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POSSIBILISTIC ALTERNATIVES OF ELEMENTARY NOTIONS AND RELATIONS OF THE THEORY OF BELIEF FUNCTIONS[1]

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The elementary notions and relations of the so called Dempster–Shafer theory, introducing belief functions as the basic numerical characteristic of uncertainty, are modified to the case when probabilistic measures and basic probability assignments are substituted by possibilistic measures and basic possibilistic assignments. It is shown that there exists a high degree of formal similarity between the probabilistic and the possibilistic approaches including the role of the possibilistic Dempster combination rule and the relations concerning the possibilistic nonspecificity degrees.

1. INTRODUCTION – CLASSICAL BELIEF FUNCTIONS

First of all, let us explicitate the following methodological principle: this paper is conceived as a mathematical and theoretical one, so that the reader interested in the intuition and possible interpretations behind the notions introduced and statements claimed and proved below is kindly invited to consult appropriate sources from an already rich list of works dealing with belief functions and Dempster–Shafer theory. For the same reasons we shall not go into details when analyzing the intuition and interpretation behind the possibilistic alternatives of the notions and relations of the theory of belief functions, which are introduced, investigated and deduced below.

The most simple combinatorial definition of classical non-normalized belief function over a finite nonempty space S reads as follows. *Basic probability assignment* (b.p.a.) over S is a mapping m which takes the power-set $\mathcal{P}(S)$ of all subsets of S into the unit interval $[0,1]$ of real numbers in such a way that $\sum_{A\subset S} m(A) = 1$. Hence, m is nothing else than a probability distribution over the power-set $\mathcal{P}(S)$. The *non-normalized degree of belief* bel_m generated by the b.p.a. m and ascribed to a subset A of S is defined by

$$bel_m(A) = \sum_{\emptyset\neq B\subset A} m(B), \tag{1.1}$$

setting $bel_m(\emptyset) = 0$ for the empty subset $\emptyset$ of S.

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An alternative way how to arrive at (1.1) reads as follows. Let S be taken as the set of all possible internal states of a system (answers to a question, solutions to a problem, medical or technical diagnoses, etc.), let E be the space of empirical values (observations, symptoms, hints) concerning the system in question, and let $\rho : S \times E \to \{0,1\}$ (or $\rho \subset S \times E$) be a *compatibility relation* which defines the subjects's knowledge as far as the system is concerned. Namely, if $\rho(s,x) = 0$ for some $s \in S$ and $x \in E$, then x cannot be the actual internal state of the system supposing that x was observed, hence, s and x are *incompatible*. If $\rho(s,x) = 1$, then the state s and the empirical value x are *compatible*. So, for each $x \in E$, the set $U_\rho(x) = \{s \in S : \rho(s,x) = 1\}$ of states compatible with x is defined.

Let the observed empirical value x be of statistical (stochastical) nature, formally, let x be the value taken by a random variable X defined on a fixed probability space $\langle \Omega, \mathcal{A}, P\rangle$ with the values in a measurable space $\langle E, \mathcal{E}\rangle$ over E. Under some reasonable measurability conditions, e. g., when E is finite and $\mathcal{E} = \mathcal{P}(E)$, the composed mapping $U(X(\cdot)) : \Omega \to \mathcal{P}(S)$ is a set-valued (generalized) random variable. Setting $m(A) = P(\{\omega \in \Omega : U(X(\omega)) = A\})$ for each $A \subset S$, (1.1) transforms into

$$bel_m(A) = P\left(\{\omega \in \Omega : \emptyset \neq U(X(\omega)) \subset A\}\right). \tag{1.2}$$

Also the *Dempster combination rule*, which enables to combine the degrees of belief ascribed by two or more subjects to the same system charged with uncertainty, can be most easily defined at the algebraic combinatorial level. Let m_1, m_2 be b.p.a.'s on the same finite space S. Define the mapping $m_1 \oplus m_2 : \mathcal{P}(S) \to (-\infty, \infty)$ setting for each $A \subset S$

$$(m_1 \oplus m_2)(A) = \sum\nolimits_{B,C \subset S,\, B \cap C = A} m_1(B)\, m_2(C). \tag{1.3}$$

An easy calculation yields that $m_1 \oplus m_2$ is also a b.p.a. on S, so that $(m_1 \oplus m_2)(A) \in [0,1]$ holds for each $A \subset S$ and $\sum_{A \subset S} (m_1 \oplus m_2)(A) = 1$. The b.p.a. $m_1 \oplus m_2$ is called the *Dempster product* of the b.p.a.'s m_1 and m_2 and the operation $\oplus$ is called the *Dempster combination rule*. For more b.p.a.'s $m_1, m_2, \dots, m_n$ on the same S their Dempster product $m_1 \oplus m_2 \oplus \cdots \oplus m_n$, or simply $\bigoplus_{i=1}^n m_i$, is defined by recursion, i. e., by $(m_1 \oplus m_2 \oplus \cdots \oplus m_{n-1}) \oplus m_n$. As the operation $\oplus$ is commutative and associative, the bracketing is irrelevant and can be omitted. The so called *vacuous* b.p.a. m_S is defined by $m_S(S) = 1$, so that $m_S(A) = 0$ for every $A \subset S$, $A \neq S$, and plays the role of the unit element with respect to the operation $\oplus$ in the sense that

$$(m \oplus m_S) \equiv (m_S \oplus m) \equiv m \tag{1.4}$$

holds for each b.p.a. m on S, here $\equiv$ denotes the equality of the corresponding values for all subsets of S.

The Dempster combination rule for belief functions, denoted with a certain tolerance also by $\oplus$, is defined by the Dempster product of the corresponding b.p.a.'s in the following way. If m_1, m_2 are b.p.a.'s on S and bel_{m_1}, bel_{m_2} are the corresponding belief functions (as S is finite, there exists a one-to-one relation between basic probability assignments and belief functions), then the Dempster product $bel_{m_1} \oplus bel_{m_2}$ of bel_{m_1} and bel_{m_2} is defined by

$$bel_{m_1} \oplus bel_{m_2} \equiv_{\mathrm{df}} bel_{m_1 \oplus m_2}. \tag{1.5}$$

Also within the space of belief functions the Dempster combination rule is commutative and associative, so that the way in which Dempster product of more b.p.a.'s is defined applies also to the case of belief functions.

The approach to Dempster combination rule through random sets reads as follows. Consider two subjects observing and investigating the same system, hence, S is the same for both of them. Also the observational space E, probability space $\langle \Omega, \mathcal{A}, P \rangle$ and random variable X are supposed to be the same for both the subjects. This assumption may be accepted without a too great loss of generality, as E can be a many-dimensional vector space $\mathbb{X}_{i=1}^{n} E_i$, so that observations of different nature, possibly made by different subjects, can be described by values from different E_i's. What matters are possibly different compatibility relations ρ_1, $\rho_2 : S \times E \to \{0,1\}$, defining the perhaps different kinds and degrees of knowledge of both the subjects as far as the decision or testing problem in question is concerned. The basic idea, when combining the pieces of knowledge of both the subjects, reads that any assertion of one of them, claiming that a particular $s \in S$ and $x \in E$ are *incompatible* is taken as valid and, consequently, accepted by the other subject. In symbols, the combined knowledge of both the subjects is defined by a new compatibility relation $\rho_{12} : S \times E \to \{0,1\}$ such that

$$\rho_{12}(s,x) = \rho_1(s,x) \wedge \rho_2(s,x) \tag{1.6}$$

holds for every $s \in S$ and $x \in E$; here $\wedge$ denotes the usual operation of infimum (minimum, in this particular case) in the unit interval of reals. Hence, for sets of compatible states (1.6) yields that

$$U_{\rho_{12}}(x) = U_{\rho_1}(x) \cap U_{\rho_2}(x) \tag{1.7}$$

and, supposing that $x = X(\omega)$ for a random variable, as above, we obtain the set-valued mapping $U_{\rho_{12}}(X(\cdot))$ defined, for each $\omega \in \Omega$, by

$$U_{\rho_{12}}(X(\omega)) = U_{\rho_1}(X(\omega)) \cap U_{\rho_2}(X(\omega)). \tag{1.8}$$

If this mapping is measurable with respect to the σ-field $\mathcal{P}(\mathcal{P}(S))$, we can define, for each $A \subset S$, the value $m_{12}(A)$ by

$$m_{12}(A) = P\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) \cap U_{\rho_2}(X(\omega)) = A\}\right). \tag{1.9}$$

As S and, consequently, also $\mathcal{P}(S)$ are finite, $m_{12}(A)$ can be written as

$$m_{12}(A) = \sum\nolimits_{B,C \subset S,\, B \cap C = A} P\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right). \tag{1.10}$$

If the set-valued random variables $U_{\rho_1}(X(\cdot))$ and $U_{\rho_2}(X(\cdot))$ are statistically independent in the sense that the equality

$$\begin{aligned} & P\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \qquad (1.11) \\ = \ & P\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}\right) \cdot P\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \end{aligned}$$

holds for every B, $C \subset S$, and if we set

$$m_i(A) = P\left(\{\omega \in \Omega : U_{\rho_i}(X(\omega)) = A\}\right) \tag{1.12}$$

for each $A \subset S$ and both $i = 1$, 2, (1.10) implies that

$$\begin{aligned} m_{12}(A) &= \sum\nolimits_{B,C \subset S,\, B \cap C = A} P\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}\right) \\ &\qquad P\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \\ &= \sum\nolimits_{B,C \subset S,\, B \cap C = A} m_1(B)\, m_2(C), \end{aligned} \tag{1.13}$$

hence,

$$m_{12}(A) = (m_1 \oplus m_2)(A) \tag{1.14}$$

according to the definition of Dempster product by (1.3).

In what follows, a common and important feature of the formal model explained above consists in the fact that all the values ascribed to a subset A of S by various basic probability assignments and belief functions are defined by probabilities, i. e., values of the probability measure P given by the probability space $\langle \Omega, \mathcal{A}, P \rangle$ in question and ascribed to random events (measurable subsets of Ω, i. e., subsets from the σ-field $\mathcal{A}$) appropriately induced by the subset A of S under consideration. In their turn, probability measures are particular cases of numerically quantified sizes of sets (or at least of certain subsets of the universe of discourse), namely those fulfilling the demands of normalization (values from the unit interval of reals) and σ-additivity. So, an immediate idea arises: to modify the model of belief functions from above in such a way that the sizes of corresponding sets of elementary random events (subsets of Ω) will be quantified by set functions alternative to probability measures. In the rest of this paper we shall try to do so replacing probability measures by the so called *possibilistic measures.* It could and should be a matter of further investigation to generalize the approach developed below also to the case of so called *fuzzy measures.*

2. POSSIBILISTIC MEASURES – DEFINITION AND PRELIMINARIES

Possibilistic measures were introduced by L. Zadeh in [12] and have been widely developed since, let us recall the works by D. Dubois and H. Prade ([2, 3, 4]), or the detailed work [1] by G. de Cooman dealing with non-numerical (lattice-valued) possibilistic measures. It is just this work [1] which discovers and proves far going and deep *formal* analogies between possibilistic and probabilistic measures, for which possibilistic measures deserve to be taken into consideration when looking for possible alternatives to probability measures. In the sequel we shall very often take profit of these analogies.

The most simple definition of possibilistic measures reads as follows.

Definition 2.1. Let Ω be a nonempty set, let $\mathcal{P}(\Omega)$ denote the set of all subsets of Ω. A *possibilistic measure* on Ω is a mapping $\Pi : \mathcal{P}(\Omega) \to [0,1]$ such that

$$\Pi(\emptyset) = 0, \quad \Pi(\Omega) = 1 \tag{2.1}$$

$$\Pi(A \cup B) = \Pi(A) \vee \Pi(B) \tag{2.2}$$

for every A, $B \subset \Omega$, here $\vee$ denotes the standard supremum in the unit interval (and, dually, $\wedge$ denotes infimum). A possibilistic measure Π on Ω is called *distributive* if the equality

$$\Pi(A) = \bigvee_{\omega \in A} \Pi(\{\omega\}) \tag{2.3}$$

holds for all $\emptyset \neq A \subset \Omega$ (it holds for $A = \emptyset$ as well, if we apply the convention according to which the supremum over the empty set of nonnegative items equals zero. As a matter of fact, we shall apply this convention in what follows). If (2.3) holds, then the mapping $\pi : \Omega \to [0,1]$ defined by $\pi(\omega) = \Pi(\{\omega\})$ for every $\omega \in \Omega$ is called the *possibilistic distribution* induced by and generating the possibilistic measure Π.

By induction, (2.2) yields that $\Pi\left(\bigcup_{i=1}^{n} A_i\right) = \bigvee_{i=1}^{n} \Pi(A_i)$ is valid for every finite sequence $A_1, A_2, \ldots, A_n$ of subsets of Ω, consequently, if Ω is finite, every possibilistic measure on Ω is distributive. If Ω is infinite, this need not be the case: consider the possibilistic measure Π_0 such that $\Pi_0(A) = 0$ for finite subsets A of S (including the empty one) and $\Pi_0(A) = 1$, if A is infinite. If Π is distributive, then for every nonempty system $\mathcal{R}$ of subsets of Ω the following identity holds:

$$\Pi\left(\bigcup_{R \in \mathcal{R}} R\right) = \bigvee \left\{\Pi(\{\omega\}) : \omega \in \bigcup_{R \in \mathcal{R}} R\right\} \tag{2.4}$$

$$= \bigvee_{R \in \mathcal{R}} \bigvee_{\omega \in R} \Pi(\{\omega\}) = \bigvee_{R \in \mathcal{R}} \Pi(R).$$

A possibilistic measure Π is called *compact*, if there exists $\omega \in \Omega$ such that $\Pi(\{\omega\}) = 1$, if Π is distributive, its possibilistic distribution is also called compact. If Ω is finite, every possibilistic measure on Ω is compact. Any function $f : \Omega \to [0,1]$ such that $\bigvee_{\omega \in \Omega} f(\omega) = 1$ induces the distributive possibilistic measure Π_f on Ω such that $\Pi_f(A) = \bigvee_{\omega \in A} f(\omega)$ for every $\emptyset \neq A \subset S$ and $\Pi_f(\emptyset) = 0$; its possibilistic distribution coincides with f. If $f(\omega) = 1$ for some $\omega \in \Omega$, Π_f is compact.

A possibilistic measure Π on Ω is called *two-valued*, if $\Pi(A) \in \{0,1\}$ for every $A \subset \Omega$. If a two-valued possibilistic measure on Ω is distributive and if $K_\Pi = \{\omega \in \Omega : \Pi(\{\omega\}) = 1\}$, then the *kernel* K_Π of Π uniquely defines Π in the sense that, for every $A \subset \Omega$, $\Pi(A) = 1$, if $A \cap K_\Pi \neq \emptyset$, $\Pi(A) = 0$ otherwise. Obviously, the kernel K_Π is nonempty. A two-valued possibilistic measure Π on Ω is *single*, if $K_\Pi = \{\omega_\Pi\}$ for some $\omega_\Pi \in \Omega$. Hence, for every $A \subset \Omega$, $\Pi(A) = 1$, if $\omega_\Pi \in A$, $\Pi(A) = 0$ otherwise.

The conception of possibilistic measure has been modified in several directions.

(a) Only distributive possibilistic measures are considered.

(b) Also non-normalized possibilistic measures with $\Pi(\Omega) < 1$ are considered.

(c) Possibilistic measures are defined as *partial* mappings, i.e., $\Pi(A)$ is defined for every $A \in \mathcal{R} \subset \mathcal{P}(\Omega)$, $\mathcal{R} \neq \mathcal{P}(\Omega)$. Most often, $\mathcal{R}$ is supposed to be a so called *ample field*, which is closed with respect to the operations of set-theoretic complement and unions (and intersections, consequently) of any nonempty systems of subsets of Ω. Hence, ample field strengthens the notion of σ-field which is closed with respect to finite and countable unions and intersections. As a matter of fact, an ample field $\mathcal{R} \subset \mathcal{P}(\Omega)$ can be identified with the power-set $\mathcal{P}(\Omega| \approx)$ of all subsets of the factor-space $\Omega| \approx$ induced in Ω by an equivalence relation $\approx$ on Ω. Introducing a term analogous to that used in probability theory, the triple $\langle \Omega, \mathcal{R}, \Pi \rangle$ can be called the *possibilistic space* induced in the set Ω by the ample field $\mathcal{R}$ and by the possibilistic measure Π defined on $\mathcal{R}$.

(d) Also possibilistic measures with non-numerical values are considered. The space of these values must be equipped by a structure rich enough to define the supremum and infimum operations and to process them. Most often, the space of values is supposed to be equipped by a complete lattice.

What is important in our context is the fact that possibilistic measures can simulate the probabilistic ones also when introducing a possibilistic analogy of the notion of integral or, in a more probability theory-like terms, the notion of expected value. This can be done due to the notion of Sugeno integral, the most simple but sufficient for our purposes definition of which reads as follows.

Definition 2.2. Let $\langle \Omega, \mathcal{P}(\Omega), \Pi \rangle$ be a possibilistic space, let $f : \Omega \to [0,1]$ be a function. The *Sugeno integral of f over Ω and with respect to* Π is defined by

$$\oint_{\Omega} f \, \mathrm{d}\Pi = \bigvee_{\alpha \in [0,1]} [\alpha \wedge \Pi(\{\omega \in \Omega : f(\omega) \geq \alpha\})]. \tag{2.5}$$

As proved in [9], if Π is distributive, then

$$\oint_{\Omega} f \, \mathrm{d}\Pi = \bigvee_{\omega \in \Omega} [f(\omega) \wedge \Pi(\{\omega\})] = \bigvee_{\omega \in \Omega} [f(\omega) \wedge \pi(\omega)]. \tag{2.6}$$

In [1], where only distributive possibilistic measures are considered, (2.6) plays immediately the role of the definition of Sugeno integral. Replacing, just as a formal construction, $\bigvee_{\omega \in \Omega}$ by $\sum_{\omega \in \Omega}$ and $\wedge$ by product, we arrive at the common definition of integral for the most simple case of function defined on a finite probability space.

It is, perhaps, worth saying explicitly, that the roles of supremum and infimum operations for possibilistic measures are not completely dual. Possibilistic measures are monotonous with respect to the set-theoretic inclusion, as for any $A \subset B \subset \Omega$ the inequality

$$\Pi(A) \leq \Pi(A) \vee \Pi(B - A) = \Pi(A \cup (B - A)) = \Pi(B) \tag{2.7}$$

easily follows. Consequently, for each A, $B \subset \Omega$, $\Pi(A \cap B) \leq \Pi(A)$, $\Pi(A \cap B) \leq \Pi(B)$, so that

$$\Pi(A \cap B) \leq \Pi(A) \wedge \Pi(B). \tag{2.8}$$

However, the equality in (2.8) does not hold in general. Indeed, take Ω such that $\text{card}(\Omega) \geq 2$ and consider the most trivial possibilistic measure Π on Ω such that $\Pi(\emptyset) = 0$, $\Pi(A) = 1$ for every $\emptyset \neq A \subset \Omega$. Then, for every $\emptyset \neq A \subset \Omega$, $A \neq \Omega$, we obtain that

$$0 = \Pi(\emptyset) = \Pi(A \cap (S - A)) < 1 = 1 \wedge 1 = \Pi(A) \wedge \Pi(S - A). \tag{2.9}$$

If the possibilistic measure Π is single, the equality $\Pi(A \cap B) = \Pi(A) \wedge \Pi(B)$ holds for every A, $B \subset \Omega$. Evidently, the only thing we have to prove is that if $\Pi(A \cap B) = 0$, then either $\Pi(A) = 0$ or $\Pi(B) = 0$. But, $\Pi(A \cap B) = 0$ iff $\omega_\Pi \notin A \cap B$, where $\{\omega_\Pi\}$ is the singleton kernel of the single possibilistic measure Π. Consequently, $\omega_\Pi \notin A \cap B$ implies that either $\omega_\Pi \notin A$ or $\omega_\Pi \notin B$, so that either $\Pi(A) = 0$ or $\Pi(B) = 0$.

If the relation $\Pi(A \cap B) = \Pi(A) \wedge \Pi(B)$ is valid, the sets A and B (random events A and B, when preferring the probabilistic terminology) are called *possibilistically independent* or, more correctly, *independent with respect to minimum-based relation of possibilistic independence* (cf. [5, 7, 8, 12] for a more detailed discussion), even if also alternative definitions of the notion of possibilistic independence are suggested and investigated. We shall take profit of this notion later in this paper.

3. BASIC POSSIBILISTIC ASSIGNMENTS AND POSSIBILISTIC BELIEF FUNCTIONS – COMBINATORIC MODEL

In this chapter we shall try to develop a possibilistic analogy to the notions and constructions developed in Chapter 1 following, as far as possible, the methodological pattern consisting in a more or less routine substitution of summations by suprema and products by infima in our considerations from above.

Definition 3.1. *Basic possibilistic assignment* (b.poss.a.) on a nonempty set S is a mapping $\pi : \mathcal{P}(S) \to [0,1]$ such that $\bigvee_{A \subset S} \pi(A) = 1$, hence, b.poss.a. on S is a possibilistic distribution on $\mathcal{P}(S)$. *Possibilistic belief function BEL_π* induced by the b.poss.a. π on S is a mapping such that, for all $A \subset S$,

$$BEL_\pi(A) = \bigvee_{\emptyset \neq B \subset A} \pi(B) = \Pi(\mathcal{P}(A) - \{\emptyset\}), \tag{3.1}$$

where Π is the possibilistic measure induced by π on S. Hence, $BEL_\pi(\emptyset) = \Pi(\mathcal{P}(\emptyset) - \{\emptyset\}) = \Pi(\emptyset) = 0$.

Contrary to the case of probabilistic belief functions over finite set S, there is, in general, no one-to-one relation between b.poss.a.'s and possibilistic belief functions.

Lemma 3.1. If $\pi(S) < BEL_\pi(S)$, then there exists a b.poss.a. π^* such that $\pi^*(S) \neq \pi(S)$, hence, $\pi^* \not\equiv \pi$, but $BEL_{\pi^*}(A) = BEL_\pi(A)$ for every $A \subset S$, so that $BEL_{\pi^*} \equiv BEL_\pi$ holds.

Proof. Set $\pi^*(A) = \pi(A)$ for every $A \subset S$, $A \neq S$, set $\pi(S) < \pi^*(S) < BEL_\pi(S)$. Then, for every $A \subset S$, $A \neq S$, if $B \subset A$, then $B \neq S$, so that

$$BEL_{\pi^*}(A) = \bigvee_{\emptyset \neq B \subset A} \pi^*(B) = \bigvee_{\emptyset \neq B \subset A} \pi(B) = BEL_\pi(A). \tag{3.2}$$

As $\pi(S) < \pi^*(S) < BEL_\pi(S)$, we obtain that

$$BEL_\pi(S) = \bigvee_{\emptyset \neq B \subset S} \pi(B) = \bigvee_{\emptyset \neq B \subset S,\, B \neq S} \pi(B) \tag{3.3}$$

$$BEL_{\pi^*}(S) = \bigvee_{\emptyset \neq B \subset S} \pi^*(B) = \bigvee_{\emptyset \neq B \subset S,\, B \neq S} \pi^*(B), \tag{3.4}$$

but for all $B \subset S$, $B \neq S$, π and π^* coincide, and for some $B \neq S$, $\pi^*(B) = \pi(B) > \pi(S)$, so that $BEL_\pi(S) = BEL_{\pi^*}(S)$. □

Lemma 3.2. If there exists $\emptyset \neq B \subset S$ such that $\pi(B) \geq \pi(\emptyset)$ holds (in particular, if $\pi(\emptyset) < 1$), then $BEL_\pi(S) = 1$.

Proof. If $\pi(B) \geq \pi(\emptyset)$ for some $\emptyset \neq B \subset S$, then

$$BEL_\pi(S) = \bigvee_{\emptyset \neq B \subset S} \pi(B) = \bigvee_{B \subset S} \pi(B) = 1, \tag{3.5}$$

as π is a b.poss.a., hence, a possibilistic distribution on $\mathcal{P}(S)$. If $\pi(\emptyset) < 1$, then there must exist $B \subset S$, $B \neq \emptyset$, such that $\pi(B) > \pi(\emptyset)$, as $\bigvee_{B \subset S} \pi(B) = 1$. □

Lemma 3.3. Let π be a basic possibilistic assignment on S. Then

(a) $BEL_\pi(A) \leq BEL_\pi(B)$ for every $A \subset B \subset S$,

(b) $BEL_\pi(A \cup B) \geq BEL_\pi(A) \vee BEL_\pi(B)$ for every A, $B \subset S$,

(c) $BEL_\pi(A \cap B) \leq BEL_\pi(A) \wedge BEL_\pi(B)$ for every A, $B \subset S$.

Proof. All the statements follow directly from the fact that possibilistic measures are monotonous with respect to the set-theoretic inclusion. In more detail, if $A \subset B \subset S$, then $\mathcal{P}(A) \subset \mathcal{P}(B)$ and (a) follows by

$$BEL_\pi(A) = \bigvee_{\emptyset \neq C \subset A} \pi(C) \leq \bigvee_{\emptyset \neq C \subset B} \pi(C) = BEL_\pi(B). \tag{3.6}$$

The assertions (b) and (c) follow immediately. □

Let $A \subset S$, let π_A be such a b.poss.a. on S that $\pi_A(A) = 1$, $\pi_A(B) = 0$ for every $B \subset S$, $B \neq A$ (note that in the case of b.poss.a.'s the later condition does not follow from the former one). In this case, the b.poss.a. π_S is called *vacuous* and the b.poss.a. $\pi_\emptyset$ is called *inconsistent* and we shall prove below, that they play the same roles with respect to the Dempster combination rule and with respect to the corresponding compatibility relations as the basic probability assignments labeled by the same adjectives. Moreover, let us define the *trivial* b.poss.a. π_* setting $\pi_*(A) = 1$ for every $A \subset S$. Obviously there is no analogy to this notion within the space of basic probability assignments. Denoting by BEL_A the possibilistic belief function BEL_{π_A}, we obtain that $BEL_A(B) = \bigvee_{\emptyset \neq C \subset B} \pi_A(C) = 1$, if $A \subset B$, and $BEL_A(B) = 0$ otherwise. Hence, $BEL_S(B) = 0$ for every $B \subset S$, $B \neq S$, and $BEL_S(S) = 1$. For $BEL_\emptyset$ we obtain that $BEL_\emptyset(A) = \bigvee_{\emptyset \neq B \subset A} \pi_\emptyset(B) = 0$ for every $A \subset S$ (for $A \neq \emptyset$ by convention accepted above). Finally,

$$BEL_*(A) = BEL_{\pi_*}(A) = \bigvee_{\emptyset \neq B \subset A} \pi_*(B) = 1 \tag{3.7}$$

for every $\emptyset \neq A \subset S$, $BEL_*(\emptyset) = 0$ by convention.

Also the well-known relation between probabilistic belief and plausibility functions can be modified to the case of possibilistic measures. Given a basic probability assignment on a finite set S, the corresponding *plausibility function* pl_m takes $\mathcal{P}(S)$ into $[0, 1]$ in such a way that

$$pl_m(A) = \sum_{B \subset S,\, B \cap A \neq \emptyset} m(B) \tag{3.8}$$

for every $\emptyset \neq A \subset S$, $pl_m(\emptyset) = 0$ by convention. In the terms of random sets we can write that

$$pl_m(A) = P\left(\{\omega \in \Omega : U_\rho(X(\omega)) \cap A \neq \emptyset\}\right). \tag{3.9}$$

As can be easily proved, the relation

$$pl_m(A) = bel_m(S) - bel_m(S - A) \tag{3.10}$$

holds for every $A \subset S$, sometimes (3.10) is immediately taken as the definition of pl_m.

For a *possibilistic* distribution π on $\mathcal{P}(S)$, i. e., for a basic poss.a. on S, we can define the *possibilistic plausibility function* PL_π, setting

$$PL_\pi(A) = \bigvee_{B \subset S,\, B \cap A \neq \emptyset} \pi(B), \tag{3.11}$$

so that $PL_\pi(\emptyset) = 0$ by convention.

As for every $\emptyset \neq A$, $B \subset S$ either $B \subset S - A$ or $B \cap A \neq \emptyset$ hold, we obtain that

$$PL_\pi(A) \vee BEL_\pi(S-A) = \left(\bigvee_{B \cap A \neq \emptyset} \pi(B)\right) \vee \left(\bigvee_{\emptyset \neq B \subset S-A} \pi(B)\right) \tag{3.12}$$
$$= \bigvee_{\emptyset \neq B} \pi(B) = BEL_\pi(S);$$

this relation obviously corresponds to (3.10) above.

Before examining the alternative approach to possibilistic belief functions through set-valued random (or rather possibilistic) mappings, let us introduce the possibilistic alternative to the Dempster combination rule at the abstract algebraic and combinatoric level. For the sake of simplicity we shall use the same symbol $\oplus$ also for the possibilistic Dempster combination rule hoping that it will be always clear from the context, whether it denotes the classical Dempster rule or its possibilistic modification. Considering b.poss.a.'s π_1 and π_2 on S and replacing, in a routine way, summations by suprema and products by infima, we obtain, for any $A \subset S$, that

$$(\pi_1 \oplus \pi_2)(A) =_{\text{df}} \bigvee\nolimits_{B,C \subset S,\, B \cap C = A} [\pi_1(B) \wedge \pi_2(C)]. \tag{3.13}$$

As π_1 and π_2 are b.poss.a.'s on S, the relation $\bigvee_{A \subset S} \pi_1(A) = \bigvee_{A \subset S} \pi_2(A) = 1$ holds. Hence, for each $\varepsilon > 0$ there exist B_0, $C_0 \subset S$ such that $\pi_1(B_0) > 1 - \varepsilon$ and $\pi_2(C_0) > 1 - \varepsilon$ is valid, so that, setting $A_0 = B_0 \cap C_0$, we obtain that

$$(\pi_1 \oplus \pi_2)(A_0) = \bigvee\nolimits_{B,\, C \subset S,\, B \cap C = A_0} [\pi_1(B) \wedge \pi_2(C)] \geq \pi_1(B_0) \wedge \pi_2(C_0) > 1 - \varepsilon, \tag{3.14}$$

consequently, $\bigvee_{A \subset S} (\pi_1 \oplus \pi_2)(A) = 1$ and $\pi_1 \oplus \pi_2$ is a b.poss.a. on S. The operation $\oplus$ is evidently commutative, so that $(\pi_1 \oplus \pi_2)(A) = (\pi_2 \oplus \pi_1)(A)$ for every $A \subset S$. It is also associative, as for every $A \subset S$,

$$\begin{aligned} ((\pi_1 \oplus \pi_2) \oplus \pi_3)(A) &= \bigvee\nolimits_{B \cap C = A} [(\pi_1 \oplus \pi_2)(B) \wedge \pi_3(C)] \qquad (3.15) \\ &= \bigvee\nolimits_{B \cap C = A} \left[\left[\bigvee\nolimits_{D \cap E = B} (\pi_1(D) \wedge \pi_2(E)) \right] \wedge \pi_3(C) \right] \\ &= \bigvee\nolimits_{B,E,C,D \cap E \cap C = A} (\pi_1(D) \wedge \pi_2(E) \wedge \pi_3(C)). \end{aligned}$$

However, we arrive at the same expression when analyzing, analogously, the expression $(\pi_1 \oplus (\pi_2 \oplus \pi_3))(A)$, so that the equality

$$((\pi_1 \oplus \pi_2) \oplus \pi_3)(A) = (\pi_1 \oplus (\pi_2 \oplus \pi_3))(A) \tag{3.16}$$

holds for every $A \subset S$. Hence, when defining recurrently, for b.poss.a.'s $\pi_1, \pi_2, \ldots, \pi_n$ on S,

$$\oplus_{i=1}^{n} \pi_i = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n =_{\text{df}} (\pi_1 \oplus \cdots \oplus \pi_{n-1}) \oplus \pi_n, \tag{3.17}$$

the definition is correct, as the bracketing is irrelevant.

So, the most elementary properties of possibilistic Dempster operation $\oplus$ are the same as in the classical probabilistic case. Also the roles of the vacuous b.poss.a. π_S as the unit element and the inconsistent b.poss.a. $\pi_\emptyset$ as the zero element with respect to $\oplus$ are the same as in the probabilistic case (if taking $\oplus$ as product; if taking it as summation, the roles of π_S and $\pi_\emptyset$ are interchanged). Or, for every b.poss.a. π on S and for every $A \subset S$,

$$(\pi \oplus \pi_S)(A) = \bigvee\nolimits_{B \cap C = A} (\pi(B) \wedge \pi_S(C)) = \pi(A) \wedge \pi_S(S) = \pi(A), \tag{3.18}$$

as $\langle B, C \rangle = \langle A, S \rangle$ is the only pair of subsets of S such that $B \cap C = A$ and $\pi_S(C) > 0$, hence, $\pi \oplus \pi_S \equiv \pi$. Dually, for $\emptyset \neq A \subset S$,

$$(\pi \oplus \pi_\emptyset)(A) = \bigvee\nolimits_{B \cap C = A} \pi(B) \wedge \pi_\emptyset(C) = 0, \tag{3.19}$$

as for $A \neq \emptyset$ there is no B, $C \subset S$ such that $B \cap C = A$ and $\pi_{\emptyset}(C) > 0$. For $A = \emptyset$ we obtain that

$$(\pi \oplus \pi_{\emptyset})(A) = \bigvee_{B \subset S} \pi(B) = 1, \tag{3.20}$$

as $B \cap C = \emptyset$ holds for every $B \subset S$ supposing that $C = \emptyset$. Consequently, $\pi \oplus \pi_{\emptyset} \equiv \pi_{\emptyset}$. The trivial b.poss.a. π_* (let us recall that $\pi_*(A) = 1$ for every $A \subset S$) is not so trivial with respect to the possibilistic Dempster rule, as for every $A \subset S$

$$(\pi \oplus \pi_*)(A) = \bigvee_{B \cap C = A} [\pi(B) \wedge \pi_*(C)] = \bigvee_{B \supset A} \pi(B) = Q(A), \tag{3.21}$$

as the last expression could be taken as the possibilistic analogy of the commonality function defined, for probabilistic b.p.a. m, by $q(A) = \sum_{B \supset A} m(B)$.

4. COMPATIBILITY RELATIONS AND BASIC POSSIBILISTIC ASSIGNMENTS

The following construction copies, in its first steps, the pattern briefly outlined above in the case of basic probabilistic assignments and belief functions. Let S be a nonempty set of possible states of a system (alternative interpretations are above), let E be a nonempty set of possible values of empirical data (observations, e. g.) concerning the system in question and its environment. Let $\rho : S \times E \to \{0, 1\}$ be a compatibility relation and let $U_\rho(x) = \{s \in S : \rho(s, x) = 1\}$ be the set of states compatible with an empirical value $x \in E$.

In order to describe the random or at least nondeterministic, nature of the empirical data we shall suppose that $x = X(\omega)$, where X is a measurable mapping which takes the measurable space $\langle \Omega, \mathcal{P}(\Omega) \rangle$ into a measurable space $\langle E, \mathcal{E} \rangle$ generated in E by a nonempty σ-field $\mathcal{E}$ of subsets of E; if E is finite, we take as a rule $\mathcal{E} = \mathcal{P}(E)$. Combining together the mappings $U : E \to \mathcal{P}(S)$ and $X : \Omega \to E$, we obtain a set-valued mapping $U_\rho(X(\cdot)) : \Omega \to \mathcal{P}(S)$ ascribing to each $\omega \in \Omega$ the subset $U_\rho(X(\omega))$ of S. Given $A \subset S$, we can define its inverse image $\{\omega \in \Omega : U_\rho(X(\omega)) = A\}$ with respect to this mapping and we may quantify somehow the size of this subset of Ω. Contrary to the model explained in Chapter 2 above we shall not use, for these sakes, a probability measure P, but rather a *possibilistic measure* Π_0 defined on the power-set $\mathcal{P}(\Omega)$. In other terms, we shall define a *possibilistic space* $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$ and we also define, for every $A \subset S$, the value $\pi(A)$ by

$$\pi(A) = \Pi_0 (\{\omega \in \Omega : U_\rho(X(\omega)) = A\}). \tag{4.1}$$

As can be easily seen,

$$\begin{aligned} \bigvee_{A \subset S} \pi(A) &= \bigvee_{A \subset S} \Pi_0 (\{\omega \in \Omega : U_\rho(X(\omega)) = A\}) \qquad (4.2) \\ &= \Pi_0 \left(\bigcup_{A \subset S} \{\omega : \omega \in \Omega : U_\rho(X(\omega)) = A\} \right) = \Pi_0(\Omega) = 1, \end{aligned}$$

as $U_\rho(X(\cdot))$ is total on Ω and Π_0 is a possibilistic measure on $\mathcal{P}(\Omega)$. Hence, the mapping $\pi : \mathcal{P}(S) \to [0, 1]$ is a basic possibilistic assignment on S.

As a matter of fact, *every* b.poss.a. on S can be defined by (4.1) as the following statement proves.

Theorem 4.1. For every b.poss.a. on S there exist possibilistic space $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$, empirical space E, mapping $X : \Omega \to E$ and compatibility relation $\rho : S \times E \to \{0,1\}$ such that (4.1) holds for every $A \subset S$.

Proof. Let $\pi : \mathcal{P}(S) \to [0,1]$ be a b.poss.a. on S, let Π be the possibilistic measure on $\mathcal{P}(\Omega)$ induced by π. Define the possibilistic space $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$ such that $\Omega = \mathcal{P}(S)$ and $\Pi_0 = \Pi$, hence, consider the possibilistic space $\langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S)), \Pi \rangle$. Set $E = \mathcal{P}(S)$ and suppose that X is the identity mapping on $\Omega = \mathcal{P}(S)$, so that $X(A) = A$ for every $A \subset S$. Define, finally, the compatibility relation $\rho : S \times E \to \{0,1\}$ by the characteristic function (identifier) χ_x of a subset x of S, so that $\rho(s,x) = 1$, if $s \in x$, $\rho(s,x) = 0$ otherwise. Then we obtain, for every $A \subset S$, that

$$\begin{aligned} & \Pi_0 \left(\{\omega \in \Omega : U_\rho(X(\omega)) = A\} \right) = \Pi \left(\{B \subset S : U_\rho(X(B)) = A\} \right) \qquad (4.3) \\ = \; & \Pi \left(\{B \subset S : U_\rho(B) = A\} \right) = \Pi \left(\{B \subset S : \{s \in S : \rho(s,B) = 1\} = A\} \right) \\ = \; & \Pi \left(\{B \subset S : \{s \in S : \chi_B(s) = 1\} = A\} \right) = \Pi \left(\{B \subset S : B = A\} \right) = \Pi(\{A\}) \\ = \; & \bigvee\nolimits_{B,\, B=A} \pi(B) = \pi(A). \end{aligned}$$

The assertion is proved. □

Analogously to the case of basic probability assignments, partial cases of b.poss.a.'s picked out above can be easily seen to be defined by particular compatibility relations. Let $\rho_A : S \times E \to \{0,1\}$ be such that $\rho_A(s,x) = 1$ iff $s \in A \subset S$ holds, no matter which the value of x may be. Then $U_\rho(x) = A$ for every $x \in E$, consequently, $U_\rho(X(\omega)) = A$ for every $\omega \in \Omega$. So,

$$\pi(A) = \Pi_0 \left(\{\omega \in \Omega : U_\rho(X(\omega)) = A\} \right) = \Pi_0(\Omega) = 1, \qquad (4.4)$$

and $\pi(B) = \Pi_0(\emptyset) = 0$ for every $B \subset S$, $B \neq A$, so that $\pi \equiv \pi_A$. The particular cases $\pi_\emptyset$ and π_S are obviously defined by compatibility relations $\rho_\emptyset(s,x) \equiv 0$ and $\rho_S(s,x) \equiv 1$ for every $s \in S$, $x \in E$. The trivial b.poss.a. π_* such that $\pi_*(A) = 1$ for every $A \subset S$ can be obtained by the construction presented above in the proof of Theorem 4.1, supposing that $\Omega = \mathcal{P}(S)$ and $\Pi_0(\{\omega\}) = 1$ for every $\omega \in \Omega$.

For the possibilistic Dempster combination rule we may also proceed in a way copying as close as possible our reasonings for the probabilistic case presented above. Let ρ_1, ρ_2 be two compatibility relations taking $S \times E$ into $\{0,1\}$ and let $\rho_{12} = S \times E \to \{0,1\}$ be defined by

$$\rho_{12}(s,x) = \rho_1(s,x) \wedge \rho_2(s,x) \qquad (4.5)$$

for every $s \in S$, $x \in E$. So, for every $x \in E$,

$$U_{\rho_{12}}(x) = \{s \in S : \rho_{12}(s,x) = 1\} = U_{\rho_1}(x) \cap U_{\rho_2}(x). \qquad (4.6)$$

Let $\langle \Omega, \mathcal{P}(\Omega), \Pi_0 \rangle$ be a possibilistic space, let $X : \Omega \to E$ be a measurable mapping, let

$$\pi_i(A) = \Pi_0 \left(\{\omega \in \Omega : U_{\rho_i}(X(\omega)) = A\} \right) \qquad (4.7)$$

for $i = 1$, 2 and 12 and for every $A \subset S$. Then

$$\begin{aligned} &\pi_{12}(A) = \Pi_0\left(\{\omega \in \Omega : U_{\rho_{12}}(X(\omega)) = A\}\right) \qquad (4.8)\\ &= \Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) \cap U_{\rho_2}(X(\omega)) = A\}\right)\\ &= \Pi_0\left(\bigcup_{B,C\subset S,\, B\cap C=A}\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right)\right)\\ &= \bigvee_{B,C\subset S,\, B\cap C=A} \Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right)\\ &\le \bigvee_{B\cap C=A}\left[\Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}\right) \wedge \Pi_0\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right)\right]\\ &= \bigvee_{B\cap C=A}\left[\pi_1(B) \wedge \pi_2(C)\right]. \end{aligned}$$

In general, the inequality on the fifth line in (4.8) cannot be replaced by equality. This can be done at least in the two following cases: if the possibilistic measure Π_0 is single (cf. the end of Chapter 2), or if the set-valued variables $U_{\rho_1}(X(\cdot))$ and $U_{\rho_2}(X(\cdot))$ are *possibilistically (minimum-based) independent* in the sense that the equality

$$\begin{aligned} &\Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\} \cap \{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \qquad (4.9)\\ = \; &\Pi_0\left(\{\omega \in \Omega : U_{\rho_1}(X(\omega)) = B\}\right) \wedge \Pi_0\left(\{\omega \in \Omega : U_{\rho_2}(X(\omega)) = C\}\right) \end{aligned}$$

holds for every B, $C \subset S$. If (4.9) holds, then (4.8) yields that

$$\pi_{12}(A) = \bigvee_{B\cap C=A}\left[\pi_1(B) \wedge \pi_2(C)\right] = \left(\pi_1 \oplus \pi_2\right)(A) \qquad (4.10)$$

according to (4.9). Hence, as in the case of probabilistic Dempster combination rule, also its possibilistic modification is based on two hidden assumptions: (i) minimum-based combination of compatibility relations with the same semantics as above, and (ii) possibilistic independence of the sets of compatible states taken as set-valued variables. Cf. [6] for a more detailed analysis of the probabilistically based Dempster combination rule.

5. POSSIBILISTIC NONSPECIFICITY DEGREES AND DEMPSTER COMBINATION RULE

The intuition behind the Dempster combination rule can be read as follows. The pieces of knowledge of different subjects are such that any of these pieces enable to eliminate some states from the set of possible actual internal states of the investigated system. In other terms, the subject can focus her/his attention to a proper subset of S so approaching, partially, the desired final state of reasoning when only one state $s_0 \in S$ remains as possible so that, consequently, the actual state of the system is identified. The way in which these pieces of knowledge are shared by two or more subjects is such that all the states which can be eliminated by at least one of the subjects are eliminated by all of them so that the cardinality of the remaining subset of S is as small as possible.

Hence, our informal feelings are that the better is a basic probability assignments, the smaller are, at least in average, its focal elements (elements to which positive probabilities are ascribed). So we can define, given a b.p.a. m on a finite set S, its *(probabilistic) nonspecificity degree* $W(m)$ by the expected value of the relative (i. e., normalized to one) cardinalities of all subsets of S (including the non-focal ones when the probability is 0 so that the expected value remains untouched). Hence, we set

$$W(m) = \sum\nolimits_{A\subset S} (\|A\|/\|S\|)\, m(A) \tag{5.1}$$

where $\|A\|$ denotes the cardinality of a subset A of S; as S is finite, $\|A\|$ denotes simply the number of elements in A.

At least for the extremum cases this definition agrees with the intuition behind as sketched above. Indeed, $W(m) = 1$ (the maximum possible value for m) iff m is the vacuous b.p.a. m_S which does not contain any information concerning the actual value of s beyond the apriori accepted closed world assumption according to which all possible states of the system in question are supposed to be elements of the space S. On the other side, $W(m) = 1/\|S\|$ (the minimum possible positive value of W) iff $m = m_{\{s\}}$ for some $s \in S$, hence, iff $m(\{s\}) = 1$; in this case m uniquely determines the actual state of the investigated system. Of course, $W(m_\emptyset) = 0$ for the totally inconsistent b.p.a. $m_\emptyset$, but this b.p.a. does not yield any information concerning the actual state s and will be avoided from our classification.

As analyzed in more detail and proved in [10], Dempster combination rule improves the qualities of the composed b.p.a.'s in the sense of reduction of the values of the nonspecificity degree W defined by (5.1). Indeed, for any basic probability assignments m_1, m_2 defined on the same finite set S the inequality

$$W(m_1 \oplus m_2) \leq W(m_1) \wedge W(m_2) \tag{5.2}$$

holds with $\wedge$ denoting, as above, the standard infimum operation within the unit interval $[0, 1]$ of reals. Moreover, let $\otimes$ be the combination rule dual to the Dempster one and defined by

$$(m_1 \otimes m_2)\,(A) = \sum\nolimits_{B,C\subset S,\, B\cup C=A} m_1(B)\, m_2(C) \tag{5.3}$$

for any b.p.a.'s m_1, m_2 on S and any $A \subset S$. This rule can be defined also through compatibility relations ρ_1, ρ_2 and random sets, setting

$$\rho_{12}(s, x) = \rho_1(s, x) \vee \rho_2(s, x) \tag{5.4}$$

for every $s \in S$ and $x \in E$. Consequently, $U_{\rho_{12}}(x) = U_{\rho_1}(x) \cup U_{\rho_2}(x)$ for every $x \in E$ so that, setting $x = X(\omega)$ and supposing that the set-valued random variables $U_{\rho_1}(X(\cdot))$ and $U_{\rho_2}(X(\cdot))$ are statistically independent, we arrive easily at (5.3). As could be expected, for any b.p.a.'s m_1, m_2 on S the inequality

$$W(m_1 \oplus m_2) \geq W(m_1) \vee W(m_2) \tag{5.5}$$

dual to (5.2) holds with $\vee$ as the supremum in $[0, 1]$ (cf., again, [10] for more details and proof).

In the rest of this chapter we shall try to "translate" the formulas (5.2) and (5.4) into the possibilistic terms, following the more or less routine pattern applied above, and to prove the resulting statements. Let us note that (5.1) is nothing else than the definition of the integral of the random variable $\|A\|/\|S\|$ over the probability space $\langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S)), M \rangle$, where $M(\mathcal{A}) = \sum_{A \in \mathcal{A}} m(A)$ for every $\mathcal{A} \subset \mathcal{P}(S)$, hence, (5.1) turns into

$$W(m) = \int_{\mathcal{P}(S)} (\|A\|/\|S\|)\, \mathrm{d}M. \tag{5.6}$$

The idea immediately arises to replace (5.6) by the corresponding Sugeno integral.

Theorem 5.1. Let S be a nonempty finite set, let π_1, $\pi_2 : \mathcal{P}(S) \to [0,1]$ be basic possibilistic assignments on S. Then the following inequality holds for both $i = 1, 2$

$$\begin{aligned} & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B, C \subset S,\, B \cap C = A} (\pi_1(B) \wedge \pi_2(C)) \right] \right) \\ \leq\ & \bigvee_{A \subset S} ((\|A\|/\|S\|) \wedge \pi_i(A)) . \end{aligned} \tag{5.7}$$

P r o o f. The proofs for both $i = 1, 2$ are evidently analogous, so that we can limit ourselves to the case when $i = 1$. Let π_* be the trivial b.poss.a. on S defined above by the identity $\pi_*(A) = 1$ for each $A \subset S$. Consequently, $\pi_1(A) \leq \pi_*(A)$ holds for every $A \subset S$. Replacing π_2 by π_* in the left-hand side of the inequality (5.7) we obtain that

$$\begin{aligned} & \bigvee_{B \cap C = A} (\pi_1(B) \wedge \pi_2(C)) \leq \bigvee_{B \cap C = A} (\pi_1(B) \wedge \pi_*(C)) \\ =\ & \bigvee_{B, C \subset S,\, B \cap C = A} (\pi_1(B) \wedge 1) \\ =\ & \bigvee_{B,\, B \cap C = A \text{ for some } C \subset S} \pi_1(B) = \bigvee_{B,\, B \supset A} \pi_1(B). \end{aligned} \tag{5.8}$$

So we obtain that

$$\begin{aligned} & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B \cap C = A} (\pi_1(B) \wedge \pi_2(C)) \right] \right) \\ \leq\ & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B,\, B \supset A} \pi_1(B) \right] \right) . \end{aligned} \tag{5.9}$$

As S and, consequently, also $\mathcal{P}(S)$ are finite sets there exists, for each $A \subset S$, a set $B_A \subset S$ such that $B_A \supset A$ and $\pi_1(B_A) = \bigvee_{B \supset A} \pi_1(B)$; if there are more such $B \supset A$, no matter which of them will be chosen. Then

$$\begin{aligned} & \bigvee_{A \subset S} \left(\bigvee_{B \supset A} ((\|A\|/\|S\|) \wedge \pi_1(B)) \right) \\ =\ & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left(\bigvee_{B \supset A} \pi_1(B) \right) \right) \\ =\ & \bigvee_{A \subset S} ((\|A\|/\|S\|) \wedge \pi_1(B_A)) \\ \leq\ & \bigvee_{A \subset S} ((\|B_A\|/\|S\|) \wedge \pi_1(B_A)) , \end{aligned} \tag{5.10}$$

as $B_A \supset A$ implies that $\|B_A\| \geq \|A\|$. Set

$$\mathcal{P}_0(S) = \{B \subset S : B = B_A \text{ for some } A \subset S\}. \tag{5.11}$$

Obviously, $\mathcal{P}_0(S) \subset \mathcal{P}(S)$, so that

$$\begin{aligned} & \bigvee_{A \subset S} ((\|B_A\|/\|S\|) \wedge \pi_1(B_A)) \\ = & \bigvee_{B \in \mathcal{P}_0(S)} ((\|B\|/\|S\|) \wedge \pi_1(B)) \\ \leq & \bigvee_{B \subset S} ((\|B\|/\|S\|) \wedge \pi_1(B)). \end{aligned} \tag{5.12}$$

Combining (5.9), (5.10) and (5.12) together, we obtain that

$$\begin{aligned} & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B \cap C = A} (\pi_1(B) \wedge \pi_2(C))\right]\right) \\ \leq & \bigvee_{A \subset S} ((\|A\|/\|S\|) \wedge \pi_1(A)) \end{aligned} \tag{5.13}$$

holds, so that the assertion is proved. □

As a matter of fact, (5.7) is nothing else than (5.2) modified to the case of possibilistic measures. Indeed, let π be a b.poss.a. on S, let Π be the induced possibilistic measure on $\mathcal{P}(\mathcal{P}(S))$, let $\Omega = \mathcal{P}(S)$, let $f : \Omega \to [0,1]$ be defined by $f(A) = \|A\|/\|S\|$ for every $A \subset S$. Then, setting

$$W^*(\pi) = \oint_{\Omega} f(\omega)\, \mathrm{d}\Pi = \oint_{\mathcal{P}(S)} (\|A\|/\|S\|)\, \mathrm{d}\Pi, \tag{5.14}$$

we obtain by (2.6), as Π is distributive by definition, that

$$W^*(\pi) = \bigvee_{A \subset S} [(\|A\|/\|S\|) \wedge \pi(A)]. \tag{5.15}$$

The value $W^*(\pi)$ is the possibilistic analogy of $W(m)$ and can be called the *possibilistic nonspecificity degree* ascribed to the b.poss.a. π. The relation (5.7) then reads as

$$W^*(\pi_1 \oplus \pi_2) \leq W^*(\pi_i),\ i = 1, 2, \tag{5.16}$$

for the possibilistic Dempster product $\pi_1 \oplus \pi_2$, so that the analogy of (5.2) follows immediately.

The following assertion is dual to (5.7).

Theorem 5.2. Let S be a nonempty finite set, let $\pi_1, \pi_2 : \mathcal{P}(S) \to [0,1]$ be b.poss.a.'s on S. Then the following inequality holds for $i = 1, 2$.

$$\begin{aligned} & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B,C \subset S,\, B \cup C = A} (\pi_1(B) \wedge \pi_2(C))\right]\right) \\ \geq & \bigvee_{A \subset S} ((\|A\|/\|S\|) \wedge \pi_i(A)). \end{aligned} \tag{5.17}$$

P r o o f. Again, the proof for $i = 1$ is quite sufficient. As π_2 is a b.poss.a. on a finite set S, there exists $C_0 \subset S$ such that $\pi_2(C_0) = 1$ (if there are more such subsets of S, denote by C_0 no matter which one of them). Denoting, for each $A \subset S$, by A_1 the subset $A \cup C_0 \subset S$, we obtain that

$$\bigvee_{B \cup C = A_1} (\pi_1(B) \wedge \pi_2(C)) \geq \pi_1(A) \wedge \pi_2(C_0) = \pi_1(A). \tag{5.18}$$

As $A_1 \supset A$, the inequality $\|A_1\| \geq \|A\|$ follows, so that

$$(\|A_1\|/\|S\|) \wedge \left(\bigvee_{B \cup C = A_1} (\pi_1(B) \wedge \pi_2(C))\right) \geq (\|A\|/\|S\|) \wedge \pi_1(A) \tag{5.19}$$

holds as well. As such an A_1 exists for every $A \subset S$, we can set

$$\mathcal{P}_1(S) = \{B \subset S : B = A \cup C_0 \text{ for some } A \subset S\} \subset \mathcal{P}(S) \tag{5.20}$$

and we obtain that

$$\begin{aligned} & \bigvee_{A \subset S} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B,C \subset S,\, B \cup C = A} (\pi_1(B) \wedge \pi_2(C))\right]\right) \qquad (5.21) \\ \geq \ & \bigvee_{A \in \mathcal{P}_1(S)} \left((\|A\|/\|S\|) \wedge \left[\bigvee_{B,C \subset S,\, B \cup C = A} (\pi_1(B) \wedge \pi_2(C))\right]\right) \\ \geq \ & \bigvee_{A \subset S} ((\|A\|/\|S\|) \wedge \pi_1(A)). \end{aligned}$$

The assertion is proved. □

If we define the possibilistic dual combination rule $\otimes$ in the way copying the dual Dempster rule, i. e., if we set for every b.poss.a.'s π_1, π_2 and each $A \subset S$,

$$(\pi_1 \otimes \pi_2)(A) = \bigvee_{B,C \subset S,\, B \cup C = A} (\pi_1(B) \wedge \pi_2(C)), \tag{5.22}$$

the inequality (5.17) can be rewritten in the form

$$W^*(\pi_1 \otimes \pi_2) \geq W^*(\pi_1) \vee W^*(\pi_2) \tag{5.23}$$

dual to (5.16) and analogous to (5.5).

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