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PERIODIC SYSTEMS LARGELY SYSTEM EQUIVALENT TO PERIODIC DISCRETE–TIME PROCESSES[1]

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In this paper, the problem of obtaining a periodic model in state-space form of a linear process that can be modeled by linear difference equations with periodic coefficients is considered. Such a problem was already studied and solved in [20] on the basis of the notion of system equivalence, but under the assumption that the process has no null characteristic multiplier. In this paper such an assumption is removed in order to generalize the results in [20] to linear periodic processes with possibly the null characteristic multiplier (e. g., multirate sampled-data systems). Large system equivalence between two linear periodic models of such processes is introduced and analyzed. For a given linear periodic process the necessary and sufficient conditions are found for the existence of a linear periodic system (i. e., a linear periodic model in state-space form) that is largely system equivalent to the given model of the process, together with an algorithm for deriving such a system when these conditions are satisfied.

In addition, the significance of the periodic system thus obtained for describing the original periodic process that is largely system equivalent to the system, is clarified by showing that the controllability, the reconstructibility, the stabilizability, the detectability, the stacked transfer matrix, the asymptotic stability, the rate of convergence of the free motions, and even the number and the dimensions of the Jordan blocks of the monodromy matrix corresponding to each nonnull characteristic multiplier of the periodic system, are determined by the original periodic process (although the order of the periodic system is not, in general, as well as its reachability and observability properties, because of some possible additional or removed null characteristic multipliers).

1. INTRODUCTION

In order to find a state-space model for processes that can be modeled by linear differential or difference equations with constant coefficients, Rosenbrock made use of the following pair of vector equations [30]:

$$T(s)\xi = U(s)\,u, \tag{1}$$
$$y = V(s)\,\xi + W(s)\,u, \tag{2}$$

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where u is the input vector of the process, y is the output vector, ξ is the vector of the internal variables, called here pseudo-state, and $T(s)$, $U(s)$, $V(s)$ and $W(s)$ are polynomial matrices in the indeterminate s; if s is replaced either by differentiation or by the one-step forward shift operator, the time-domain model of the process under consideration is obtained. He showed that under polynomial transformations on (1) and (2) that he called strict system equivalence and possibly an additional extension or reduction of the dimension of the pseudo-state (which will be recalled in detail in the next section), it is possible to obtain a description of process (1), (2) in state-space form, i. e., of the type:

$$sx(t) = Ax(t) + Bu(t) \tag{3}$$

$$y(t) = Cx(t) + Du(t) \tag{4}$$

(where s means either differentiation or one-step forward shift operator), provided that $T(s)$ is square and nonsingular and the input-output transfer function matrix corresponding to (1) and (2) is proper. It is stressed that for the linear system (3), (4) thus obtained, not only the transfer function matrix is the same as that of the given process described by (1) and (2), but also the Smith forms of $sI - A$, $[\,sI - A \quad B\,]$, $[\,sI - A^T \quad C^T\,]^T$ and of the Rosenbrock system matrix of system (3), (4), coincide (possibly apart from some unit invariant polynomials) with those of $T(s)$, $[\,T(s) \quad U(s)\,]$, $[\,T(s)^T \quad V(s)^T\,]^T$, and of the system matrix of (1), (2), respectively; this means that the asymptotic stability of system (3), (4), the rate of convergence of its free motions, and even the whole Jordan form of A, together with the whole Jordan forms of its non-reachable and unobservable parts, if any, and its invariant zeros with their ordered sets of structural indices, are determined by the original process described by (1), (2). Since then, several authors studied this kind of problem (see, e. g., [3, 7, 11, 21, 24, 32]), which is different from the well-known realization problem; this, in the time-invariant case, is the problem of finding a linear system of the form (3), (4) whose transfer matrix or impulse response matrix coincide with a given one (see, e. g., [23]), so that the datum of this problem consists of the input-output map that characterizes the mere zero-state output responses of the system to be found, whereas no complete information is given on the input-state map of the same system, nor on its free responses. This makes non-unique the solution of the realization problem, if no a priori information is available about the non-existence of non-reachable and/or unobservable states of the system to be found; hence, arbitrary unobservable and/or nonreachable subsystems can be added to a found minimal realization, still obtaining a solution of the realization problem.

The study of the problem of finding a state-space representation of a process that can be modeled by equations (1), (2) was extended in [20] to processes that can be modeled by linear difference equations with periodic coefficients (whose period will be denoted by ω) of the following form:

$$\sum_{i=0}^{r} T_i(k)\xi(k+i) = \sum_{i=0}^{r} U_i(k)\,u(k+i), \tag{5}$$

$$y(k) = \sum_{i=0}^{r} V_i(k)\xi(k+i) + \sum_{i=0}^{r} W_i(k)\,u(k+i), \tag{6}$$

for some integer $r \geq 0$, where $k \in \mathbf{Z}, \xi(k+i) \in \mathbf{R}^m$ is the vector of the internal variables or pseudo-state, $u(k+i) \in \mathbf{R}^p$ is the input, $y(k) \in \mathbf{R}^q$ is the output, $T_i(k)$, $U_i(k)$, $V_i(k)$ and $W_i(k)$, $i = 0, \ldots, r$, are real periodic matrices of period ω (briefly, ω-periodic), and the $T_i(k)$, $i = 0, \ldots, r$, were assumed to be square. It was shown that, within the class of transformations of (5), (6) that was called system equivalence [20], it is possible to obtain a description of (5), (6) in state-space form, i. e. a description of the form

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k), \tag{7}$$

$$y(k) = C(k)\,x(k) + D(k)\,u(k), \tag{8}$$

where $x(k) \in \mathbf{R}^n$ is the state, and $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$, are real ω-periodic matrices, provided that the process described by equations (5), (6) satisfies some causality conditions and, in addition, no null characteristic multiplier is associated with the left-hand side of (5) (see the next section for a formal definition of the mentioned notion of characteristic multiplier); moreover, most of the properties and features of the ω-periodic system (7), (8) thus obtained are determined by the original process (5), (6), as well as in the solution (3), (4) proposed by Rosenbrock [30] to the original problem of giving a state-space description of (1), (2).

The interest of obtaining a description of (5), (6) in state-space form is motivated by the variety of processes that can be modeled by linear difference equations with periodic coefficients (e. g., multirate sampled-data systems, periodically time-varying digital filters, seasonal phenomena [1, 2, 27]) and the resulting amount of contributions devoted to solve control problems for linear periodic discrete-time systems – including eigenvalue assignment, state and output dead-beat control, disturbance decoupling, optimal control, robust tracking and regulation, and input-output block decoupling (see [4, 5, 9, 14, 15, 16, 17, 19, 28, 25, 22]) – since most of these contributions are based on a state-space description. For similar reasons, the different problem of finding discrete-time linear periodic realizations of input-output linear maps was studied by several authors [8, 12, 26, 29, 31]; in particular, a necessary and sufficient condition for the existence of a linear periodic minimal realization was introduced in [31] in terms of the Hankel matrix associated with an input-output periodic application. However, in general a periodic minimal (i. e., reachable and observable at all times) realization may have a time-varying dimension (see [6, 12], also for algorithms for its computation) – although "quasi-minimal" (i. e., reachable and observable at least at one time instant) realizations with a constant dimension [6, 26] can be obtained – ; this is because the zero-state output responses of system (7), (8), which are its only features that have to be matched with the datum of the realization problem, do not depend on its unobservable and/or non-reachable parts, which, however, may have time-varying dimensions [13].

On the contrary, the number of arbitrary and independent initial conditions on which the solutions of (5), (6) depend (which is the natural characterization of the dimension of the state of the corresponding system (7), (8) to be found) is constant with time, namely it does not depend on the initial time under consideration [20], while process (5), (6) could actually have some pseudo-state free motions that are unobservable from y and/or some subvector of ξ that cannot be influenced by u, so that the equations (5), (6) (which are assumed to be a complete description of the process under consideration) could actually be equivalent (in some sense) to a complete description of an arbitrary periodic system of the form (7), (8), and not only to that of its zero-state output responses. Therefore it seems to be of interest to find a ω-periodic system of such a form that preserves most of the properties and features of the original ω-periodic process (5), (6), without the assumption about the null characteristic multiplier that was used in [20]. This is just the purpose of this paper, in order to give a solution to the same kind of problem that was studied in [20] also for the class of periodic processes of the form (5), (6) that actually have the null characteristic multiplier (e. g., multirate sampled-data systems [1] always have it).

Preliminarily, in Section 2 a polynomial time-invariant characterization of such a process, and some related notions and results, will be recalled, including the notion of system equivalence [20] between two models of the form (5), (6). In Section 3 the more general notion of large system equivalence between two such models will be introduced, and the properties and features that are invariant under it will be analyzed. In Section 4 the necessary and sufficient existence conditions of a periodic system described by (7), (8) that is largely system equivalent to a periodic process described by (5), (6) will be given, together with an algorithm for deriving such a periodic system from the given process (5), (6).

2. NOTATIONS AND SOME BACKGROUND MATERIAL

Henceforth, the identity matrix of dimension ν will be denoted either by I_ν, or simply by I; Δ will denote the ω-steps forward-shift operator, and Δ^{-1} its inverse; in addition, $R_\nu(\Delta)$, $\nu \in \mathbf{Z}^+$ will denote the following matrix operator:

$$R_\nu(\Delta) := \begin{bmatrix} 0 & I_{(\omega-1)\nu} \\ \Delta I_\nu & 0 \end{bmatrix}, \tag{9}$$

where $\mathbf{Z}^+$ is the set of positive integers.

Let a vector function $z(t) \in \mathbf{R}^\nu$ be given, with $t \in \mathbf{Z}$; for any $k \in \mathbf{Z}$, the *ω-stacked form of $z(t)$ at time k* is defined by

$$z_k(h) := \left[z^T(k + h\omega) \;\; z^T(k + h\omega + 1) \;\ldots\; z^T(k + h\omega + \omega - 1)\right]^T, \quad h \in \mathbf{Z}.$$

From now on, whenever the operator $R_\nu(\Delta)$ will be applied to $z_k(h)$, the operator Δ will have the meaning of an ω-steps forward-shift in the k variable, or, equivalently, a one-step forward-shift in the h variable. Notice that [20]:

$$R_\nu(\Delta) z_k(h) = z_{k+1}(h). \tag{10}$$

Let an ω-periodic matrix $F(t) \in \mathbf{R}^{\nu\times\mu}$ be given, with $t \in \mathbf{Z}$, representing the linear map $z(t) = F(t)w(t)$; for any $k \in \mathbf{Z}$, the *ω-stacked form of $F(t)$ at time k* is defined by $\mathcal{F}_k := \operatorname{diag}\{F(k), F(k+1), \ldots, F(k+\omega-1)\}$, and represents the induced linear map between the ω-stacked forms at time k of the vector functions $z(t)$ and $w(t)$, i. e. $z_k(h) = \mathcal{F}_k w_k(h), h \in \mathbf{Z}$.

By introducing the ω-stacked forms $\xi_{k_0}(h), u_{k_0}(h), y_{k_0}(h)$ at time k_0 of vectors $\xi(k), u(k), y(k)$ and the ω-stacked forms $\mathcal{T}_{i,k_0}, \mathcal{U}_{i,k_0}, \mathcal{V}_{i,k_0}$ and $\mathcal{W}_{i,k_0}$ at time k_0 of matrices $T_i(k), U_i(k), V_i(k), W_i(k)$, $i = 0, \ldots, r$, the model (5), (6) of the process under consideration can be expressed in the following form, which is called the *ω-stacked form at time k_0 of model* (5), (6) [20]:

$$\mathcal{T}_{k_0}(\Delta)\xi_{k_0}(h) = \mathcal{U}_{k_0}(\Delta)u_{k_0}(h), \tag{11}$$

$$y_{k_0}(h) = \mathcal{V}_{k_0}(\Delta)\xi_{k_0}(h) + \mathcal{W}_{k_0}(\Delta)u_{k_0}(h), \tag{12}$$

where $\mathcal{T}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{T}_{i,k_0} R_m^i(\Delta)$, $\mathcal{U}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{U}_{i,k_0} R_p^i(\Delta)$, $\mathcal{V}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{V}_{i,k_0} R_m^i(\Delta)$, $\mathcal{W}_{k_0}(\Delta) := \sum_{i=0}^{r} \mathcal{W}_{i,k_0} R_p^i(\Delta)$. The following polynomial matrix of Δ:

$$S_{k_0}^M(\Delta) := \begin{bmatrix} -\mathcal{T}_{k_0}(\Delta) & \mathcal{U}_{k_0}(\Delta) \\ \mathcal{V}_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{bmatrix} \tag{13}$$

is termed the *ω-stacked system matrix at time k_0 of model* (5), (6), thus extending the time-invariant Rosenbrock system matrix [30]. It will be the main tool for deriving a state-space representation (7), (8) of process (5), (6).

The following assumption is justified by Proposition 2.2 of [20], and will be assumed to hold throughout the paper. In fact, if it is not satisfied, either the number of scalar equations contained in (5) can be trivially reduced, without modifying the set of pseudo-state and output solutions of (5), (6) for any given input function $u(\cdot)$, or such solutions do not depend on a *finite* number of arbitrary and independent initial conditions or even no solution exists for some $u(\cdot)$ [20].

Assumption 1. The polynomial matrix $\mathcal{T}_{k_0}(\Delta)$ is square and nonsingular.

If Assumption 1 holds for $k_0 = \bar{k}_0 \in \mathbf{Z}$, then it holds for any $k_0 \in \mathbf{Z}$, and the degree of $\det \mathcal{T}_{k_0}(\Delta)$ is independent of the time k_0 [20]. Therefore, the degree of $\det \mathcal{T}_{k_0}(\Delta)$ for an arbitrary $k_0 \in \mathbf{Z}$ is called the *order of model* (5), (6), since it coincides with the number of arbitrary and independent initial conditions on which the solutions of (5), (6) depend [20].

Moreover, under Assumption 1 and for a fixed time k_0, the application of the z-transform to both sides of (11), (12), with zero initial conditions both for $\xi_{k_0}(h)$ and $u_{k_0}(h)$, yields $y_{k_0}(z) = G_{k_0}^M(z)\, u_{k_0}(z)$, where $G_{k_0}^M(z) := \mathcal{V}_{k_0}(z)\mathcal{T}_{k_0}^{-1}(z)\mathcal{U}_{k_0}(z) + \mathcal{W}_{k_0}(z)$ is called the *ω-stacked transfer matrix at time k_0 of model* (5), (6).

For the linear ω-periodic system described by (7), (8), equations (11), (12) reduce to the following ones:

$$R_n(\Delta)\, x_{k_0}(h) = \mathcal{A}_{k_0}\, x_{k_0}(h) + \mathcal{B}_{k_0}\, u_{k_0}(h), \tag{14}$$

$$y_{k_0}(h) = \mathcal{C}_{k_0}\, x_{k_0}(h) + \mathcal{D}_{k_0}\, u_{k_0}(h), \tag{15}$$

(where $x_{k_0}(h)$, $\mathcal{A}_{k_0}, \mathcal{B}_{k_0}, \mathcal{C}_{k_0}$ and $\mathcal{D}_{k_0}$ are the ω-stacked forms at time k_0 of $x(k)$, $A(k), B(k), C(k)$ and $D(k)$, respectively), which are termed the *ω-stacked form at* *time* k_0 *of system* (7), (8); if the symbol $S^S_{k_0}(\Delta)$ is used in this case instead of $S^M_{k_0}(\Delta)$, relation (13) reduces to

$$S^S_{k_0}(\Delta) := \begin{bmatrix} \mathcal{A}_{k_0} - R_n(\Delta) & \mathcal{B}_{k_0} \\ \mathcal{C}_{k_0} & \mathcal{D}_{k_0} \end{bmatrix}. \tag{16}$$

$S^S_{k_0}(\Delta)$ is called the *ω-stacked system matrix at time* k_0 *of system* (7), (8). In a similar way, the *ω-stacked transfer matrix* $G^S_{k_0}(z)$ *at time* k_0 *of system* (7), (8) is expressed by $G^S_{k_0}(z) := \mathcal{C}_{k_0}(R_n(z) - \mathcal{A}_{k_0})^{-1}\mathcal{B}_{k_0} + \mathcal{D}_{k_0}$. Further, the state transition matrix over a period $\Phi(k_0 + \omega, k_0)$ of system (7), (8) at the initial time k_0, expressed by $\Phi(k_0 + \omega, k_0) := A(k_0 + \omega - 1) \cdots A(k_0 + 1)A(k_0)$, will be called the *monodromy matrix at time* k_0 *of system* (7), (8).

The basic relation that will be used here between two $(m\omega + q\omega) \times (m\omega + p\omega)$ polynomial matrices $S^1(\Delta)$ and $S^2(\Delta)$ with real coefficients will still be the same relation that was introduced by Rosenbrock through strict system equivalence in the time-invariant case [30]. Namely, two $(m\omega + q\omega) \times (m\omega + p\omega)$ polynomial matrices $S^1(\Delta)$ and $S^2(\Delta)$ with real coefficients are said to be *strictly system equivalent* if a relation of the following form holds:

$$S^2(\Delta) = \begin{bmatrix} M(\Delta) & 0 \\ Y(\Delta) & I_{q\omega} \end{bmatrix} S^1(\Delta) \begin{bmatrix} N(\Delta) & X(\Delta) \\ 0 & I_{p\omega} \end{bmatrix}, \tag{17}$$

where $M(\Delta), N(\Delta), X(\Delta)$ and $Y(\Delta)$ are polynomial matrices in Δ with real coefficients, and $M(\Delta), N(\Delta)$ are square and unimodular [20]; in addition, if matrix $S^1(\Delta)$ in (17) is an ω-stacked system matrix at some time k_0, then the matrix $S^2(\Delta)$ that is obtained by (17) for some $M(\Delta)$, $N(\Delta)$, $X(\Delta)$ and $Y(\Delta)$ of the type defined above, will be referred to as an ω-stacked system matrix at the same time, with the same abuse of terminology used in [20].

The meaning of the strict system equivalence relation between two ω-stacked system matrices is similar to the meaning of the strict system equivalence relation between two system matrices corresponding to two time-invariant processes described by a pair of equations of the form (1), (2). In the latter case, in order to obtain a pair of equations of the form (3), (4) from the original model of the form (1), (2), an extension of the dimension of ξ was needed whenever such a dimension was lower than the degree of $\det T(s)$ in (1) [30]; it was obtained by introducing some additional scalar components ξ^j into ξ, characterized by the scalar equations $\xi^j = 0$, which had to be added to the scalar components of equation (1), so as to obtain a new pair of vector equations of the form (1), (2) in such an extended pseudo-state and with the matrices

$$T_1(s) := \begin{bmatrix} I & 0 \\ 0 & T(s) \end{bmatrix}, \quad U_1(s) := \begin{bmatrix} 0 \\ U(s) \end{bmatrix}, \quad V_1(s) := [0 \quad V(s)], \tag{18}$$

instead of matrices $T(s)$, $U(s)$ and $V(s)$, respectively. For similar reasons, the same kind of extension of $\xi(k)$ could be needed in equation (5) [20]; this can be obtained by introducing some new zero components into $\xi(k)$ and by introducing the corresponding trivial scalar equations into vector equation (5). Denoting by $\xi^j(k)$,

$j = m+1, \ldots, m+\nu$, such new zero components of $\xi(k)$, where the nonnegative integer ν denotes their number, such an extension of $\xi(k)$ gives rise to the following operation to be performed on the original model (5), (6):

(a) for $\nu \geq 0$, $\nu \in \mathbf{Z}$, add the following ν scalar equations to equation (5):

$$\xi^j(k) = 0, \tag{19}$$

for $j = m+1, \ldots, m+\nu$, so that, defining

$$\xi^e(k) := [\, \xi^T(k) \quad \xi^{m+1}(k) \quad \ldots \quad \xi^{m+\nu}(k) \,]^T, \tag{20}$$

a new model of the form (5), (6) is obtained, with $\xi^e(k) \in \mathbb{R}^{m+\nu}$ instead of $\xi(k)$.

Notice that for $\nu = 0$ such an operation leaves unchanged the original model (5), (6).

As in [30], in order to find a state-space description (7), (8) of process (5), (6) having the order of this as the dimension n of the state $x(k)$, the converse of operation (a) is actually needed whenever the dimension m of $\xi(k)$ is greater than the order of process (5), (6) [20], since strict system equivalence does not alter the dimension of the pseudostate. The converse of operation (a) can be formally defined as follows:

(b) if, for some $\nu \geq 0$, $\nu \in \mathbf{Z}$, vector $\xi(k)$ can be partitioned as follows:

$$\xi(k) = [\, \xi^\ell(k)^T \quad \xi^{m-\nu+1}(k) \quad \ldots \quad \xi^m(k) \,]^T, \tag{21}$$

where $\xi^j(k)$, $j = m-\nu+1, \ldots, m$, are scalar functions satisfying (19), and vector $\xi^\ell(k)$ satisfies an $(m-\nu)$-dimensional vector equation of the form (5) and a q-dimensional vector equation of the form (6), with $\xi^\ell(k) \in \mathbf{R}^{m-\nu}$ instead of $\xi(k)$, then remove equations (19) from the given model for each $j = m-\nu+1, \ldots, m$.

Operation (b) too leaves unchanged the original model (5), (6) for $\nu = 0$.

The most general relation between two ω-periodic models of the form (5), (6) that was used in [20] can be obtained by putting together strict system equivalence and operations (a) and (b), and is now recalled, since the contribution of this paper is based on enlarging such relation. Then, two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the form (5), (6), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, are said to be *system equivalent at time* k_0 if there exist an operation of the type (a) or (b) to be carried out on $\mathcal{M}_1$ and an operation of the type (a) or (b) to be carried out on $\mathcal{M}_2$ such that the ω-stacked system matrices at time k_0 of the resulting models are strictly system equivalent.

System equivalence at time k_0 is an equivalence relation [20] under which many features and properties of process (5), (6) remain unchanged; among them, the ω-stacked transfer matrix at any time and the order [20]. For the following developments it is useful to recall all of them, together with the definition of those which are non-standard. Then, in view of the results in [20], under Assumption 1, the zeros of the polynomial $\det \mathcal{T}_{k_0}(z)$ are called the *characteristic multipliers of model* (5), (6) *at time* k_0 with corresponding (*ordered sets of*) *structural indices* at the same time defined as their (nondecreasing sequences of) multiplicities as zeros of the invariant polynomials of $\mathcal{T}_{k_0}(z)$. The polynomial $\det \mathcal{T}_{k_0}(z)$ is independent of k_0 [20], and is called the *characteristic polynomial of model* (5), (6). Further, under the same Assumption 1, the *invariant zeros, input decoupling zeros,* and *output decoupling zeros of model* (5), (6) *at time* k_0 are defined to be the zeros of the invariant polynomials of $S^M_{k_0}(z)$, $[-\mathcal{T}_{k_0}(z) \quad \mathcal{U}_{k_0}(z)]$, $[-\mathcal{T}^T_{k_0}(z) \quad \mathcal{V}^T_{k_0}(z)]^T$, respectively, with *ordered sets of structural indices* at the same time defined as their nondecreasing sequences of multiplicities as zeros of such polynomials.

About these notions, it can now be stressed that not only the ω-stacked transfer matrix at any time of a model of the form (5), (6), and its order, but also its whole characteristic polynomial (apart from some nonnull scalar constant), the nonnull structural indices of its nonnull characteristic multipliers at any time, and all types of the nonnull zeros at any time, together with their nonnull structural indices, are invariant under system equivalence at time k_0 [20], as well as its null characteristic multiplier, if any, and its null zeros of all types at time k_0, if any, together with their nonnull structural indices at the same time.

In [20] this was discussed in order to clarify its meaning in terms of the very strong properties that are not altered by system equivalence at time k_0, and therefore the significance of finding an ω-periodic system of the form (7), (8) that is system equivalent at time k_0 to a given model of the form (5), (6) satisfying Assumption 1. In fact, in [20] such a problem was solved under the additional assumption that the given model has no null characteristic multiplier. In the following sections similar results will be obtained without such an additional assumption.

3. LARGE SYSTEM EQUIVALENCE

In order to make easier to find a solution to the problem of obtaining an ω-periodic state-space description of the form (7), (8) of an ω-periodic process modeled by a pair of equations of the form (5), (6), without any assumption about the null characteristic multiplier of the original model, system equivalence at time k_0 will be now suitably enlarged.

Specifically, it will be convenient to allow to record the values of $u(k-\omega+1)$, $u(k-\omega+2), \ldots, u(k-1)$ in a suitable extension $\xi^L(k)$ of the pseudo-state $\xi(k)$, and to allow to correspondingly increase the order of the given model by introducing the new factor $z^{p(\omega-1)}$ into its characteristic polynomial (as will soon be clear). This will be obtained by allowing the use of the following extra operation on the ω-periodic model (5), (6):

(c) add the following $\omega - 1$ vector equations to equation (5):

$$\begin{aligned} \zeta_1(k+1) &= \zeta_2(k), \\ &\vdots \\ \zeta_{\omega-2}(k+1) &= \zeta_{\omega-1}(k), \\ \zeta_{\omega-1}(k+1) &= u(k), \end{aligned} \tag{22}$$

so that, defining

$$\xi^L(k) := [\zeta_1^T(k)\ \zeta_2^T(k)\ \ldots\ \zeta_{\omega-1}^T(k)\ \xi^T(k)]^T, \tag{23}$$

a new model of the form (5), (6) is obtained, with $\xi^L(k) \in \mathbf{R}^{m+(\omega-1)p}$ instead of $\xi(k)$.

In addition, the converse of operation (c) has to be allowed in order that system equivalence at time k_0 remains an equivalence relation after the enlargement thus obtained. The converse of operation (c) can be formally defined as follows:

(d) if vector $\xi(k)$ can be partitioned as follows:

$$\xi(k) := [\zeta_1^T(k)\ \zeta_2^T(k)\ \ldots\ \zeta_{\omega-1}^T(k)\ \xi^{0^T}(k)]^T, \tag{24}$$

where $\zeta_i(k) \in \mathbf{R}^p, i = 1, \ldots, \omega-1$, satisfy (22) and $\xi^0(k)$ satisfies an $[m-(\omega-1)\,p]$-dimensional vector equation of the form (5), and a q-dimensional vector equation of the form (6), with $\xi^0(k) \in \mathbf{R}^{m-(\omega-1)p}$ instead of $\xi(k)$, then remove equations (22) from the given model.

It is clear from (22) that the solutions $y(k)$ of (5), (6) are not altered by the operations of the types (c) and (d).

Notice also that, by (10), the ω-stacked system matrix at time k_0 of the model obtained after that an operation of the type (c) has been carried out on model (5), (6), is strictly system equivalent to the following one:

$$S_{k_0}^{ML}(\Delta) = \left[\begin{array}{cccccc|c} -R_p(\Delta) & I_{\omega p} & \ldots & 0 & 0 & 0 \\ 0 & -R_p(\Delta) & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I_{\omega p} & 0 & 0 \\ 0 & 0 & \ldots & -R_p(\Delta) & 0 & I_{\omega p} \\ 0 & 0 & \ldots & 0 & -\mathcal{T}_{k_0}(\Delta) & \mathcal{U}_{k_0}(\Delta) \\ \hline 0 & 0 & \ldots & 0 & \mathcal{V}_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{array}\right]$$

$$=: \begin{bmatrix} -\mathcal{T}_{k_0}^L(\Delta) & \mathcal{U}_{k_0}^L(\Delta) \\ \mathcal{V}_{k_0}^L(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{bmatrix}, \tag{25}$$

having $\omega - 1$ block rows and columns in addition to $S_{k_0}^M(\Delta)$. Hence, the following relation holds:

$$\det \mathcal{T}_{k_0}^L(z) = \pm z^{p(\omega-1)} \det \mathcal{T}_{k_0}(z), \tag{26}$$

as it was previously mentioned.

Then, two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the form (5), (6), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, are said to be *largely system equivalent at time* k_0 if there exist a (possibly null) finite number of operations of the type (c) or (d) to be carried out on $\mathcal{M}_1$ and a (possibly null) finite number of operations of the type (c) or (d) to be carried out on $\mathcal{M}_2$, such that the resulting models, $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$, respectively, are system equivalent at time k_0.

Proposition 1. The relation of large system equivalence at time k_0 between two ω-periodic models of the form (5), (6) is an equivalence relation.

Proof. The reflexivity and symmetry properties are obvious. As regards transitivity, given three ω-periodic models $\mathcal{M}_i, i = 1,2,3$, having inputs and outputs of the same dimensions p and q, respectively, assume that the pairs $\mathcal{M}_1$ and $\mathcal{M}_2$, and, respectively, $\mathcal{M}_2$ and $\mathcal{M}_3$, are largely system equivalent at time k_0. That is, there exist pairs of integers $\bar{j}_1, \bar{j}_2$, and $\tilde{j}_2, \tilde{j}_3$, such that the pairs of models $\overline{\mathcal{M}}_1, \overline{\mathcal{M}}_2$ and $\widetilde{\mathcal{M}}_2, \widetilde{\mathcal{M}}_3$, respectively, are system equivalent at time k_0, where $\overline{\mathcal{M}}_i$ is obtained from $\mathcal{M}_i$ after that $\bar{j}_i$ operations of the type (c) ($-\bar{j}_i$ operations of the type (d)) have been carried out on $\mathcal{M}_i$, if $\bar{j}_i \geq 0$ (if $\bar{j}_i < 0$), $i = 1,2$, and $\widetilde{\mathcal{M}}_i$ is obtained from $\mathcal{M}_i$ after that $\tilde{j}_i$ operations of the type (c) [$-\tilde{j}_i$ operations of the type (d)] have been carried out on $\mathcal{M}_i$, if $\tilde{j}_i \geq 0$ (if $\tilde{j}_i < 0$), $i = 2,3$. Without loss of generality (apart from a renumbering of the three models), assume that $\bar{j}_2 \geq \tilde{j}_2$, and define $\bar{j}_2 - \tilde{j}_2 =: j_2 \geq 0$. It can be easily checked that the system equivalence at time k_0 of $\widetilde{\mathcal{M}}_2$ and $\widetilde{\mathcal{M}}_3$ implies the system equivalence at time k_0 of $\widetilde{\widetilde{\mathcal{M}}}_2$ and $\widetilde{\widetilde{\mathcal{M}}}_3$, where $\widetilde{\widetilde{\mathcal{M}}}_i$ is the model obtained from $\widetilde{\mathcal{M}}_i$ after that j_2 operations of the type (c) have been carried out on $\widetilde{\mathcal{M}}_i, i = 2,3$. Since $\overline{\mathcal{M}}_2 = \widetilde{\widetilde{\mathcal{M}}}_2$, models $\overline{\mathcal{M}}_1$ and $\widetilde{\widetilde{\mathcal{M}}}_3$ are system equivalent at time k_0, by virtue of the transitivity of system equivalence at time k_0 [20]. This proves that $\mathcal{M}_1$ and $\mathcal{M}_3$ are largely system equivalent at time k_0. □

The following proposition and remark stress that most of the features and properties of a given model that are invariant under system equivalence at time k_0, are still invariant under large system equivalence at time k_0.

Proposition 2. Given two ω-periodic models $\mathcal{M}_1$ and $\mathcal{M}_2$ of the form (5), (6), satisfying Assumption 1 and having inputs and outputs of the same dimensions p and q, respectively, pseudo-states of dimensions $m_i, i = 1,2$, and ω-stacked system matrices at time k_0 $S^M_{k_0,i}(\Delta), i = 1,2$, if they are largely system equivalent at time k_0, then:

(α) the ω-stacked transfer matrices of $\mathcal{M}_1$ and $\mathcal{M}_2$ at any time coincide;

(β) $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same nonnull characteristic multipliers, nonnull input decoupling zeros, and nonnull output decoupling zeros at all times and the same corresponding ordered sets of structural indices (apart from $\omega|m_1 - m_2|$ null structural indices);

(γ) $S^M_{k_0,1}(\Delta)$ has full row-rank if and only if $S^M_{k_0,2}(\Delta)$ has full row-rank;

(δ) if $S^M_{k_0,i}(\Delta)$, $i = 1,2$, have full row-rank, then $S^M_{k,i}(\Delta)$, $i = 1,2$, has full row-rank for all $k \in \mathbf{Z}$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same nonnull invariant zeros at all times and the same corresponding ordered sets of structural indices (apart from $\omega|m_1 - m_2|$ null structural indices).

Proof. By relation (25) and by the properties of strict system equivalence, the ω-stacked transfer matrix at time k_0 of the model obtained after that an operation of the type (c) has been carried out on model (5), (6), coincides with that of the same model (5), (6). This, together with Proposition 3.3 and relation (22) of [20], proves (α).

Item (γ) trivially follows from (9) and (25), together with Proposition 3.3 of [20]. The former assertion in the item (δ) follows from Proposition 2.4 of [20].

As regards nonnull input decoupling zeros and their ordered sets of structural indices in item (β), using the notations in (25), notice that there exist unimodular matrices $M(z)$ and $R(z)$ such that

$$M(z)[-\mathcal{T}_{k_0}^L(z) \quad \mathcal{U}_{k_0}^L(z)]R(z) = \begin{bmatrix} -R_p(z) & I_{\omega p} & 0 & \dots & 0 & 0 & 0 \\ 0 & -R_p(z) & I_{\omega p} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -R_p(z) & I_{\omega p} & 0 \\ 0 & 0 & 0 & \dots & 0 & -R_p(z) & U(z) \\ 0 & 0 & 0 & \dots & 0 & 0 & S(z) \end{bmatrix}, \tag{27}$$

where $S(z)$ is the Smith form of $[-\mathcal{T}_{k_0}(z) \quad \mathcal{U}_{k_0}(z)]$ and $U(z)$ is some polynomial matrix. Then, the meaning of the invariant polynomials of the matrix in the right-hand side of (27) in terms of the g.c.d. of its minors, and Propositions 2.4 and 3.3 of [20], prove the assertion in item (β) concerning nonnull input decoupling zeros and their ordered sets of structural indices. The other assertions in item (β), as well as the latter assertion in item (δ), can be proved in a similar way. □

Remark 1. By Proposition 2, if a system $\mathcal{M}_2$ in the state-space form (7), (8) is largely system equivalent at some time k_0 to a given model $\mathcal{M}_1$ of the form (5), (6), then the ω-stacked transfer matrix of $\mathcal{M}_2$ at any initial time and all the features of $\mathcal{M}_2$ that are listed in items (β), (γ) and (δ) of Proposition 2 are specified by the original model $\mathcal{M}_1$. Their meaning and significance with reference to the structural properties and to the free motions of system $\mathcal{M}_2$ is analyzed in detail in [18] (where it is shown that, in particular, the characteristic multipliers of system $\mathcal{M}_2$ coincide with the eigenvalues of its monodromy matrix). This connection allows to deduce from Proposition 2 that, for example, such a system $\mathcal{M}_2$ is controllable (resp., reconstructible) if and only if $\mathcal{M}_1$ has no nonnull input (resp., output) decoupling zeros, it is stabilizable (resp., detectable), if and only if $\mathcal{M}_1$ has no input (resp., output) decoupling zeros outside the open disk of unit radius [18]; moreover, not only the ω-stacked transfer matrix at any time $\overline{k}_0$ and all the nonnull characteristic multipliers of system $\mathcal{M}_2$ – and therefore the asymptotic stability [10], and the rate of convergence of the free motions –, but even the number and the dimensions of the Jordan blocks corresponding to each nonnull characteristic multiplier, in the Jordan form of the monodromy matrix of system $\mathcal{M}_2$, at any time $\overline{k}_0$, are determined by the properties of the original model $\mathcal{M}_1$ (in fact, in [20] a short analysis was developed about the role of the characteristic multipliers of model $\mathcal{M}_1$ for its pseudo-state free motions). In addition, the relevance of the property that $S^M_{\overline{k}_0,2}(\overline{z})$ has full row-rank for any $\overline{k}_0 \in \mathbf{Z}$ and for any nonnull $\overline{z} \in \mathbf{C}$ if and only if $S^M_{\overline{k}_0,1}(\overline{z})$ has full row-rank (which is implied by items (γ) and (δ) of Proposition 2) is clarified by recalling that such a condition on the ω-stacked system matrix $S^M_{\overline{k}_0,2}(z)$ of the ω-periodic system $\mathcal{M}_2$, is necessary and sufficient, together with stabilizability and detectability, for the existence of a solution of the robust tracking and regulation problem for system $\mathcal{M}_2$ when the ω-stacked forms of reference signals and disturbance functions have a time dependence characterized by $\bar{z}^h$, $|\bar{z}| \geq 1$ [17].

The above analysis, if it is compared with a similar one concerning with system equivalence between a system of the form (7), (8) and a model $\mathcal{M}_1$ of the form (5), (6) (i. e., Remark 3.4 in [20]), shows that most of the features and properties of system (7), (8) that are determined by the features and properties of the given model $\mathcal{M}_1$, which is system equivalent at time k_0 to the system, are preserved in the system $\mathcal{M}_2$ of the same form (7), (8) that is obtained through large system equivalence at time k_0 from the same given model $\mathcal{M}_1$. However, it is worth to stress that the order and the characteristic polynomial of the system $\mathcal{M}_2$ can be different from those of the original model $\mathcal{M}_1$ (since factors $z^{p(\omega-1)}$ can be introduced or removed – see (26)), and that, in addition, the reachability and the observability at time k_0 of the system $\mathcal{M}_2$, the Smith form of its stacked system matrix at time k_0, and its nonnull invariant zeros at all times, are not specified, in general, by the original model $\mathcal{M}_1$, whereas in system equivalence they are.

On the other hand, it is emphasized that, by (22) and Remark 3.2 of [20], the output solutions of $\mathcal{M}_1$ and $\mathcal{M}_2$ over the interval $[k_0, \infty)$, for each given input function $u(\cdot)_{[k_0,+\infty)}$, are exactly the same, and their pseudo-state solutions over the same interval are biuniquely related apart from transient terms, which become equal to zero from time $\bar{k} = k_0 + \omega - 1$. □

In view of the previous discussion, it seems reasonable to look for an ω-periodic system of the form (7), (8) that is largely system equivalent at time k_0 to a given ω-periodic model of the form (5), (6).

4. MAIN RESULT

The conditions for the existence of an ω-periodic system (7), (8) that is largely system equivalent at time k_0 to the given ω-periodic model (5), (6), under Assumption 1, are expressed by the following theorem.

Theorem 1. For the ω-periodic model (5), (6), under Assumption 1, there exists an ω-periodic system of the form (7), (8) that is largely system equivalent at time k_0 to the model (5), (6), if and only if its ω-stacked transfer matrix $G_{k_0}^M(z)$ satisfies the following conditions:

(i) $G_{k_0}^M(z)$ is a proper rational matrix;

(ii) if $G_{k_0}^M(z)$ is rewritten as $G_{k_0}^M(z) = F_{k_0}(z) + Q_{k_0}$, with $F_{k_0}(z)$ strictly proper and Q_{k_0} constant, and Q_{k_0} is decomposed into blocks of dimensions $q \times p$, then Q_{k_0} is lower block triangular.

If conditions (i) and (ii) hold for $k_0 = \bar{k}_0, \bar{k}_0 \in \mathbf{Z}$, then (i) and (ii) hold for all $k_0 \in \mathbf{Z}$.

Proof. The last statement is contained in Proposition 2.5 of [20].

Since system (7), (8) satisfies conditions (i) and (ii) restated in terms of the same system (see Theorem 3.1 and relations (2.2) and (2.3) in [18]), then the necessity follows from Proposition 2.

As regards the sufficiency, denote by $\mathcal{M}$ the given ω-periodic model (5), (6), and by $\tilde{n}$ its order, i.e. the degree in Δ of $\det \mathcal{T}_{k_0}(\Delta)$. Denote by $\widetilde{\mathcal{M}}$ the ω-periodic model that is system equivalent to $\mathcal{M}$ at time k_0, and is obtained from $\mathcal{M}$ by an operation of the type (a) with $\nu := \tilde{n} - m$, if $m \leq \tilde{n}$, and with $\nu := 0$, if $m > \tilde{n}$; denote by $\tilde{m} := m + \nu \geq \tilde{n}$ the dimension of the pseudo-state of $\widetilde{\mathcal{M}}$, by $\tilde{S}^M_{k_0}(\Delta)$ its ω-stacked system matrix at time k_0, and by $\widetilde{\mathcal{T}}_{k_0}(\Delta)$, $\widetilde{\mathcal{U}}_{k_0}(\Delta)$, $\widetilde{\mathcal{V}}_{k_0}(\Delta)$ and $\widetilde{\mathcal{W}}_{k_0}(\Delta)$ the four blocks constituting $\tilde{S}^M_{k_0}(\Delta)$.

Call $\overline{\mathcal{M}}$ the ω-periodic model that is largely system equivalent to $\widetilde{\mathcal{M}}$ at time k_0, and is obtained from $\widetilde{\mathcal{M}}$ by an operation of the type (c). The ω-stacked system matrix of $\overline{\mathcal{M}}$ at time k_0 is strictly system equivalent to the matrix $S^{ML}_{k_0}(\Delta)$ expressed by (25) rewritten with $\widetilde{\mathcal{T}}_{k_0}(\Delta)$, $\widetilde{\mathcal{U}}_{k_0}(\Delta)$, $\widetilde{\mathcal{V}}_{k_0}(\Delta)$ and $\widetilde{\mathcal{W}}_{k_0}(\Delta)$ instead of $\mathcal{T}_{k_0}(\Delta)$, $\mathcal{U}_{k_0}(\Delta)$, $\mathcal{V}_{k_0}(\Delta)$ and $\mathcal{W}_{k_0}(\Delta)$, respectively.

Since, by Proposition 3.3 of [20], the hypotheses on $\mathcal{M}$ still holds on $\widetilde{\mathcal{M}}$, and $\tilde{m}\omega \geq \tilde{n}$, there exist [30] unimodular matrices $M(\Delta)$ and $N(\Delta)$, and polynomial matrices $Y(\Delta)$ and $X(\Delta)$ such that:

$$\hat{S}^{ML}_{k_0}(\Delta) := \left[\begin{array}{cc|c} I_{\omega p(\omega-1)} & 0 & 0 \\ 0 & M(\Delta) & 0 \\ \hline 0 & Y(\Delta) & I_{q\omega} \end{array}\right] S^{ML}_{k_0}(\Delta) \left[\begin{array}{cc|c} I_{\omega p(\omega-1)} & 0 & 0 \\ 0 & N(\Delta) & X(\Delta) \\ \hline 0 & 0 & I_{p\omega} \end{array}\right]$$

$$= \left[\begin{array}{cccccc|c} -R_p(\Delta) & I_{\omega p} & \dots & 0 & 0 & 0 & 0 \\ 0 & -R_p(\Delta) & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I_{\omega p} & 0 & 0 & 0 \\ 0 & 0 & \dots & -R_p(\Delta) & 0 & 0 & I_{\omega p} \\ 0 & 0 & \dots & 0 & -I_{\tilde{m}\omega-\tilde{n}} & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & E^M_{k_0} - \Delta I_{\tilde{n}} & J^M_{k_0} \\ \hline 0 & 0 & \dots & 0 & 0 & L^M_{k_0} & P^M_{k_0} \end{array}\right] \tag{28}$$

where $E^M_{k_0}$, $J^M_{k_0}$, $L^M_{k_0}$ and $P^M_{k_0}$ are constant and, if $P^M_{k_0}$ is decomposed into blocks of dimensions $q \times p$, then $P^M_{k_0}$ is lower block triangular by Proposition 3.3 in [20] and condition *(ii)*. Relation (28) implies that $\hat{S}^{ML}_{k_0}(\Delta)$ is strictly system equivalent to $S^{ML}_{k_0}(\Delta)$. Let $J^M_{k_0}, L^M_{k_0}$, and $P^M_{k_0}$ be partitioned as follows:

$$J^M_{k_0} = [\, J^M_{k_0,0} \;\; J^M_{k_0,1} \;\; \cdots \;\; J^M_{k_0,\omega-1} \,], \tag{29}$$

$$P^M_{k_0} = \begin{bmatrix} P^M_{k_0,0,0} & 0 & 0 & \dots & 0 \\ P^M_{k_0,1,0} & P^M_{k_0,1,1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ P^M_{k_0,\omega-1,0} & P^M_{k_0,\omega-1,1} & P^M_{k_0,\omega-1,2} & \cdots & P^M_{k_0,\omega-1,\omega-1} \end{bmatrix}, \tag{30}$$

$$L_{k_0}^M = \begin{bmatrix} L_{k_0,0}^M \\ L_{k_0,1}^M \\ \cdots \\ L_{k_0,\omega-1}^M \end{bmatrix}, \tag{31}$$

where all the blocks in (29), (30) (and, respectively, (30), (31)) have p columns (and, respectively, q rows). Then, it is possible to check that the following strict system equivalence relation holds:

$$\begin{bmatrix} \hat{M} & 0 \\ \hat{Y} & I_{q\omega} \end{bmatrix} \hat{S}_{k_0}^{ML}(\Delta) \begin{bmatrix} \hat{N} & \hat{X} \\ 0 & I_{p\omega} \end{bmatrix} = \hat{\hat{S}}_{k_0}^{ML}(\Delta), \tag{32}$$

where $\hat{M}, \hat{N}, \hat{X}$ and $\hat{Y}$ are constant (with $\hat{M}$ and $\hat{N}$ being nonsingular) and expressed by

$$\hat{M} = \begin{bmatrix} 0 & -I_{(\tilde{m}-\tilde{n})\omega} & 0 \\ I_{\omega p(\omega-1)} & 0 & 0 \\ \hat{M}_{31} & 0 & \hat{M}_{33} \end{bmatrix}, \tag{33}$$

$$\hat{Y} = [\hat{Y}_1 \;\; 0 \;\; \hat{Y}_3], \tag{34}$$

$$\hat{N} = \begin{bmatrix} 0 & I_{\omega p(\omega-1)} & 0 \\ -I_{(\tilde{m}-\tilde{n})\omega} & 0 & 0 \\ 0 & 0 & \hat{N}_{33} \end{bmatrix}, \tag{35}$$

$$\hat{X} = 0, \tag{36}$$

and

$$\hat{\hat{S}}_{k_0}^{ML}(\Delta) =$$

$$\left[\begin{array}{ccccccc|c} -I_{(\tilde{m}-\tilde{n})\omega} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -R_p(\Delta) & I_{\omega p} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_p(\Delta) & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -R_p(\Delta) & I_{\omega p} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -R_p(\Delta) & 0 & I_{\omega p} \\ 0 & -\mathcal{T}_{\omega-1,0} & -\mathcal{T}_{\omega-1,1} & \cdots & -\mathcal{T}_{\omega-1,\omega-3} & -\mathcal{T}_{\omega-1,\omega-2} & -\mathcal{T}_{\omega-1,\omega-1}(\Delta) & \mathcal{U}_{\omega-1} \\ \hline 0 & \mathcal{V}_0 & \mathcal{V}_1 & \cdots & \mathcal{V}_{\omega-3} & \mathcal{V}_{\omega-2} & \mathcal{V}_{\omega-1} & \mathcal{W} \end{array}\right], \tag{37}$$

with

$$\hat{M}_{31} = [\hat{M}_{310} \;\; \hat{M}_{311} \;\; \ldots \;\; \hat{M}_{31,\omega-2}] \tag{38}$$

$$\hat{M}_{31i} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -J_{k_0,-\omega+i+2}^M & -J_{k_0,-\omega+i+3}^M & \cdots & -J_{k_0,i}^M & 0 \end{bmatrix}, \; i = 0, \ldots, \omega-2, \tag{39}$$

$$J_{k_0,j}^M := 0, \; j = -\omega+2, -\omega+3, \ldots, -1, \; (\text{if } \omega > 2), \tag{40}$$

$$\hat{M}_{33} = \begin{bmatrix} -I_{\tilde{n}(\omega-1)} & 0 \\ [E_{k_0}^M \;\; E_{k_0}^M \;\ldots\; E_{k_0}^M] & I_{\tilde{n}} \end{bmatrix}, \tag{41}$$

$$\hat{Y}_1 = [\hat{Y}_{10} \quad \hat{Y}_{11} \quad \ldots \quad \hat{Y}_{1,\omega-2} \quad], \tag{42}$$

$$\hat{Y}_{1i} = \begin{bmatrix} 0 & \ldots & 0 & 0 & 0 \\ 0 & \ldots & 0 & \overline{Y}_{1,i} & 0 \end{bmatrix}, \; i = 0, \ldots, \omega - 3, \tag{43}$$

$$\hat{Y}_{1,\omega-2} = \begin{bmatrix} 0 & 0 \\ \overline{Y}_{1,\omega-2} & 0 \end{bmatrix}, \tag{44}$$

$$\overline{Y}_{1,i} = \begin{bmatrix} -P^M_{k_0,\omega-1-i,0} & 0 & \ldots & 0 & 0 \\ -P^M_{k_0,\omega-i,0} & -P^M_{k_0,\omega-i,1} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -P^M_{k_0,\omega-2,0} & -P^M_{k_0,\omega-2,1} & \ldots & -P^M_{k_0,\omega-2,i-1} & 0 \\ -P^M_{k_0,\omega-1,0} & -P^M_{k_0,\omega-1,1} & \ldots & -P^M_{k_0,\omega-1,i-1} & -P^M_{k_0,\omega-1,i} \end{bmatrix}, \; i = 0, \ldots, \omega - 2, \tag{45}$$

$$\hat{Y}_3 = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ L^M_{k_0,1} & 0 & \ldots & 0 & 0 \\ L^M_{k_0,2} & L^M_{k_0,2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ L^M_{k_0,\omega-2} & L^M_{k_0,\omega-2} & \ldots & 0 & 0 \\ L^M_{k_0,\omega-1} & L^M_{k_0,\omega-1} & \ldots & L^M_{k_0,\omega-1} & 0 \end{bmatrix}, \tag{46}$$

$$\hat{N}_{33} = \begin{bmatrix} I_{\tilde{n}} & -I_{\tilde{n}} & 0 & \ldots & 0 & 0 \\ 0 & I_{\tilde{n}} & -I_{\tilde{n}} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & I_{\tilde{n}} & -I_{\tilde{n}} \\ I_{\tilde{n}} & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{47}$$

$$\mathcal{T}_{\omega-1,i} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & -J^M_{k_0,i} \end{bmatrix}, \; i = 0, \ldots, \omega - 2, \tag{48}$$

$$\mathcal{T}_{\omega-1,\omega-1}(\Delta) = R_{\tilde{n}}(\Delta) - \operatorname{diag}\left\{I_{\tilde{n}}, \ldots, I_{\tilde{n}}, E^M_{k_0}\right\}, \tag{49}$$

$$\mathcal{U}_{\omega-1} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & J^M_{k_0,\omega-1} \end{bmatrix}, \tag{50}$$

$$\mathcal{V}_i = \begin{bmatrix} 0 & 0 \\ 0 & \operatorname{diag}\left\{P^M_{k_0,\omega-1-i,0}, P^M_{k_0,\omega-i,1}, \ldots, P^M_{k_0,\omega-1,i}\right\} \end{bmatrix}, \; i = 0, \ldots, \omega - 2, \tag{51}$$

$$\mathcal{V}_{\omega-1} = \operatorname{diag}\{L^M_{k_0,0}, L^M_{k_0,1}, \ldots, L^M_{k_0,\omega-1}\}, \tag{52}$$

$$\mathcal{W} = \operatorname{diag}\{P^M_{k_0,0,0}, P^M_{k_0,1,1}, \ldots, P^M_{k_0,\omega-1,\omega-1}\}; \tag{53}$$

in (39) there are ω block columns, each of which with p scalar columns; each block column $\hat{Y}_{1i}$ in (42) has ωp scalar columns; the null block columns in (43) and (44) have p scalar columns; the null block column in (46) has $\tilde{n}$ scalar columns. Now, it is easy to see that, by a further strict system equivalence on $\hat{\hat{S}}^{ML}_{k_0}(\Delta)$, consisting of suitably interchanging its first $(\omega-1)p\omega + \tilde{m}\omega$ rows and columns, the following system matrix is obtained:

$$\tilde{\hat{S}}^{ML}_{k_0}(\Delta) =$$

$$\left[\begin{array}{cc|c} -I_{(\tilde{m}-\tilde{n})\omega} & 0 & 0 \\ 0 & \operatorname{diag}\{A_0, A_1, \ldots, A_{\omega-1}\} - R_{(\omega-1)p+\tilde{n}}(\Delta) & \operatorname{diag}\{B_0, B_1, \ldots, B_{\omega-1}\} \\ \hline 0 & \operatorname{diag}\{C_0, C_1, \ldots, C_{\omega-1}\} & \operatorname{diag}\{D_0, D_1, \ldots, D_{\omega-1}\} \end{array}\right], \tag{54}$$

where

$$A_i = \begin{bmatrix} 0 & I_{(\omega-2)p} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\tilde{n}} \end{bmatrix}, \; i = 0, \ldots, \omega-2, \tag{55}$$

$$A_{\omega-1} = \begin{bmatrix} 0 & I_p & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_p & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & I_p & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ J^M_{k_0,0} & J^M_{k_0,1} & J^M_{k_0,2} & \ldots & J^M_{k_0,\omega-2} & E^M_{k_0} \end{bmatrix}, \tag{56}$$

$$B_i = \begin{bmatrix} 0 \\ I_p \\ 0 \end{bmatrix}, \; i = 0, \ldots, \omega-2, \tag{57}$$

$$B_{\omega-1} = \begin{bmatrix} 0 \\ I_p \\ J^M_{k_0,\omega-1} \end{bmatrix}, \tag{58}$$

$$C_0 = [0 \;\ldots\; 0 \;\; L^M_{k_0,0}], \tag{59}$$

$$C_i = [0 \;\ldots\; 0 \;\; P^M_{k_0,i,0} \;\ldots\; P^M_{k_0,i,i-1} \;\; L^M_{k_0,i}], \; i = 1, \ldots, \omega-2, \tag{60}$$

$$C_{\omega-1} = [P^M_{k_0,\omega-1,0} \;\ldots\; P^M_{k_0,\omega-1,\omega-2} \;\; L^M_{k_0,\omega-1}], \tag{61}$$

$$D_i = P^M_{k_0,i,i}, \; i = 0, \ldots, \omega-1; \tag{62}$$

in (55)–(61) the square matrices $A_i, i = 0, \ldots, \omega-1$, have the same dimensions, as well as the matrices $B_i, i = 0, \ldots, \omega - 1$, and the matrices $C_i, i = 0, \ldots, \omega - 1$; the numbers of scalar rows of the row blocks of B_i in (57) are the same as those of the corresponding row blocks of A_i in (55). Lastly, by a further strict system equivalence on $\hat{\tilde{S}}_{k_0}^{ML}(\Delta)$ — merely consisting of proper interchanges of the first $(\omega - 1)\,p\omega + \tilde{m}\omega$ rows and columns of $\hat{\tilde{S}}_{k_0}^{ML}(\Delta)$ —, the ω-stacked system matrix at time k_0 is obtained of an ω-periodic model of the form (5), (6) having $\tilde{m} - \tilde{n}$ identically null components of the pseudo-state; since the hypotheses required by operation (b) are satisfied, an operation of the type (b) with $\nu := \tilde{m} - \tilde{n}$ can be performed, so that the ω-stacked system matrix at time k_0 is obtained of an ω-periodic system of the type (7), (8), with $n = \tilde{n} + (\omega - 1)\,p$ and

$$
\begin{aligned}
A(k_0 + i + h\omega) &= A_i,\ i = 0, \ldots, \omega - 1, \forall h \in \mathbf{Z}, && (63)\\
B(k_0 + i + h\omega) &= B_i,\ i = 0, \ldots, \omega - 1, \forall h \in \mathbf{Z}, && (64)\\
C(k_0 + i + h\omega) &= C_i,\ i = 0, \ldots, \omega - 1, \forall h \in \mathbf{Z}, && (65)\\
D(k_0 + i + h\omega) &= D_i,\ i = 0, \ldots, \omega - 1, \forall h \in \mathbf{Z}. && (66)
\end{aligned}
$$

Thus, such a system is largely system equivalent to $\mathcal{M}$. □

Now, the constructive procedure that is contained in the sufficiency proof of Theorem 1 will be given in full details for the simplest case $\omega = 2$ and $\tilde{n} = m$, where $\tilde{n}$ denotes the degree of $\det \mathcal{T}_{k_0}(\Delta)$.

Then, consider a 2-periodic model $\mathcal{M}$ of the form (5), (6) satisfying Assumption 1 and conditions (i) and (ii) of Theorem 1, with the degree $\tilde{n}$ of $\det \mathcal{T}_{k_0}(\Delta)$ equal to m. In this case no preliminary operation of the type (a) is needed. After an operation of the type (c), the ω-stacked system matrix at time k_0 of the model $\overline{\mathcal{M}}$ thus obtained is strictly system equivalent to the matrix $S_{k_0}^{ML}(\Delta)$ expressed by

$$
S_{k_0}^{ML}(\Delta) = \left[\begin{array}{cc|c} -R_p(\Delta) & 0 & I_{2p} \\ 0 & -\mathcal{T}_{k_0}(\Delta) & \mathcal{U}_{k_0}(\Delta) \\ \hline 0 & \mathcal{V}_{k_0}(\Delta) & \mathcal{W}_{k_0}(\Delta) \end{array}\right].
$$

Then, by standard strict system equivalence, polynomial matrices $M(\Delta)$, $N(\Delta)$, $Y(\Delta)$ and $X(\Delta)$ are found [30], with $M(\Delta)$ and $N(\Delta)$ being unimodular, such that (28) holds, where $E_{k_0}^M$, $J_{k_0}^M$, $L_{k_0}^M$ and $P_{k_0}^M$ are constant, and matrix $\hat{S}_{k_0}^{ML}(\Delta)$ reduces to

$$
\hat{S}_{k_0}^{ML}(\Delta) = \left[\begin{array}{cccc|cc} 0 & -I_p & 0 & 0 & I_p & 0 \\ -\Delta I_p & 0 & 0 & 0 & 0 & I_p \\ 0 & 0 & -I_{\tilde{n}} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{k_0}^M - \Delta I_{\tilde{n}} & J_{k_0,0}^M & J_{k_0,1}^M \\ \hline 0 & 0 & 0 & L_{k_0,0}^M & P_{k_0,0,0}^M & 0 \\ 0 & 0 & 0 & L_{k_0,1}^M & P_{k_0,1,0}^M & P_{k_0,1,1}^M \end{array}\right], \quad (67)
$$

where the partitions (29), (30) and (31) of $J_{k_0}^M$, $P_{k_0}^M$ and $L_{k_0}^M$ have been used, with the last two block rows having q scalar rows.

For $\hat{S}_{k_0}^{ML}(\Delta)$ the following strict system equivalence relation holds:

$$\left[\begin{array}{cccc|cc} I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{\tilde{n}} & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & 0 & 0 \\ -J_{k_0,0}^M & 0 & E_{k_0}^M & I_{\tilde{n}} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_q & 0 \\ -P_{k_0,1,0}^M & 0 & L_{k_0,1}^M & 0 & 0 & I_q \end{array}\right] \cdot \hat{S}_{k_0}^{ML}(\Delta)$$

$$\cdot \left[\begin{array}{cccc|cc} I_p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 & 0 \\ 0 & I_{\tilde{n}} & 0 & -I_{\tilde{n}} & 0 & 0 \\ 0 & I_{\tilde{n}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & 0 & I_p \end{array}\right]$$

$$= \left[\begin{array}{cccc|cc} 0 & 0 & -I_p & 0 & I_p & 0 \\ 0 & I_{\tilde{n}} & 0 & -I_{\tilde{n}} & 0 & 0 \\ -\Delta I_p & 0 & 0 & 0 & 0 & I_p \\ 0 & -\Delta I_{\tilde{n}} & J_{k_0,0}^M & E_{k_0}^M & 0 & J_{k_0,1}^M \\ \hline 0 & L_{k_0,0}^M & 0 & 0 & P_{k_0,0,0}^M & 0 \\ 0 & 0 & P_{k_0,1,0}^M & L_{k_0,1}^M & 0 & P_{k_0,1,1}^M \end{array}\right],$$

$$:= \tilde{\hat{S}}_{k_0}^{ML}(\Delta). \tag{68}$$

Matrix $\tilde{\hat{S}}_{k_0}^{ML}(\Delta)$ is the ω-stacked system matrix at time k_0 of a 2-periodic system of the form (7), (8), with:

$$A(k_0 + h\omega) = \begin{bmatrix} 0 & 0 \\ 0 & I_{\tilde{n}} \end{bmatrix}, \forall h \in \mathbf{Z}, \tag{69}$$

$$A(k_0 + h\omega + 1) = \begin{bmatrix} 0 & 0 \\ J_{k_0,0}^M & E_{k_0}^M \end{bmatrix}, \forall h \in \mathbf{Z}, \tag{70}$$

$$B(k_0 + h\omega) = \begin{bmatrix} I_p \\ 0 \end{bmatrix}, \forall h \in \mathbf{Z}, \tag{71}$$

$$B(k_0 + h\omega + 1) = \begin{bmatrix} I_p \\ J_{k_0,1}^M \end{bmatrix}, \forall h \in \mathbf{Z}, \tag{72}$$

$$C(k_0 + h\omega) = [0 \quad L_{k_0,0}^M], \forall h \in \mathbf{Z}, \tag{73}$$

$$C(k_0 + h\omega + 1) = [P_{k_0,1,0}^M \quad L_{k_0,1}^M], \forall h \in \mathbf{Z}, \tag{74}$$

$$D(k_0 + h\omega + i) = P_{k_0,i,i}^M, \; i = 0, 1, \forall h \in \mathbf{Z}. \tag{75}$$

5. CONCLUSIONS

In this paper a description in state-space form of a discrete-time linear periodic process has been obtained within the class of models which are largely system equivalent at some time k_0 to the given one.

It has been shown that the ω-stacked transfer matrix at any initial time and the nonnull characteristic multipliers of the periodic system thus obtained coincide with those of the original periodic model (although their orders do not coincide), and the asymptotic stability, the controllability, the reconstructibility, the stabilizability, the detectability, and even the number and the dimensions of the Jordan blocks, in the Jordan form of the monodromy matrix of such a system, corresponding to each nonnull characteristic multiplier, are determined by the original periodic model, as well as the existence of a solution of the robust tracking and regulation problem.

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