case, $f_* = f^*$, and (G,u) possesses a unique state.

Proof. Let f_* be a state on (G,u). Then by (d) of Proposition 3.2, $f^*(x+y) = -f_*(-(x+y)) = -f_*(-y-x) = -f_*(-x) - f_*(-y) = f^*(x) + f^*(y)$ for all $x, y \in G$.

In a similar way we prove that if f^* is state on (G,u) so is f_*. In addition, $f_*(x) = -f^*(-x) = f^*(x)$ for any $x \in G$. □

Proposition 3.7. Let $(G,u) = (G_1,u_1) \times (G_2,u_2)$, where (G_i,u_i) is a unital po-group for $i = 1,2$. Then, for all $(x_1,x_2) \in G_1 \times G_2$,

$$\begin{aligned} f^*(x_1,x_2) &= \max\{f_1^*(x_1), f_2^*(x_2\}, \\ f_*(x_1,x_2) &= \min\{f_*^1(x_1), f_*^2(x_2\}. \end{aligned}$$

Proof. Let $n(x_1,x_2) \le l(u_1,u_2)$, where $n \ge 1$ and $l \in \mathbb{Z}$. Then $nx_1 \le lu_1$ and $nx_2 \le lu_2$, i.e., $f_i^*(x_i) \le l/n$. Hence $f_i^*(x_i) \le f^*(x_1,x_2)$. Now assume that $f_1^*(x_1) \le f_2^*(x_2)$. Then there exist integers $n_2 \ge 1$ and $l_2 \in \mathbb{Z}$ such that $n_2x_2 \le l_2u_2$ and $f_2^*(x_2) \le l_2/n_2$. Since $f_1^*(x_1) \le f_2^*(x_2)$, there exist integers $n_1 \ge 1$ and $l_1 \in \mathbb{Z}$ such that $n_1x_1 \le l_1u_1$ and $f_1^*(x_1) \le l_1/n_1 \le l_2/n_2$. Then $l_1n_2 \le l_2n_1$, and $n_1n_2x_2 \le n_1l_2u_2$, whence $n_1n_2x_1 \le n_2l_1u_1 \le n_1l_2u_1$. Therefore, $n_1n_2(x_1,x_2) \le n_1l_2(u_1,u_2)$ so that $f^*(x_1,x_2) \le l_2/n_2$, which proves $f^*(x_1,x_2) = f_2^*(x_2)$.

Using (d) of Proposition 3.2, we have

$$\begin{aligned} f_*(x_1,x_2) &= -f^*(-x_1,-x_2) = -\max\{f_1^*(-x_1), f_2^*(-x_2)\} \\ &= \min\{-f_1^*(-x_1), -f_2^*(-x_2)\} = \min\{f_*^1(x_1), f_*^2(x)\}. \end{aligned}$$

□

4. THE STATE INTERPOLATION PROPERTY

In this section, we introduce an important notion, the state interpolation property, holding in any Abelian unital po-groups and exhibit conditions of the existence of states and of a unique state.

We say that a unital po-group (G,u) satisfies the *state interpolation property* if, given $x \in G$ and a real number q such that $f_*(x) \le q \le f^*(x)$, there exists a state s on (G,u) such that $s(x) = q$. For example, any linearly ordered unital po-group (Proposition 3.5) or any Abelian unital po-group ([15, Prop. 4.7]) possesses the state interpolation property. On the other hand, there is an ℓ-group (G,u) having no state on it [6, Cor. 7.4]. Another example of non-Abelian unital ℓ-group is as follows.[3]

Example 4.1. Let $G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}; +, (0,0,0), \le)$ be the Scrimger 2-group, i.e.,

$$(k_1,m_1,n_1) + (k_2,m_2,n_2) := \begin{cases} (k_1+m_2, m_1+k_2, n_1+n_2), & \text{if } n_2 \text{ is odd} \\ (k_1+k_2, m_1+m_2, n_1+n_2), & \text{if } n_2 \text{ is even.} \end{cases}$$

Then $0 = (0,0,0)$ is the neutral element, and

$$-(k,m,n) = \begin{cases} (-m,-k,-n), & \text{if } n \text{ is odd} \\ (-k,-m,-n), & \text{if } n \text{ is even,} \end{cases}$$

and G is a non-Abelian ℓ-group with the positive cone $G^+ = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+_{>0} \cup \mathbb{Z}^+ \times \mathbb{Z}^+ \times \{0\}$, or equivalently, $(k_1,m_1,n_1) \le (k_2,m_2,n_2)$ iff (i) $n_1 < n_2$, or (ii) $n_1 = n_2$, $k_1 \le k_2$, $m_1 \le m_2$.

[3]This is a special case of a general statement of Proposition 7.2.

Then $u = (1,1,1)$ is a strong unit in G. According to [6], we can show that (G,u) possesses a unique state s, namely $s(k,m,n) = n$ for $(k,m,n) \in G$.

We show that (G,u) possesses the state interpolation property; for that we show that $f_*(x) = s(x) = f^*(x)$ for any $x \in G$. Let $x = (k_1,m_1,n_1)$ and let either $f_*(x) \le s(x) = n_1 < f^*(x)$ or $f_*(x) < s(x) = n_1 \le f^*(x)$. In the first case, we may choose two integers k and $n \ge 1$ such that $n_1 < k/n < f^*(x)$. Since $k/n < f^*(x)$, we have $nx \not\le ku$. Hence $n(k_1,m_1,n_1) \not\le (k,k,k)$ which gives three possibilities: (a) $nn_1 < k$, (b) $nn_1 = k$, and (c) $nn_1 > k$ which all give a contradiction. For the second case, use (d) of Proposition 3.2 and the first case.

Proposition 4.2. Let $(G,u) = (G_1,u_1) \times \cdots \times (G_n,u_n)$, where each (G_i,u_i) is a unital po-group. If s is a state on (G,u), then there exists a unique integer k $(1 \le k \le n)$ and only states $s_{i_1},\dots,s_{i_k}$ on $(G_{i_1}u_{i_1}),\dots,(G_{i_k},u_{i_k})$ such that

$$s(x_1,\dots,x_n) = \sum_{j=1}^{k} \lambda_{i_j} s_{i_j}(x_{i_j}), \quad (x_1,\dots,x_n) \in G, \tag{4.1}$$

where $\lambda_{i_j} > 0$ and $\sum_{j=1}^k \lambda_{i_j} = 1$.

Conversely, if $s_{i_1},\dots,s_{i_k}$ are states on $(G_{i_1},u_{i_1}),\dots,(G_{i_k},u_{i_k})$ and $\lambda_{i_j} > 0$ for $j = 1,\dots,k$, $1 \le k \le n$, with $\sum_{j=1}^k \lambda_{i_j} = 1$, then the right-hand side of (4.1) defines a state on (G,u).

Proof. Suppose that s is a state on (G,u). Define $\lambda_i := s(0,\dots,0,1,0,\dots,0)$, $i = 1,\dots,n$. For nonzero λ_i we define $s_i(x_i) := \frac{1}{\lambda_i} s(0,\dots,0,x_i,0,\dots,0)$ which is a state on (G_i,u_i), and (4.1) holds.

The converse is evident. $\square$

Proposition 4.3. Let unital po-groups (G_1,u_1) and (G_2,u_2) satisfy the state interpolation property. Then so does $(G,u) = (G_1,u_1) \times (G_2,u_2)$.

Proof. Fix $x = (x_1,x_2) \in G$ and let q be a real number such that $f_*(x_1,x_2) \le q \le f^*(x_1,x_2)$. According to Proposition 3.7, there are two possibilities. Either $f_*^i(x_i) \le q \le f_i^*(x_i)$ for some $i = 1,2$, or $q \notin [f_*^i(x_i), f_i^*(x_i)]$ for $i = 1,2$.

In the first one, there exists a state s_i on (G_i,u_i) such that $s_i(x_i) = q$. Define $s(y_1,y_2) := s(y_1)$, $(y_1,y_2) \in G$. Then $s(x_1,x_2) = q$.

In the second one, there exist two real numbers q_1 and q_2 such that $q_1 < q < q_2$ and $q_i \in [f_*^i(x_i), f_i^*(x_i)]$. There are two states s_1 and s_2 on (G_1,u_1) and (G_2,u_2), respectively, such that $s_i(x_i) = q_i$. Take a real number $\lambda \in (0,1)$ such that $q = \lambda q_1 + (1-\lambda)q_2$. By Proposition 4.2, the mapping s on G defined by

$$s(y_1,y_2) := \lambda s_1(y_1) + (1-\lambda)s_2(y_2), \quad (y_1,y_2) \in G,$$

is a state on (G,u). Moreover, $s(x_1,x_2) = q$ which was needed. $\square$

For po-groups satisfying the state interpolation property, we extend Proposition 3.6.

Proposition 4.4. Let (G, u) be a unital po-group satisfying the state interpolation property. Then the following statements are equivalent:

(a) There is a unique state on (G, u).

(b) $f_* = f^*$.

(c) f_* is a state on (G, u).

(d) f^* is a state on (G, u).

Proof. (a) $\Rightarrow$ (b). The definition of the state interpolation property implies that, given $x \in G$, there exist states s and t on (G, u) such that $s(x) = f_*(x)$ and $t(x) = f^*(x)$. As $s = t$, we have $f_*(x) = f^*(x)$.

(b)$\Rightarrow$ (c). By the assumption, given $x \in G$, there exists a state s on (G, u) such that $f_*(x) = t(x) = f^*(x)$. Hence f^* is a state on (G, u).

By Proposition 3.6, (c) and (d) are equivalent, and they imply (a). $\square$

Now we show that if (G, u) does not satisfy the state interpolation property, then the equivalence of (a) – (d) in Proposition 4.4 can fail.

Example 4.5. Let $(G, u) = (\mathbb{R}, 1) \times (G_2, u_2)$, where (G_2, u_2) is a unital ℓ-group possessing no state, [6, Cor. 7.4]. Then (G, u) possesses a unique state s, namely $s(t, x) := t$, $(t, x) \in \mathbb{R} \times G_2$. Then by Proposition 3.7, we have $f^*(1/2, u_2) = \max\{1/2, 1\} = 1$ while $f_*(1/2, u_2) = \min\{1/2, 1\} = 1/2$, so that $f_*(1/2, u_2) < f^*(1/2, u_2)$, and (G, u) does not satisfy the state interpolation property.

Theorem 4.6. Let a unital po-group (G, u) satisfy the state interpolation property, and let $x \in G$. Then $s(x) > 0$ for all states s on (G, u) if and only if there exists an integer $i \geq 1$ such that ix is a strong unit in G.

Proof. If ix is a strong unit in G for some $i \geq 1$, then $kix \geq u$ for some $k \geq 1$. Then, for any state s on (G, u), we have $kis(x) \geq 1$, and thus $s(x) > 0$.

Conversely, let $s(x) > 0$ for any state s on (G, u). By hypothesis, there exists a state s on (G, u) such that $f_*(x) = s(x)$. Then there exist $k \in \mathbb{Z}$ and $i \geq 1$ such that $ku \leq ix$ and $k/i > 0$. Then $k > 0$ and $ix \geq ku \geq u > 0$. For any element $y \in G$, there exists an integer $n \geq 1$ such that $y \leq nu$ so that $y \leq nix$ which proves that ix is a strong unit in G. $\square$

A po-group G is said to be *unperforated* if $kx \geq 0$ for some $k \geq 1$ entails $x \geq 0$. Every ℓ-group is unperforated.

Corollary 4.7. Let an unperforated unital po-group (G, u) satisfy the state interpolation property. Then $s(x) > 0$ for all states s on (G, u) if and only if x is a strong unit in G.

Proof. By Theorem 4.6, there exists an integer $i \geq 1$ such that ix is a strong unit in G. The inequality $ix > 0$ yields $x > 0$. $\square$

Corollary 4.8. Let a unital ℓ-group (G, u) satisfy the state interpolation property, and let $x \in G$. Then $s(x) > 0$ for all states s on (G, u) if and only if x is a strong unit in G.

Proof. Since any ℓ-group is unperforated, we apply Corollary 4.7. $\square$

Remark 4.9. If a unital ℓ-group does not satisfy the state interpolation property, then the statement of Corollary 4.8 can fail. Indeed, take (G, u) from Example 4.5. Then $s(1/2, 0) = 1/2$ but $(1/2, 0)$ is not a strong unit in G.

We say that a net $\{s_\alpha\}$ of states on a unital po-group (G, u) *converges weakly* to a state s iff $s_\alpha(x) \to s(x)$ for all $x \in G$. Let $\mathcal{S}(G, u)$ and $Ext(\mathcal{S}(G, u))$ denote the system of all states and extremal states, respectively, on (G, u) Then $\mathcal{S}(G, u)$ is a convex compact Hausdorff space in the weak topology which can be the empty set [6, Cor. 7.4]. By the Krein–Mil'man theorem, any state on (G, u) is a weak limit of a net of convex combinations of $Ext(\mathcal{S}(G, u))$.

Theorem 4.10. Let a unital po-group (G, u) satisfy the state interpolation property, and let $x \in G$. Then $s(x) > 0$ for all extremal states s on (G, u) if and only if there exists an integer $i \geq 1$ such that ix is a strong unit in G.

Proof. Define a mapping $p : \mathcal{S}(G, u) \to \mathbb{R}$ by $p(s) := s(x)$, $s \in \mathcal{S}(G, u)$. Then p is continuous and affine,[5] and $p(s) > 0$ for all $s \in Ext(\mathcal{S}(G, u))$. By [15, Cor. 5.20 (b)], $p(s) > 0$ for all $s \in \mathcal{S}(G, u)$. Applying Theorem 4.6, we obtain the assertion in question. $\square$

Corollary 4.11. Let an unperforated unital po-group (ℓ-group) (G, u) satisfy the state interpolation property. Then $s(x) > 0$ for all extremal states s on (G, u) if and only if x is a strong unit in G.

Proof. This follows from Theorem 4.10 and Corollaries 4.7 and 4.8. $\square$

We say that a po-group is *Archimedean* if the equality $nx \leq y$ for all $n \geq 1$ $(x, y \in G)$ implies $x \leq 0$. For example, an ℓ-group G is Archimedean iff, for $x \in G^+$, $nx \leq y$ for any $n \geq 1$ $(y \in G)$ implies $x = 0$. It is well-known that any directed Archimedean po-group is commutative, [12, Cor. V.20].

Theorem 4.12. Let a unital po-group (G, u) have at least one state. The following statements are equivalent:

(a) $G^+ = \{x \in G : \ s(x) \geq 0 \text{ for all states } s \text{ on } (G, u)\}$.

(b) G is Archimedean.

(c) G is isomorphic (as an ordered group) to a subgroup of $\mathbb{R}^X$ for some nonempty set X.

[5] We say that a function p from a convex set K_1 onto a convex set K_2 is *affine* if it preserves convex combinations.

Proof. (a) $\Rightarrow$ (b). Let $nx \le y$ for any $n \ge 1$. Then, for any state s on (G,u), we have $s(x) \le 1/n$, i. e., $s(x) \le 0$, whence $x \le 0$ which proves that G is Archimedean.

(b) $\Rightarrow$ (c). Since (G,u) is directed and Archimedean, by [12, Cor. V.20] G is commutative. Let X be the set of all states on (G,u) and let $\phi: G \to \mathbb{R}^X$ be the evaluation mapping, so that $\phi(x)(s) := s(x)$ for all $x \in G$ and $s \in X$. Then $x \le y$ iff $\phi(x) \le \phi(y)$ which proves that ϕ is injective and order preserving, and $\phi(G) \subseteq \mathbb{R}^X$ as an ordered group.

(c) $\Rightarrow$ (a). If $x \in G^+$, then $s(x) \ge 0$ for any state s on (G,u). Let now $s(x) \ge 0$ for any state s on (G,u). Then $\phi(x) \ge 0$, where ϕ is an order preserving mapping of G into $\mathbb{R}^X$ for some $X \ne \emptyset$, which means that $x \ge 0$. □

Corollary 4.13. Let a unital po-group (G,u) have at least one state. The following statements are equivalent:

(a) $G^+ = \{x \in G : s(x) \ge 0 \text{ for all extremal states } s \text{ on } (G,u)\}$.

(b) G is Archimedean.

(c) G is isomorphic (as an ordered group) to a subgroup of $\mathbb{R}^X$ for some nonempty set X.

Proof. Define a map $p: \mathcal{S}(G,u) \to \mathbb{R}$ by $p(s) := s(x)$, $s \in \mathcal{S}(G,u)$. Then p is continuous and affine, and $p(s) \ge 0$ for all $s \in Ext(\mathcal{S}(G,u))$. By [15, Cor. 5.20], $p(s) \ge 0$ for all states $s \in \mathcal{S}(G,u)$ which proves that all conditions of Theorem 4.12 are satisfied. □

5. IDEALS AND STATES

In this section, we exhibit a close connection among states and ideals of unital po-groups. We show the main results, Theorem 5.2 and Theorem 5.8 saying that the existence of states on a unital po-group (G,u) is closely connected with the existence of a special ideal H such that $(G/H, u/H)$ is an Abelian unital po-group, and in addition any state on (G,u) can be shifted to $(G/H, u/H)$ and vice-versa.

An *o-ideal* of a directed po-group G is any normal convex directed subgroup H of G. A subgroup H of an ℓ-group G is an o-ideal of G iff H is an ℓ-ideal of G.

If H is an o-ideal of G, then G/H is a po-group, where $x/H \le y/H$ iff there exists $h \in H$ such that $x \le h + y$. Moreover, if (G,u) is a unital po-group, so is $(G/H, u/H)$.

Proposition 5.1. Let H be a proper convex and normal ideal of a unital po-group (G,u). Then, for any $x \in G$,

$$f_*(x) \le f_*^H(x/H) \le f_H^*(x/H) \le f^*(x), \tag{5.1}$$

where f_*^H and f_H^* are the corresponding functions on G/H defined by Proposition 3.2.

Proof. It is clear that $u \notin H$. Let $nx \le lu$ for some $n \ge 1$ and $l \in \mathbb{Z}$. Then $nx/H \le lu/H$ which proves $f^*_H(x/H) \le f^*(x)$. In a similar way we prove the second inequality. $\square$

Let s be a state on a unital po-group (G, u). Then the set

$$Ker(s) := \{g \in G : \ s(g) = 0\} \tag{5.2}$$

is a normal convex subgroup of G. We recall that not every $Ker(s)$ is an o-ideal of G. Indeed, take $(\mathbb{Z}^2, (1,1))$ and for a state s on it defined by $s(x,y) = (x+y)/2$, we have $Ker(s) = \{(n, -n) : \ n \in \mathbb{Z}\}$, which is a normal convex subgroup of $\mathbb{Z}^2$ but not an ℓ-subgroup.

For any state s on a unital po-group (G, u), we define

$$\mathrm{K}(s) := \{x - y : \ x, y \in G^+ \cap Ker(s)\}. \tag{5.3}$$

Then $\mathrm{K}(s)$ is an o-ideal of (G,u). Moreover, the mapping $\tilde{s} : \ G/\mathrm{K}(s) \to \mathbb{R}$ defined via

$$\tilde{s}(x/\mathrm{K}(s)) := s(x), \quad x/\mathrm{K}(s) \in G/\mathrm{K}(s), \tag{5.4}$$

is a state on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$, and $\mathrm{K}(\tilde{s}) = \{0\}$.

Similarly, we define $\mathrm{C}(s)$ as the convex subgroup of G generated by $Ker(s)$. Then $\mathrm{C}(s)$ is a normal convex subgroup of G, and $\mathrm{K}(s) \subseteq \mathrm{C}(s)$, and $G/\mathrm{C}(s)$ is an Abelian po-group. Moreover, if s is a state on (G,u), then the mapping $\bar{s}$ on $G/\mathrm{C}(s)$ defined by

$$\bar{s}(x/\mathrm{C}(s)) := s(x) \quad (x \in G) \tag{5.5}$$

is a state on $(G/\mathrm{C}(s), u/\mathrm{C}(s))$. Indeed, if $x - y \in \mathrm{C}(s)$, then by [12, (d), p. 18], $\mathrm{C}(s) := \{z \in G : \ \exists\, a_1, a_2 \in Ker(s), \ b_1, b_2 \in G^+, a_1 + b_1 = z = a_2 - b_2\}$, and hence $s(x-y) = s(b_1) = s(-b_2)$ which entails $s(x-y) = 0$ and $s(x) = s(y)$.

Let G be a po-group. We denote by $\mathcal{CN}_c(G)$ the set of all convex normal subgroups I of G such that G/I is a commutative po-group. It is clear that $G \in \mathcal{CN}_c(G)$.

The following important result shows the state space of a (non-Abelian) unital po-group is affinely isomorphic with the state space of some Abelian unital po-group.

Theorem 5.2. Let G be a directed po-group. Denote by

$$I_c := \bigcap \{I : \ I \in \mathcal{CN}_c(G)\}. \tag{5.6}$$

Then $I_c \in \mathcal{CN}_c(G)$, and G is an Abelian po-group if and only if $I_c = \{0\}$.

If, in addition, (G,u) is a unital po-group, then (G,u) possesses at least one state if and only if $I_c \ne G$. Moreover, if s is a state on (G,u), then the mapping $\hat{s}$ on G/I_c defined by

$$\hat{s}(x/I_c) := s(x) \ (x \in G) \tag{5.7}$$

is a state on $(G/I_c, u/I_c)$. Conversely, if t is a state on $(G/I_c, u/I_c)$, then the mapping s on (G,u) defined by $s(x) := t(x/I_c)$, $x \in G$, is a state on (G,u).

P r o o f . It is evident that I_c is a normal and convex ideal of G. Let $x, y \in G$. Then $x + y - (y + x) \in I$ for any $I \in \mathcal{CN}_c(G)$. Hence $x + y - (y + x) \in I_c$, which entails $I_c \in \mathcal{CN}_c(G)$.

Suppose that s is a state on (G, u). Then $\mathrm{C}(s) \in \mathcal{CN}_c(G)$, so that $I_c \subseteq \mathrm{C}(s)$ for any state s which proves that $I_c \neq G$ while $u \notin \mathrm{C}(s)$. In addition, the mapping $\hat{s}$ defined by $\hat{s}(x/I_c) := s(x)$ ($x \in G$) is a well-defined mapping while if $x/I_c = y/I_c$, then $x - y \in I_c \subseteq \mathrm{C}(s)$ which means that $s(x) = s(y)$. Hence, $\hat{s}$ is a state on $(G/I_c, u/I_c)$.

If $I_c \neq G$, then G/I_c is Abelian. It is well-known that every Abelian unital po-group possesses at least one state [15, Cor. 4.4]. Suppose that t is a state on $(G/I_c, u/I_c)$. Then the mapping $s(x) := t(x/I_c)$, $x \in G$, is a state on (G, u). □

Now we prove that if s is a state on a unital ℓ-group (G, u), then $G/\mathrm{K}(s)$ is an Abelian ℓ-group.

Proposition 5.3. Let s be a state on a unital ℓ-group (G, u), then $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ is an Abelian and Archimedean ℓ-group, and $\mathrm{K}(s) \in \mathcal{CN}_c(G)$.

P r o o f . According to the definition, $\mathrm{K}(s)$ is an ℓ-group, so that $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ is a non-trivial ℓ-group. We show that $G/\mathrm{K}(s)$ is Archimedean. Assume that $x/\mathrm{K}(s)$ is a positive element in $G/\mathrm{K}(s)$ such that $nx/\mathrm{K}(s) \leq y/\mathrm{K}(s)$ for any integer $n \geq 1$ and some element $y \in G$. Then $0 \leq n\tilde{s}(x/\mathrm{K}(s)) \leq \tilde{s}(y/\mathrm{K}(s))$, where $\tilde{s}$ is defined by (5.4). Then $ns(x) \leq s(y)$, so that $s(x) = 0$, which implies $\tilde{s}(x/\mathrm{K}(s)) = 0$, i.e., $x/\mathrm{K}(s) = 0$. Since any Archimedean ℓ-group is commutative, [12, Cor. V.20], we have that $\mathrm{K}(s) \in \mathcal{CN}_c(G)$. □

In what follows, we show that if s is a state on a unital po-group (G, u) which is not an ℓ-group with $Ker(s) = \{0\}$, then (G, u) is not necessary Archimedean even if it is commutative, compare with Theorem 4.10.

Example 5.4. Let $G = \mathbb{Z}$ with $G^+ := \{0, 2, 3, \dots\}$, then $u = 2$ is a strong unit in G and (G, u) possesses a unique state s, namely $s(n) := n/2$, $n \in G$. Then $Ker(s) = \{0\}$, but G is not Archimedean as well as not unperforated ($2 \cdot 1 \in G^+$, but $1 \notin G^+$); we have $\{x \in G : \ s(x) \geq 0\} = \{0, 1, 2, \dots\} \neq G^+$.

Proposition 5.5. Let s be a state on a unital ℓ-group (G, u). Then the following statements are equivalent:

(i) s is extremal.

(ii) $s(x \wedge y) = \min\{s(x), s(y)\}$ for all $x, y \in G^+$.

(iii) s is a lattice homomorphism.

(iv) $Ker(s)$ is a normal maximal ℓ-ideal of G.

In such the case, $\mathrm{K}(s) = Ker(s)$.

P r o o f . The equivalence of (i) – (iv) follows from [8, Thm. 4.1].

It is clear that $\mathrm{K}(s) \subseteq Ker(s)$. Suppose that $g \in Ker(s)$. Then by (iii), $s(g^+) = s(g \vee 0) = \max\{s(g), s(0)\} = 0$. In a similar way, $s(g^-) = 0$, so that $s(|g|) = 0$, which yields that for $g = g^+ - g^-$ we have $g^+, g^- \in Ker(s) \cap G^+$, i. e., $Ker(s) \subseteq \mathrm{K}(s)$. □

A state s on (G, u) is said to be *discrete* if $s(G) = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$.

Proposition 5.6. Let s be a discrete state on a unital ℓ-group (G, u). Then $(G/Ker(s))$ is a sub-ℓ-group of the ℓ-group $\mathbb{R}$ of real numbers.

Proof. By Proposition 5.5, $Ker(s) = \mathrm{K}(s)$, so that $G/Ker(s)$ is by Proposition 5.3 an Abelian ℓ-group. Defining $\tilde{s}$ by (5.4), we see that $\tilde{s}$ is a discrete state on $(G/Ker(s), u/Ker(s))$ with $Ker(\tilde{s}) = \{0\}$, and there is no infinite sequence $\{x_i/Ker(s)\}$ of positive elements of $G/Ker(s)$ such that $x_1/Ker(s) > x_2/Ker(s) > \cdots > 0$. Therefore, $G/Ker(s)$ is a simplicial group, that is $G/Ker(s)$ is isomorphic (as a po-group) to $\mathbb{Z}^n$ for some integer $n \geq 1$, [15, Cor. 3.14].

Choose a simplicial basis $\{y_1, \ldots, y_n\}$ for $G/Ker(s)$. Then

$$u/Ker(s) = a_1 y_1 + \cdots + a_n y_n$$

for some integers a_i. By [15, Prop. 6.6], each $a_>0$, and any state t on $(G/Ker(s), u/Ker(s))$ equals the convex hull of the states $t_1, \ldots, t_n$ defined by

$$t_j(c_1 y_1 + \cdots + c_n y_n) = c_j / a_j.$$

Therefore, $\tilde{s}$ is a convex combination of $t_1, \ldots, t_n$. Since s is extremal, so is $\tilde{s}$ on $(G/Ker(s), u/Ker(s))$, and we have $\tilde{s} = t_j$ for some j. Since $Ker(t_j)$ is a maximal ℓ-ideal, Proposition 5.5 (iv), $Ker(t_j)$ is nonzero if $n > 1$, we must have $n = 1$.

Thus $G/Ker(s) \cong \mathbb{Z}$ (as ordered groups) which proves that $G/Ker(s)$ is linear.

Since $\tilde{s}(G/Ker(s)) = \frac{1}{m}\mathbb{Z}$ for some integer $m \geq 1$, $\tilde{s}$ is an ℓ-group homomorphism, Proposition 5.5 (iii) with $Ker(\tilde{s}) = \{0\}$, we have that $\tilde{s}$ defines an injective ℓ-group homomorphism of $G/Ker(s)$ into $\mathbb{R}$. □

Let G be an ℓ-group. We denote by $\mathcal{C}\ell(G)$ the set of all ℓ-ideals I of G such that G/I is commutative.

Proposition 5.7. Let $s_1, \ldots, s_k$ be distinct discrete extremal states on a unital ℓ-group (G, u) and set

$$H = Ker(s_1) \cap \cdots \cap Ker(s_k).$$

Then $H \in \mathcal{C}\ell(G)$ and the quotient group G/H is isomorphic with $s_1(G) \times \cdots \times s_k(G) \subset \mathbb{R}^k$.

Proof. According to (iv) of Proposition 5.5, $Ker(s_i)$ is an ℓ-ideal of G, so is H. In view of Proposition 5.6, $G/Ker(s_i)$ is commutative, so is G/H. Since $s_i \neq s_j$, by [7, Prop. 4.5., Thm. 6.1], we have $Ker(s_i) \neq Ker(s_j)$.

Define a mapping s_i^H on $(G/H, u/H)$ such that $s_i^H(x/H) := s_i(x)$ $(x \in G, i = 1, \dots, k)$. Then each s_i^H is a discrete state on $(G/H, u/H)$ and $\bigcap_{i=1}^k Ker(s_i^H) = \{0\}$. Similarly as in the proof of Proposition 5.6, G/H is simplicial. Choose a simplicial basis $\{y_1, \dots, y_n\}$ for $G/Ker(s)$ and write $u/H = a_1y_1 + \cdots + a_ny_n$ for some integers a_i. Then each s_i^H is a convex combination of the states $t_1, \dots, t_n$ defined in the proof of Proposition 5.6. Since each s_i^H is an ℓ-group homomorphism, see Proposition 5.5, s_i^H is extremal, so that every s_i^H is equal to a unique t_j. Hence, $k \le n$. Since $\bigcap_{i=1}^k Ker(s_i^H) = \{0\}$, this implies $k = n$.

By Proposition 5.6 we have $G/Ker(s_i) \cong s_i(G)$. Consequently, the mapping $s: G/H \to G/Ker(s_1) \times \cdots \times G/Ker(s_k)$ defined via $s(x/H) := (s_1(x), \dots, s_k(x))$ $(x \in G)$ is an ℓ-group isomorphism of G/H with $s_1(G) \times \cdots \times s_k(G)$. □

Now we reformulate Theorem 5.2 for unital ℓ-groups. We recall that the convex sets K_1 and K_2 are *affinely isomorphic* if there exists a one-to-one correspondence $\phi: K_1 \to K_2$ which preserves convex combinations. If K_1 and K_2 are, in addition, two compact Hausdorff spaces, we say that K_1 and K_2 are *affinely homeomorphic* if the affine isomorphism $\phi: K_1 \to K_2$ is in addition a homeomorphism.

Theorem 5.8. The set

$$I_\ell := \bigcap \{I: I \in \mathcal{C}\ell(G)\} \tag{5.8}$$

is an element of $\mathcal{C}\ell(G)$.

If (G, u) is a unital ℓ-group, then the state spaces $\mathcal{S}(G, u)$ and $\mathcal{S}(G/I_\ell, u/I_\ell)$ are affinely homeomorphic.

Proof. Similarly as in Theorem 5.2, I_ℓ is an ℓ-ideal of G. Moreover, $I_\ell \in \mathcal{C}\ell(G)$.

Suppose that $I_\ell = G$. Then (G, u) has no state, otherwise, $K(s) \supseteq I_\ell$. Consequently, $(G/I_\ell, u/I_\ell)$ is the unital ℓ-group $\{0\}$, with strong unit $u/I_\ell = 0$.

If $I_\ell \ne G$, then $(G/I_\ell, u/I_\ell)$ is a nontrivial Abelian unital ℓ-group having at least one state, say t. Then $s(x) := t(x/I_\ell)$, $x \in G$, is a state on (G, u). Conversely, if s is a state on (G, u), then the mapping s' defined on $(G/I_\ell, u/I_\ell)$ via $s'(x/I_\ell) := s(x)$, $(x \in G)$ is a state on $(G/I_\ell, u/I_\ell)$, and the mapping $\phi: \mathcal{S}(G, u) \to \mathcal{S}(G/I_\ell, u/I_\ell)$ defined by $\phi(s) := s'$, $s \in \mathcal{S}(G, u)$ defines an affine homeomorphism in question. □

6. EXTREMAL STATES AND THE RIESZ INTERPOLATION PROPERTIES

We continue with the study of extremal states on unital po-groups having a generalized form of the Riesz interpolation property and the Riesz decomposition property.

For $a, b \in G^+$, we write $a \,\mathbf{com}\, b$ iff, for all $a_1, b_1 \in G^+$ such that $a_1 \le a$ and $b_1 \le b$, we have $a_1 + b_1 = b_1 + a_1$. According to [9, 10], we introduce different types of the Riesz interpolation properties.

Let $(G; +, 0, \le)$ be a directed po-group.

(a) We say that G fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1, a_2 \leq b_1, b_2$, there is a $c \in G$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(b) We say that G fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a, b_1, b_2 \geq 0$ such that $a \leq b_1 + b_2$, there are $d_1, d_2 \in G$ such that $0 \leq d_1 \leq b_1$, $0 \leq d_2 \leq b_2$ and $a = d_1 + d_2$.

(c) We say that G fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.

(d) We say that G fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \,\mathbf{com}\, d_3$.

(e) We say that G fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \geq 0$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1, d_2, d_3, d_4 \geq 0$ such that (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

Then

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Leftrightarrow (\mathrm{RIP}),$$

and if G is Abelian, then (RDP_0) $\Leftrightarrow$ (RDP_1); if G is not Abelian, the converse implications do not hold, in general, [9, 10]. In addition, (RDP_2) holds in G iff G is an ℓ-group.

Similar notions can be introduced also for pseudo-effect algebras. Then they are in a one-to-one correspondence as the intervals $\Gamma(G, u)$ in unital po-groups with (RDP_1) or with (RDP_2), for more details see [9, 10].

Proposition 6.1. *Let H be an o-ideal of a directed po-group G. If G satisfies one of the Riesz decomposition properties (a) – (e), so does G/H.*

Proof. (RIP). Let $a_i/H \leq b_j/H$, $i, j = 1, 2$. Then there are elements $h_{ij} \in H$ such that $a_i \leq h_{ij} + b_j$. Since H is directed, there exists $h \in H$ such that $h_{ij} \leq h$ for $i, j = 1, 2$. Then $a_i \leq h + b_j$. Hence there exists $c \in G$ such that $a_i \leq c \leq h + b_j$ which gives $a_i/H \leq c/H \leq b_j/H$ for $i = 1, 2$.

(RDP_0). Let $a/H \leq b_1/H + b_1/H$ for $a/H, b_1/H, b_2/H \geq 0$. There exists $h, h_a, h_1, h_2 \in H$ such that $0 \leq h_a + a$, $0 \leq h_1 + b_1$, $0 \leq h_2 + b_2$, and $0 \leq h_a + a \leq (h + h_1 + b_1) + (h_2 + b_2)$. Since H is directed, we can assume that $h \geq 0$. (RDP_0) on G entails that there exist $a_1, a_2 \in G^+$ such that $h_a + a = a_1 + a_2$ and $a_1 \leq (h + h_1 + b_1)$ and $a_2 \leq (h_2 + b_2)$, which proves $a/H = a_1/H + a_2/H$ and $a_i/H \leq b_i/H$ for $i = 1, 2$.

(RDP). Let, for $a_i/H, b_j/H \geq 0$ we have $a_1/H + a_2/H = b_1/H + b_2/H$. Then there exist four elements $h_1, h_2, h_3, h_4 \in H$ such that $h_1 + a_1, h_2 + a_2, h_3 + b_1, h_4 + b_2 \in G^+$. Choose an element $h \in H$ such that $h_1 + a_1 + h_2 + a_2 = h + h_3 + b_1 + h_4 + b_2$. Since H is directed, there exists $h_0 \in H^+$ such that $h_i \leq h_0$ for $i = 1, 2, 3, 4$. Hence $(h_0 + h_1 + a_1) + (h_2 + a_2 + h_0) = (h_0 + h + h_3 + b_1) + (h_4 + b_2 + h_0) = h' + (h_0 + h_3 + b_1) + (h_4 + b_2 + h_0)$, where $h' \in H$. Expressing $h' = -h_1' + h_2'$, where $h_1', h_2' \in H^+$, we have $(h_1'+h_0+h_1+a_1)+(h_2+a_2+h_0) = (h_2'+h_0+h_3+b_1)+(h_4+b_2+h_0)$, where all the elements in parentheses are positive. Apply (RDP) on G to this equality, we have G/H satisfies (RDP).

(RDP$_1$). Let $a_1/H+a_2/H = b_1/H+b_2/H$. As in the proof of (RDP), we have that there exist elements $d_{11}, d_{12}, d_{21}, d_{22} \in G^+$ such that $h_1'+h_0+h_1+a_1 = d_{11}+d_{12}$, $h_2+a_2+h_0 = d_{21}+d_{22}$, $h_2'+h_0+h_3+b_1 = d_{11}+d_{21}$, $h_4+b_2+h_0 = d_{12}+d_{22}$, and $d_{12} \,\mathbf{com}\, d_{21}$. Let now $0 \le x_1/H \le d_{12}/H$ and $0 \le x_2/H \le d_{21}/H$. Then there exist elements $h_0', h_0'', h', h'' \in H$ such that $x_1 \le h'+d_{12}$, $x_2 \le h''+d_{21}$ and $0 \le h_0'+x_1$, $0 \le h_0''+x_2$, so that $0 \le h_0'+x_1 \le h_0'+h'+d_{12}$ and $0 \le h_0''+x_2 \le h_0''+h''+d_{21}$. Since H is directed, there exists an element $h_3 \in H^+$ such that $h_3 \ge h_0'+h', h_0''+h''$. Then $0 \le h_0'+x_1 \le h_3+d_{12}$, and $0 \le h_0''+x_2 \le h_3+d_{21}$. Since (RDP$_0$) holds, there exist four elements $x_{11}, x_{12}, x_{21}, x_{22} \in G^+$ such that $h_0'+x_1 = x_{11}+x_{12}$, $h_0''+x_2 = x_{21}+x_{22}$ where $0 \le x_{11}, x_{21} \le h_3$ and $0 \le x_{12} \le d_{12}$, $0 \le x_{22} \le d_{21}$. Since $d_{12} \,\mathbf{com}\, d_{21}$, we have that x_{12} and x_{22} commute. Therefore,

$$\begin{aligned} x_1/H + x_2/H &= (h_0'+x_1)/H + (h_0''+x_2)/H \\ &= (x_{11}+x_{12})/H + (x_{21}+x_{22})/H \\ &= x_{12}/H + x_{22}/H = (x_{12}+x_{22})/H \\ &= (x_{22}+x_{12})/H = x_2/H + x_1/H, \end{aligned}$$

which proves that $d_{12}/H \,\mathbf{com}\, d_{21}/H$.

(RDP$_2$). Since (RDP$_2$) holds in G iff G is an ℓ-group, the statement is trivial while in such the case H is an ℓ-ideal of G. □

Proposition 6.2. A state s on a unital po-group (G,u) is extremal if and only if $\tilde{s}$ is extremal on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$.

Proof. Let s be extremal, and suppose $\tilde{s} = \lambda t_1 + (1-\lambda)t_2$, where t_1, t_2 are states on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ and $0 < \lambda < 1$. Then $s_i(x) := t_i(x/\mathrm{K}(s))$, $x \in G$, $(i = 1,2)$ is a state on (G,u). Hence, $s(x) = \tilde{s}(x/\mathrm{K}(s)) = \lambda t_1(x/\mathrm{K}(s)) + (1-\lambda)t_2(y/\mathrm{K}(s)) = \lambda s_1(x) + (1-\lambda)s_2(x)$ which implies $s_1 = s_2$, i.e., $t_1 = t_2$ and $\tilde{s}$ is extremal on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$.

Assume $\tilde{s}$ is extremal on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$, and let $s = \lambda s_1 + (1-\lambda)s_2$, where s_1 and s_2 are states on (G,u) and $0 < \lambda < 1$. Therefore, $Ker(s)^+ \subseteq Ker(s_i)^+$ for $i = 1,2$, i.e., $\mathrm{K}(s) \subseteq \mathrm{K}(s_i)$. Hence $t_i(x/\mathrm{K}(s)) := s_i(x)$ $(x \in G)$ is a state on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ for $i = 1,2$. Consequently, $\tilde{s} = \lambda t_1 + (1-\lambda)t_2$ which yields $t_1 = t_2$, i.e., $s_1 = s_2$. □

Proposition 6.3. Let s be a state on a linear unital po-group (G,u). Then $\mathrm{K}(s) = Ker(s)$. If $Ker(s) = \{0\}$, then G is commutative.

Proof. (i) It is clear that $\mathrm{K}(s) \subseteq Ker(s)$. Take $x \in Ker(s)$. Then either $x \ge 0$ or $x \le 0$ which proves $x \in \mathrm{K}(s)$.

(ii) Let $x+y \ge y+x$. Then $s((x+y)-(y+x)) = 0$ which by (i) proves $x+y = y+x$. □

Proposition 6.4. Let s be a discrete extremal state on a unital po-group (G,u) such that $G/\mathrm{K}(s)$ is linear. Then $G/\mathrm{K}(s)$ is commutative, $\mathrm{K}(s) = Ker(s)$, and $G/Ker(s)$ is isomorphic with $s(G)$ (as ordered groups).

Proof. Define $\tilde{s}$ by (5.4) on linear $(G/\mathrm{K}(s), u/\mathrm{K}(s))$. By Proposition 6.2, $Ker(\tilde{s}) = \mathrm{K}(\tilde{s}) = \{0\}$. Therefore, $G/\mathrm{K}(s)$ is a commutative ℓ-group.

The state $\tilde{s}$ is discrete on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$. Therefore, there is no infinite sequence $x_1/\mathrm{K}(s) > x_2/\mathrm{K}(s) > \cdots > 0$ which implies that $G/\mathrm{K}(s)$ is simplicial, i.e., $G/\mathrm{K}(s) \cong \mathbb{Z}^n$ (as po-groups) for some $n \geq 1$. Since $G/\mathrm{K}(s)$ is linear, we have $n = 1$.

Assume now $s(x) = 0$. Then $\tilde{s}(x/\mathrm{K}(s)) = 0$, i.e., $x/\mathrm{K}(s) \geq 0$ or $x/\mathrm{K}(s) \leq 0$. Consequently there exists $h \in \mathrm{K}(s)$ such that either $0 \leq h + x$ or $0 \leq h - x$, which implies $x \in \mathrm{K}(s)$.

The mapping $x/Ker(s) \mapsto s(x) = \tilde{s}(x/\mathrm{K}(s))$ defines an isomorphism of $G/Ker(s)$ onto $s(G)$. □

Proposition 6.5. Let s be a discrete state on a unital po-group (G,u) such that given any $x, y \in G^+$, there exists $z \in G^+$ such that $z \leq x$, $z \leq y$ and

$$s(z) = \min\{s(x), s(y)\}.$$

Then s is extremal.

Proof. Assume s is not extremal, then $s = \lambda s_1(x) + (1-\lambda)s_2(x)$, where s_1 and s_2 are distinct states on (G,u) and $0 < \lambda < 1$. Then $s_1(a) \neq s_2(a)$ for some $a \in G$. Replacing a by $-a$, if necessary, we may assume that $s_1(a) < s_2(a)$. Choose an integer $m \geq 1$ such that $b := a + mu \geq 0$. Since $0 \leq s_1(b) < s_2(b)$, there exist positive integers k, n such that

$$s_1(b) < k/n < s_2(b).$$

Set $x = nb$ and $y = ku$. Then $x, y \in G^+$ and $s_1(x) < s_1(y)$ and $s_2(y) < s_2(x)$. For any $z \in G^+$ satisfying $z \leq x$ and $z \leq y$, we have

$$\begin{aligned} s(z) &= \lambda s_1(z) + (1-\lambda)s_2(z) \leq \lambda s_1(x) + (1-\lambda)s_2(y) \\ &< \lambda s_1(x) + (1-\lambda)s_2(x) = s(x). \end{aligned}$$

and in a similar way, $s(z) < s(y)$. Therefore, $s(z) < \min\{s(x), s(y)\}$. □

Proposition 6.6. Let s be a discrete extremal state on a unital po-group (G,u) satisfying (RDP_0) such that $G/\mathrm{K}(s)$ is linear. Then given any $x, y \in G^+$, there exists $z \in G^+$ such that $z \leq x$, $z \leq y$ and $s(z) = \min\{s(x), s(y)\}$.

Proof. Assume s is extremal and consider $x, y \in G^+$. Without loss of generality, we can assume $s(x) \leq s(y)$. Due to the assumptions, by Proposition 6.5, $x/\mathrm{K}(s) \leq y/\mathrm{K}(s)$ which implies that $x \leq h + y$ for some $h \in \mathrm{K}(s)^+$. (RDP_0) entails that $x = h_1 + z$ for $0 \leq h_1 \leq h$, $0 \leq z \leq y$. Hence $h_1 \in \mathrm{K}(s)$ and $z \leq x$, so that $s(z) = s(x) = \min\{s(x), s(y)\}$. □

Remark 6.7. We note that if s is a (discrete) state on a unital ℓ-group (G, u), then $G/\mathrm{K}(s)$ is commutative, see Proposition 5.3. We don't know whether do Proposition 6.4 and Proposition 6.6 hold without the assumption that $G/\mathrm{K}(s)$ is linear.

Proposition 6.8. Let s be a discrete extremal state on a unital po-group (G, u) satisfying (RIP) such that $G/\mathrm{K}(s)$ is commutative. Then $Ker(s) = \mathrm{K}(s)$, $G/\mathrm{K}(s)$ is linear, and $G/Ker(s)$ is isomorphic with $s(G)$.

Proof. By Proposition 6.2, $\tilde{s}$ defined on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ by (5.4) is a discrete extremal state. Since $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ satisfies by Proposition 6.1 the Riesz interpolation property, applying [15, Prop. 6.19], we have that $(G/\mathrm{K}(s))/\mathrm{K}(\tilde{s}) \cong \mathbb{Z}$. But $G/\mathrm{K}(s) = (G/\mathrm{K}(s))/\mathrm{K}(\tilde{s})$ while $\mathrm{K}(\tilde{s}) = \{0\}$. Hence $G/\mathrm{K}(s) \cong \mathbb{Z}$.

Now we show that $Ker(s) = \mathrm{K}(s)$. Let $s(x) = 0$. Hence $\tilde{s}(x/\mathrm{K}(s)) = 0$. Since either $x/\mathrm{K}(s) \geq 0$ or $x/\mathrm{K}(s) \leq 0$, we have $0 \leq h + x$ or $0 \leq h - x$ for some $h \in \mathrm{K}(s)$, which yields $x \in \mathrm{K}(s)$.

Since $G/\mathrm{K}(s)$ is isomorphic with $s(G)$, we have $G/Ker(s)$ is isomorphic with $s(G)$. □

We generalize Proposition 6.5 and Proposition 6.6 to non-discrete states.

Proposition 6.9. Let s be a discrete state on a unital po-group (G, u) such that

$$\min\{s(x), s(y)\} = \sup\{s(z) : \ 0 \leq z \leq x, y\} \tag{6.1}$$

for all $x, y \in G^+$. Then s is extremal.

Proof. Suppose that $s = \lambda s_1(x) + (1-\lambda)s_2(x)$, where s_1 and s_2 are distinct states on (G, u) and $0 < \lambda < 1$. Since G is directed, there exists $a \in G^+$ such that $s_1(a) \neq s_2(a)$. After renumbering s_1 and s_2, if necessary, we may assume $0 \leq s_1(a) < s_2(a)$. Choose two integers $m, n \leq 1$ such that $s_1(a) < m/n < s_2(a)$, and set $x = na$ and $y = mu$. Then x and y are positive elements for which $s_1(x) < s_1(y)$ and $s_2(x) > s_2(y)$.

For any $z \in G^+$ with $z \leq x$ and $z \leq y$, we have

$$s(z) = \lambda s_1(z) + (1-\lambda)s_2(z) \leq \lambda s_1(x) + (1-\lambda)s_2(y).$$

Consequently,

$$\min\{s(x), s(y)\} = \sup\{s(z) : \ 0 \leq z \leq x, y\} \leq \lambda s_1(x) + (1-\lambda)s_2(y).$$

Since $0 < \lambda < 1$ and $s_1(x) < s_1(y)$, we have $\lambda s_1(x) + (1-\lambda)s_2(y) < \lambda s_1(y) + (1-\lambda)s_2(y) = s(y)$. Similarly, $\lambda s_1(x) + (1-\lambda)s_2(y) < s(x)$, which is a contradiction. □

Theorem 6.10. Let s be a state on a unital po-group (G,u) satisfying (RIP) such that $G/\mathrm{K}(s)$ is commutative. Then s is extremal if and only if given $x,y \in G^+$,

$$\min\{s(x),s(y)\} = \sup\{s(z):\ 0 \le z \le x,y\}.$$

Proof. Let (6.1) hold. By Proposition 6.9, s is extremal.

Assume that s is extremal. By Proposition 6.2, s is extremal iff $\tilde{s}$ defined by (5.4) is extremal on $(G/\mathrm{K}(s), u/\mathrm{K}(s))$. Then $G/\mathrm{K}(s)$ is a commutative po-group satisfying (RIP), see Proposition 6.1. Therefore by [15, Thm. 12.14], we have that (6.1) applied to $\tilde{s}$ and $(G/\mathrm{K}(s), u/\mathrm{K}(s))$ is necessary and sufficient for $\tilde{s}$ to be extremal. That is, given any $x/\mathrm{K}(s), y/\mathrm{K}(s) \ge 0$, we have

$$\min\{\tilde{s}(x/\mathrm{K}(s)),\tilde{s}(y/\mathrm{K}(s))\} = \sup\{\tilde{s}(z/\mathrm{K}(s)) : 0 \le z/\mathrm{K}(s) \le x/\mathrm{K}(s), y/\mathrm{K}(s)\}.$$

Let $x,y \in G^+$ be given and $0 \le z \le x,y$. Then we have $\min\{s(x),s(y)\} = \sup\{\tilde{s}(z/\mathrm{K}(s)) :\ 0 \le z/\mathrm{K}(s) \le x/\mathrm{K}(s), y/\mathrm{K}(s)\}$. Set $K_0 := \min\{s(x),s(y)\}$. Given $\epsilon > 0$, there exists $z' \in G$ with $0 \le z'/\mathrm{K}(s) \le x/\mathrm{K}(s), y/\mathrm{K}(s)$, such that $K_0 - \epsilon < \tilde{s}(z'/\mathrm{K}(s)) = s(z')$. Hence there exist $h_1, h_2 \in \mathrm{K}(s)$ such that $0 \le h_1 + z' \le h_2 + x$ and $h_1 + z' \le h_2 + y$. Putting $h = -h_2 + h_1$, we have $h + z' \le x, y$.

Since G is directed, there exists an integer $m \ge 1$ such that $h+z'+mu \ge 0$. Then $0 \le h + z' + mu, z + mu \le x + mu, y + mu$, and due to (RIP), there exists z_0' such that $h+z'+mu, z+mu \le z_0' \le x+mu, y+mu$. Consequently, for $z_0 = z_0' - mu$ we have $h+z', z \le z_0 \le x,y$. Hence $K_0 - \epsilon < s(z') = s(h+z') \le s(z) \le \min\{s(x),s(y)\}$, which implies $K_0 = \sup\{s(z) :\ 0 \le z \le x,y\}$. □

We note that we do not know whether the condition that $G/\mathrm{K}(s)$ is commutative is necessary for the validity of Theorem 6.10.

7. NORMAL–VALUED ℓ–GROUPS AND THE STATE INTERPOLATION PROPERTY

In the present section, we show that any normal-valued unital ℓ-group possesses the state interpolation property. The example of the Scrimger 2-group from Example 4.1 is a special case of the following Proposition 7.2.

We recall some notions of ℓ-group theory. For any ℓ-group G and $g \in G$, we say that a convex ℓ-subgroup V of G is a *value* of g if $g \notin V$ yet $g \in H$ for all convex ℓ-subgroups H which properly contain V. A *cover* of a convex ℓ-subgroup A is a convex ℓ-subgroup B which properly contains A, and if, for a convex ℓ-subgroup C we have $A \subseteq C \subseteq B$, then either $A = C$ or $B = C$. An ℓ-group G is said to be *normal-valued* if every value is normal in its cover.

Let $a,b \in G$, we say that a is *infinitarily small with respect* to b (in notation $a \ll b$) if $na \le b$ for $n \in \mathbb{Z}$.

According to [6, Thm. 6.5], any normal-valued unital ℓ-group (G,u) possesses at least one state, and at least one maximal convex ℓ-subgroup which is an ℓ-ideal.

Denote by $\mathrm{Rad}(G,u)$, the *radical* of (G,u), the intersection of all maximal convex ℓ-subgroups of (G,u). By [1, Prop. 4.3.9],

$$\mathrm{Rad}(G,u) = \{g \in G : g \ll u\}, \tag{7.1}$$

and $\mathrm{Rad}(G,u)$ is an ℓ-ideal of (G,u) for any normal-valued unital ℓ-group (G,u).

Proposition 7.1. Let (G,u) be a normal-valued unital ℓ-group. For any $x \in G$ and $a \in \mathrm{Rad}(G,u)$ we have

$$f^*(x+a) = f^*(x) = f^*(a+x),$$
$$f_*(x+a) = f_*(x) = f_*(a+x).$$

Proof. *Step 1.* Let $x \in G$ and $a \in \mathrm{Rad}(G,u)$. Assume $nx \le lu$, where $n \ge 1$ and $l \in \mathbb{Z}$. Then, for any integer $t \ge 1$, we have $tnx \le tlu$. Hence $tn(x+a) = tnx + a'_{tn}$, where $a'_{tn} \in \mathrm{Rad}(G,u)$. Hence $tnx + a'_{tn} \le tlu + u$ which gives $f^*(x+a) \le (tl+1)/(tn) = l/n + 1/(tn)$ for any $t \ge 1$. Therefore, $f^*(x+a) \le l/n$, i. e. $f^*(x+a) \le f^*(x)$.

Step 2. Assume now $n(x+a) \le lu$ for $n \ge 1$ and $l \in \mathbb{Z}$. Then, for any integer $t \ge 1$, we have $tn(x+a) \le tlu$, so that $tnx + a'_{tn} \le tlu$ for some $a'_{tn} \in \mathrm{Rad}(G,u)$. This implies $tnx \le tlu - a'_{tn} \le tlu + u$ which yields $f^*(x) \le l/n + 1/(tn)$ for any $t \ge 1$, i. e. $f^*(x) \le f^*(x+a)$.

Combining Steps 1–2, we have $f^*(x) = f^*(x+a)$.

Since (G,u) is normal-valued, then $a + x = x + b$ for some $b \in \mathrm{Rad}(G,u)$. Then $f^*(a+x) = f^*(x+b) = f^*(x)$.

The rest is evident. □

Proposition 7.2. Every normal-valued unital ℓ-group (G,u) possesses the state interpolation property, and for $x \in G$

$$f^*_H(x/H) = f^*(x), \quad f^H_*(x/H) = f_*(x), \tag{7.2}$$

where $H = \mathrm{Rad}(G,u)$.

Proof. Put $H = \mathrm{Rad}(G,u)$. By Proposition 5.1, $f^*_H(x/H) \le f^*(x)$.

Assume $n(x/H) \le l(u/H)$ for some $n \ge 1$ and $l \in \mathbb{Z}$. Then there exists an element $a \in \mathrm{Rad}(G,u)$ such that $a + nx \le lu$. By Proposition 3.2 and Proposition 7.1, $f^*(a+nx) = f^*(nx) \le f^*(lu)$, i. e., $nf^*(x) \le l$ and $f^*(x) \le l/n$ which gives $f^*(x) \le f^*_H(x/H)$.

In a similar way we have $f_*(x) = f^H_*(x/H)$ which proves (7.2).

Choose a real number q such that $f_*(x) \le q \le f^*(x)$. Then by (7.2), $f^H_*(x/H) \le q \le f^*_H(x/H)$. As in Theorem 5.2, $(G/H, u/H)$ is a unital Abelian ℓ-group, and it has the state interpolation property [15, Prop. 4.7]. Therefore, there exists a state s_H on $(G/H, u/H)$ such that $q = s_H(x/H)$. A mapping $s : G \to \mathbb{R}$ defined via $s(x) := s_H(x/H)$ $(x \in G)$ is a state on (G,u) such that $q = s(x)$ which proves that (G,u) has the state interpolation property. □

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