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A SECOND ORDER APPROXIMATION FOR THE INVERSE OF THE DISTRIBUTION FUNCTION OF THE SAMPLE MEAN

JORGE M. AREVALILLO

The classical quantile approximation for the sample mean, based on the central limit theorem, has been proved to fail when the sample size is small and we approach the tail of the distribution. In this paper we will develop a second order approximation formula for the quantile which improves the classical one under heavy tails underlying distributions, and performs very accurately in the upper tail of the distribution even for relatively small samples.

1. INTRODUCTION

Since the appearance of H. E. Daniels' paper: "Saddlepoint Approximations in Statistics" [3] a great deal of literature has been written on this topic.

Until now all efforts have been focused on finding different approaches that yield to Daniels' approximation of the sample mean density, trying to extend them to the multivariate case and evaluating new ones for tail probabilities; but little research has been developed to look into sample mean quantile expansions.

This is very interesting for us, since it has a great deal of applications in statistical testing. We aim to invert one of these tail probability expansions in order to get a quantile approximation and therefore a second order expansion for the critical value of the test.

The paper is organized in four sections. In Section 2 we deal with an approximation formula for the solution of the saddlepoint equation. This expression is used in Section 3 in order to get a new formulation for the tail probability saddlepoint approximation. Its inversion, by means of an inversion technique used in [2], is achieved in Section 4 and will yield to a second order approximation for the quantile of the sample mean distribution. We will finish the paper by displaying some numerical examples which will shed light on the analytical results.

2. AN APPROXIMATION FOR THE SOLUTION OF SADDLEPOINT EQUATION

Let X be a random variable with distribution function, $F(x)$, absolutely continuous with respect to Lebesgue measure and density function $f(x)$. Suppose X has moment generating function $M(\lambda) = \int e^{\lambda x} f(x)\, dx$, finite in an interval $(-r, s)$, $r > 0$, $s > 0$; this suffices to guarantee the existence of all the moments and cumulants of the distribution.

Let $K(\lambda) = \log M(\lambda)$ be the cumulant generating function of X; we will denote the mean of X by μ, the variance by σ^2 and the cumulants by $k_j : j \geq 1$, where $k_1 = \mu$ and $k_2 = \sigma^2$.

Suppose $X_1, X_2, \cdots, X_n$ is a random sample of X; we can deduce a saddlepoint approximation for the tail probability, thus for

$$P(Z_n \geq x) = P(\bar{X} \geq \bar{x}) \quad \text{where} \quad Z_n = \frac{(\bar{X} - \mu)\sqrt{n}}{\sigma} \quad \text{and} \quad \bar{x} = \mu + \frac{x\sigma}{\sqrt{n}}.$$

If we start from Daniels' saddlepoint approximation for the sample mean density [3] and use Laplacian expansion [7] (see Chapter 3, Section 3.1), as outlined in [5], we will get a large deviation approximation for the tail probability given by

$$P(Z_n \geq x) = P(\bar{X} \geq \bar{x}) = \frac{e^{n[K(\hat{\lambda}) - \hat{\lambda}\bar{x}]}}{\hat{\lambda}\sqrt{2\pi n K''(\hat{\lambda})}} + O(n^{-3/2}), \tag{1}$$

with $\hat{\lambda}$ satisfying the saddlepoint equation

$$K'(\hat{\lambda}) = \bar{x} = \mu + \frac{x\sigma}{\sqrt{n}}. \tag{2}$$

A high order large deviation expansion for the tail probability is deduced in [1]; we have considered in (1) the first term of this expansion.

Some assumptions on the cumulant generating function are established in [3] (Section 6), so that (2) has a unique real root. These assumptions, for the general case of a distribution with support $(a, b) : -\infty \leq a < b \leq \infty$, are given by

$$\lim_{\lambda \to -r} K'(\lambda) = a, \quad \lim_{\lambda \to s} K'(\lambda) = b$$

and guarantee the existence of a unique real solution of the saddlepoint equation for every $\bar{x}$ within (a, b).

We will get an approximation of equation (2) solution by means of a one step Newton–Raphson method. Since $K'(\hat{\lambda}) = \mu + \sigma^2\hat{\lambda} + O(\hat{\lambda}^2) = \mu + \frac{x\sigma}{\sqrt{n}}$, it is natural that we use as a starting point $\hat{\lambda}_0 = \frac{x}{\sigma\sqrt{n}}$; this point will yield to an approximation of $\hat{\lambda}$ given by $\tilde{\lambda} = \hat{\lambda}_0 + \frac{\bar{x} - K'(\hat{\lambda}_0)}{K''(\hat{\lambda}_0)}$.

Expanding the previous quotient, we get

$$\tilde{\lambda} = \hat{\lambda}_0 + \frac{-\frac{k_3 x^2}{2\sigma^2 n} - \frac{k_4 x^3}{6\sigma^3\sqrt{n^3}} + O(n^{-2})}{\sigma^2 + \frac{k_3 x}{\sigma\sqrt{n}} + \frac{k_4 x^2}{2\sigma^2 n} + O(n^{-3/2})} = \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + O(n^{-2}),$$

with $a_1 = \frac{x}{\sigma}$, $a_2 = -\frac{k_3 x^2}{2\sigma^4}$ and $a_3 = \frac{k_3^2 x^3}{2\sigma^7} - \frac{k_4 x^3}{6\sigma^5}$.

Since $K'(\hat{\lambda}) - K'(\tilde{\lambda}) = O(n^{-2})$, we can conclude that $\hat{\lambda} - \tilde{\lambda} = O(n^{-2})$ and therefore

$$\hat{\lambda} = \frac{a_1}{\sqrt{n}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + O(n^{-2}). \tag{3}$$

3. A NEW FORMULATION FOR THE TAIL PROBABILITY SADDLEPOINT APPROXIMATION

In this section we aim to get a new formulation for (1) that incorporates the results of the previous one. We will face up the task in three steps: The first one will deal with an approximation formula for $n[K(\hat{\lambda}) - \hat{\lambda}\bar{x}]$; the second one is devoted to develop an analogous formula for $\hat{\lambda}\sqrt{nK''(\hat{\lambda})}$. We will finally mix up the preceding results in order to obtain an expansion for the quotient, thus for the tail probability.

3.1. Phase I

Taylor expansion of $K(\hat{\lambda}) - \hat{\lambda}\bar{x}$ yields to

$$K(\hat{\lambda}) - \hat{\lambda}\bar{x} = \mu\hat{\lambda} + \frac{\sigma^2}{2!}\hat{\lambda}^2 + \frac{k_3}{3!}\hat{\lambda}^3 + \cdots - \hat{\lambda}(\mu + \frac{x\sigma}{\sqrt{n}}) = -\hat{\lambda}\frac{x\sigma}{\sqrt{n}} + \frac{\sigma^2}{2}\hat{\lambda}^2 + \frac{k_3}{6}\hat{\lambda}^3 + \cdots.$$

Therefore

$$K(\hat{\lambda}) - \hat{\lambda}\bar{x} = -\frac{x\sigma}{\sqrt{n}}\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right) + \frac{\sigma^2}{2}\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right)^2$$

$$+\frac{k_3}{6}\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right)^3 + \frac{k_4}{24}\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right)^4 + \cdots.$$

If we replace a_1, a_2, a_3 by its values from (3) and gather together the terms of the same order we will get

$$K(\hat{\lambda}) - \hat{\lambda}\bar{x} = -\frac{x^2}{2n} + \frac{1}{n^{3/2}}\frac{k_3 x^3}{6\sigma^3} + \frac{1}{n^2}\left(\frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}\right) + O(n^{-5/2}),$$

where the coefficients of $\frac{1}{\sqrt{n^5}}$ in the expansion of $K(\hat{\lambda}) - \hat{\lambda}\bar{x}$ have been grouped together to obtain a $O(n^{-5/2})$ remainder.

Hence

$$n[K(\hat{\lambda}) - \hat{\lambda}\bar{x}] = -\frac{x^2}{2} + \frac{1}{n^{1/2}}\frac{k_3x^3}{6\sigma^3} + \frac{1}{n}\left(\frac{k_4x^4}{24\sigma^4} - \frac{k_3^2x^4}{8\sigma^6}\right) + O(n^{-3/2}). \qquad (4)$$

3.2. Phase II

Considering that

$$K''(\hat{\lambda}) = \sigma^2 + k_3\hat{\lambda} + \frac{k_4}{2}\hat{\lambda}^2 + \cdots$$

$$= \sigma^2 + k_3\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right) + \frac{k_4}{2}\left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right)^2 + \cdots,$$

we will get an expansion for $\sqrt{K''(\hat{\lambda})}$ in powers of $n^{-1/2}$, that is to say: $\sqrt{K''(\hat{\lambda})} = \sum_{i=0}^{\infty} b_i n^{-i/2}$, where the unknown coefficients are deduced by comparisons between the terms of the same order from $K''(\hat{\lambda})$ expansion and

$$K''(\hat{\lambda}) = \left(\sum_{i=0}^{\infty} b_i n^{-i/2}\right)^2.$$

For example, to obtain the values of b_0, b_1 and b_2, we need comparisons up to the n^{-1} power; in this way we will get the following equations:

— $b_0^2 = \sigma^2$

— $2b_0b_1 = k_3a_1$

— $2b_0b_2 + b_1^2 = k_3a_2 + \frac{k_4}{2}a_1^2$

from which $b_0 = \sigma$, $b_1 = \frac{k_3x}{2\sigma^2}$ and $b_2 = \frac{k_4x^2}{4\sigma^3} - \frac{3k_3^2x^2}{8\sigma^5}$.

These values and those from (3), lead to an expansion for $\hat{\lambda}\sqrt{K''(\hat{\lambda})}$:

$$\hat{\lambda}\sqrt{K''(\hat{\lambda})} = \left(\sum_{i=1}^{3} a_i n^{-i/2} + O(n^{-2})\right)\left(\sum_{i=0}^{2} b_i n^{-i/2} + O(n^{-3/2})\right)$$

$$= \frac{b_0a_1}{\sqrt{n}} + \frac{1}{n}(b_0a_2 + a_1b_1) + \frac{1}{n^{3/2}}(a_1b_2 + a_2b_1 + a_3b_0) + O(n^{-2})$$

$$= \frac{x}{\sqrt{n}} + \frac{1}{n^{3/2}}\left(\frac{k_4x^3}{12\sigma^4} - \frac{k_3^2x^3}{8\sigma^6}\right) + O(n^{-2}).$$

Therefore

$$\hat{\lambda}\sqrt{nK''(\hat{\lambda})} = c_1 + \frac{c_2}{\sqrt{n}} + \frac{c_3}{n} + O(n^{-3/2}), \qquad (5)$$

where $c_1 = x$, $c_2 = 0$ and $c_3 = \frac{k_4x^3}{12\sigma^4} - \frac{k_3^2x^3}{8\sigma^6}$.

3.3. Phase III

Including (4) and (5) into (1) we will get

$$P(\bar{X} \geq \bar{x}) = \frac{\phi(x) \exp\left(\frac{1}{n^{1/2}} \frac{k_3 x^3}{6\sigma^3} + \frac{1}{n}(\frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}) + O(n^{-3/2})\right)}{(c_1 + \frac{c_3}{n} + O(n^{-3/2}))} + O(n^{-3/2}).$$

Expanding the quotient $\frac{P(\bar{X} \geq \bar{x})}{\phi(x)}$ in powers of $n^{-1/2}$: $\frac{P(\bar{X} \geq \bar{x})}{\phi(x)} = \sum_{i=0}^{\infty} d_i n^{-i/2}$, we get

$$\exp\left(\frac{1}{n^{1/2}} \frac{k_3 x^3}{6\sigma^3} + \frac{1}{n}(\frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}) + O(n^{-3/2})\right)$$

$$= 1 + \left(\frac{1}{n^{1/2}} \frac{k_3 x^3}{6\sigma^3} + \frac{1}{n}(\frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}) + O(n^{-3/2})\right)$$

$$+ \frac{1}{2}\left(\frac{1}{n^{1/2}} \frac{k_3 x^3}{6\sigma^3} + \frac{1}{n}(\frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}) + O(n^{-3/2})\right)^2$$

$$= \left(c_1 + \frac{c_3}{n} + O(n^{-3/2})\right)\left(d_0 + \frac{d_1}{\sqrt{n}} + \frac{d_2}{n} + \cdots\right).$$

Comparisons between terms of the same order lead to the following equations:

— $c_1 d_0 = 1$

— $c_1 d_1 = \frac{k_3 x^3}{6\sigma^3}$

— $c_3 d_0 + c_1 d_2 = \frac{k_3^2 x^6}{72\sigma^6} + \frac{k_4 x^4}{24\sigma^4} - \frac{k_3^2 x^4}{8\sigma^6}$

which allow us to determine d_0, d_1, d_2 :

$$d_0 = \frac{1}{x} \text{ , } d_1 = \frac{k_3 x^2}{6\sigma^3} \text{ and } d_2 = \frac{k_3^2}{8\sigma^6}\left(\frac{x^5}{9} - x^3 + x\right) + \frac{k_4}{12\sigma^4}\left(\frac{x^3}{2} - x\right).$$

Therefore

$$P(Z_n \geq x) = P(\bar{X} \geq \bar{x}) = \frac{\phi(x)}{x} + \frac{\phi(x)\, d_1(x)}{\sqrt{n}} + \frac{\phi(x)\, d_2(x)}{n} + O(n^{-3/2}), \quad (6)$$

with $d_1(x) = \frac{k_3 x^2}{6\sigma^3}$ and $d_2(x) = \frac{k_3^2}{8\sigma^6}\left(\frac{x^5}{9} - x^3 + x\right) + \frac{k_4}{12\sigma^4}\left(\frac{x^3}{2} - x\right)$.

4. A SECOND ORDER APPROXIMATION FOR THE QUANTILE

If we put $\bar{P}(x) = \frac{\phi(x)}{x} + \frac{\phi(x)\, d_1(x)}{\sqrt{n}} + \frac{\phi(x)\, d_2(x)}{n}$, then (6) can be written as follows:

$$P(Z_n \geq x) = P(\bar{X} \geq \bar{x}) = \bar{P}(x) + O(n^{-3/2})$$

so that, a certain significance level α will almost satisfy that

$$\bar{P}(x) = \frac{\phi(x)}{x} + \frac{\phi(x)\, d_1(x)}{\sqrt{n}} + \frac{\phi(x)\, d_2(x)}{n} = \alpha. \tag{7}$$

Proper inversion of (7) will allow us to find an approximation formula for the quantile $\bar{x}$. This approximation will be deduced using the inversion technique from [2] together with the following lemma, which is a consequence of an inequality from [4].

Lemma. If Φ and ϕ are the distribution and density functions of the standard normal law, then

$$1 - \Phi(x) = \frac{\phi(x)}{x}(1 + o(1)) \text{ as } x \to \infty.$$

In our framework x is in the upper tail of the distribution. It then stands to reason that x is large enough, so $\frac{\phi(x)}{x}$ may be replaced by $1 - \Phi(x)$ in (7). This will perform very accurately when α is small enough, at least for heavy tails distributions, as will be shown in Section 5.

Under these conditions, (7) can be replaced by

$$\bar{P}(x) = 1 - \Phi(x) + \frac{\phi(x)\, d_1(x)}{\sqrt{n}} + \frac{\phi(x)\, d_2(x)}{n} = \alpha. \tag{8}$$

From now on, we will focus on inverting (8).

Assume that $x = z_\alpha + \frac{t_1}{\sqrt{n}} + \frac{t_2}{n} + \cdots$ with $z_\alpha = \Phi^{-1}(1-\alpha)$.

To begin with, we will expand $\tilde{P}(\bar{X} \geq \bar{x})$ in a Taylor series at z_α:

$$1 - \Phi(z_\alpha) + \phi(z_\alpha)\left(\frac{d_1(z_\alpha)}{\sqrt{n}} + \frac{d_2(z_\alpha)}{n}\right) + \phi(z_\alpha)\left(-1 + \frac{-z_\alpha d_1(z_\alpha) + d_1'(z_\alpha)}{\sqrt{n}}\right.$$
$$\left. + \frac{-z_\alpha d_2(z_\alpha) + d_2'(z_\alpha)}{n}\right)(x - z_\alpha) + \cdots = \alpha,$$

which implies that

$$\frac{d_1(z_\alpha)}{\sqrt{n}} + \frac{d_2(z_\alpha)}{n} + \left(-1 + \frac{d_1'(z_\alpha) - z_\alpha d_1(z_\alpha)}{\sqrt{n}} + \frac{d_2'(z_\alpha) - z_\alpha d_2(z_\alpha)}{n}\right)(x - z_\alpha) + \cdots = 0.$$

If we replace $x - z_\alpha$ by $\sum_{i=1}^{\infty} t_i n^{-i/2}$ in the previous expression and compare the terms of the same order from both members of the equality, we get

$$t_1 = d_1(z_\alpha).$$

Consequently

$$x = z_\alpha + \frac{d_1(z_\alpha)}{\sqrt{n}} + O(n^{-1}),$$

and the desired second order quantile approximation for the quantile $\bar{x}$ will be

$$\bar{x} = \mu + \frac{x\sigma}{\sqrt{n}} = \mu + \frac{\sigma z_\alpha}{\sqrt{n}} + \frac{\sigma d_1(z_\alpha)}{n} + O(n^{-3/2}) \quad \text{with} \quad d_1(z_\alpha) = \frac{k_3 z_\alpha^2}{6\sigma^3}. \qquad (9)$$

5. NUMERICAL EXAMPLES

In this section we will display some numerical examples which will allow us to compare the normal approximation with ours.

5.1. Example I

The following tables show for different significance levels, α, the values of the exact quantile, the normal quantile approximation and the second order quantile approximation, as well as the relative errors under an exponential distribution with parameter 1.

We have chosen small sample sizes for better comparisons between normal and second order approximations.

n = 3

α	e	n	s	$er1$	$er2$
0.050	2.098598	1.949657	2.250273	7.097176	7.227433
0.045	2.146482	1.978838	2.298213	7.810149	7.068842
0.040	2.199636	2.010759	2.351304	8.586726	6.895139
0.035	2.259446	2.046107	2.410887	9.442098	6.702567
0.030	2.327936	2.085877	2.478919	10.398028	6.485714
0.025	2.408229	2.131586	2.558414	11.487423	6.236336
0.020	2.505535	2.185732	2.654386	12.763828	5.940916
0.015	2.629565	2.252902	2.776157	14.324157	5.574748
0.010	2.801982	2.343118	2.944439	16.376432	5.084146
0.005	3.091264	2.487156	3.224366	19.542436	4.305761

e = exact quantile, n = normal approximation, s = second order approximation, $er1 = 100 * |e - n|/e$ and $er2 = 100 * |e - s|/e$.

Note that as we approach the upper tail of the distribution, the normal approximation based on central limit theorem breaks down, giving large relative errors: 16.37 % or 19.54 %, in comparison with the relative errors given by the second order approximation: 5 %, 4.3 %.

n = 7

α	e	n	s	$er1$	$er2$
0.050	1.691768	1.621696	1.750532	4.141935	3.473493
0.045	1.718931	1.640800	1.777675	4.545307	3.417487
0.040	1.748959	1.661697	1.807645	4.989381	3.355450
0.035	1.782599	1.684838	1.841172	5.484197	3.285824
0.030	1.820935	1.710873	1.879320	6.044231	3.206342
0.025	1.865636	1.740797	1.923723	6.691526	3.113527
0.020	1.919480	1.776244	1.977096	7.462245	3.001610
0.015	1.987625	1.820217	2.044469	8.422491	2.859922
0.010	2.081516	1.879277	2.136986	9.715943	2.664903
0.005	2.237096	1.973572	2.289519	11.779753	2.343351

e = exact quantile, n = normal approximation, s = second order approximation, $er1 = 100 * |e - n|/e$ and $er2 = 100 * |e - s|/e$.

n = 11

α	e	n	s	$er1$	$er2$
0.050	1.542020	1.495942	1.577928	2.988126	2.328679
0.045	1.562393	1.511182	1.598284	3.277722	2.297204
0.040	1.584875	1.527852	1.620728	3.597961	2.262174
0.035	1.610012	1.546312	1.645797	3.956522	2.222653
0.030	1.638597	1.567081	1.674274	4.364506	2.177277
0.025	1.671850	1.590951	1.707359	4.838875	2.123941
0.020	1.711795	1.619229	1.747043	5.407564	2.059141
0.015	1.762186	1.654307	1.797013	6.121874	1.976355
0.010	1.831334	1.701420	1.865417	7.093961	1.861083
0.005	1.945257	1.776642	1.977699	8.668021	1.667758

e = exact quantile, n = normal approximation, s = second order approximation, $er1 = 100 * |e - n|/e$ and $er2 = 100 * |e - s|/e$.

Note that, n increase reduces the relative errors corresponding to the normal and second order approximations; however the former remains larger than the later. In fact, we have thought that such differences are quite appreciable for heavy tails distributions. It would be an interesting task, analyzing them with respect to a tailing order, i. e. Loh order that is alluded in [6].

5.2. Example II

This example shows the accuracy of the normal and second order approximations when the underlying distribution is a χ_1^2 distribution. Small sample sizes have been tried in order to make the differences between both approximations clear.

We will use the same notation as in the previous examples; the results are displayed in the following tables.

n = 3

α	e	n	s	$er1$	$er2$
0.050	2.604909	2.343017	2.944249	10.05378	13.026939
0.045	2.683162	2.384286	3.023036	11.13892	12.666941
0.040	2.770390	2.429429	3.110518	12.30733	12.277265
0.035	2.868982	2.479419	3.208979	13.57846	11.850768
0.030	2.982429	2.535662	3.321747	14.97999	11.377231
0.025	3.116135	2.600304	3.453961	16.55354	10.841216
0.020	3.279136	2.676879	3.614187	18.36634	10.217636
0.015	3.488344	2.771871	3.818381	20.53904	9.461144
0.010	3.781622	2.899455	4.102098	23.32774	8.474565
0.005	4.279385	3.103156	4.577577	27.48595	6.968099

n = 5

α	e	n	s	$er1$	$er2$
0.050	2.214100	2.040297	2.401036	7.849817	8.442997
0.045	2.268461	2.072264	2.455513	8.648901	8.245807
0.040	2.328866	2.107231	2.515885	9.516874	8.030443
0.035	2.396911	2.145953	2.583689	10.470077	7.792427
0.030	2.474924	2.189518	2.661170	11.531886	7.525319
0.025	2.566500	2.239590	2.751785	12.737591	7.219332
0.020	2.677645	2.298905	2.861289	14.144509	6.858452
0.015	2.819559	2.372486	3.000391	15.856143	6.413498
0.010	3.017254	2.471312	3.192898	18.094029	5.821286
0.005	3.349920	2.629097	3.513750	21.517615	4.890561

n = 7

α	e	n	s	$er1$	$er2$
0.050	2.009591	1.879211	2.136882	6.487898	6.334151
0.045	2.052663	1.906228	2.179978	7.133887	6.202441
0.040	2.100435	1.935781	2.227676	7.839050	6.057851
0.035	2.154143	1.968507	2.281176	8.617606	5.897153
0.030	2.215584	2.005326	2.342220	9.489947	5.715693
0.025	2.287538	2.047645	2.413498	10.486950	5.506370
0.020	2.374632	2.097775	2.499478	11.658933	5.257513
0.015	2.485490	2.159962	2.608466	13.097145	4.947746
0.010	2.639330	2.243485	2.758904	14.997912	4.530476
0.005	2.896820	2.376839	3.008734	17.950073	3.863326

Observe that for very small significance levels, corresponding to quantiles far away in the upper tail of the distribution, the second order approximation is more accurate than the normal; see table for $n = 5$ in which the relative errors are 5.82 % against 18.09 % for $\alpha = 0.010$. But as α increases we move away the tail, and the differences between both approximations become smaller; in some cases the normal approximation is more accurate than ours (see the relative errors of the first table for $\alpha = 0.050$).

However, if we load the upper tail of the distribution, that is to say if we increase the degrees of freedom of the χ^2, (9) will become more accurate than the central limit approximation for every significance level. The next example, which considers a χ^2_5 distribution, corroborates this assertion.

5.3. Example III

n = 3

α	e	n	s	$er1$	$er2$
0.050	8.331920	8.003078	8.604310	3.946774	3.269232
0.045	8.461816	8.095359	8.734108	4.330719	3.217893
0.040	8.605376	8.196301	8.877390	4.753715	3.160981
0.035	8.766149	8.308081	9.037641	5.225418	3.097051
0.030	8.949302	8.433844	9.219929	5.759758	3.024005
0.025	9.162787	8.578388	9.432046	6.377957	2.938613
0.020	9.419822	8.749615	9.686923	7.114854	2.835522
0.015	9.744951	8.962025	10.00853	8.034177	2.704814
0.010	10.19263	9.247311	10.44995	9.274570	2.524571
0.005	10.93377	9.702799	11.17722	11.258457	2.226562

n = 5

α	e	n	s	$er1$	$er2$
0.050	7.530496	7.326174	7.686913	2.713251	2.077126
0.045	7.624566	7.397654	7.780904	2.976056	2.050456
0.040	7.728328	7.475844	7.884498	3.266997	2.020739
0.035	7.844286	7.562429	8.000165	3.593151	1.987166
0.030	7.976076	7.659844	8.131495	3.964761	1.948564
0.025	8.129292	7.771808	8.284002	4.397490	1.903114
0.020	8.313214	7.904440	8.466824	4.917161	1.847788
0.015	8.545037	8.068971	8.696877	5.571250	1.776941
0.010	8.862821	8.289953	9.011539	6.463719	1.678000
0.005	9.385578	8.642773	9.527426	7.914327	1.511336

n = 7

α	e	n	s	$er1$	$er2$
0.050	7.114550	6.965976	7.223647	2.088304	1.533440
0.045	7.191100	7.026388	7.300138	2.290498	1.516289
0.040	7.275443	7.092470	7.384365	2.514941	1.497122
0.035	7.369585	7.165647	7.478316	2.767289	1.475400
0.030	7.476439	7.247978	7.584872	3.055738	1.450335
0.025	7.600478	7.342605	7.708458	3.392851	1.420704
0.020	7.749119	7.454699	7.856403	3.799392	1.384466
0.015	7.936089	7.593754	8.042258	4.313650	1.337799
0.010	8.191725	7.780518	8.295936	5.019786	1.272155
0.005	8.610682	8.078705	8.710600	6.178102	1.160399

The three preceding tables display the quantile approximation under a χ^2_5 distribution. Observe that the relative errors have decreased for both the normal and second order approximation, with the former always higher than the later.

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