Friedrich Liese; Udo Lorz

Contiguity and LAN-property of sequences of Poisson processes

Kybernetika, Vol. 35 (1999), No. 3, [281]--308

Persistent URL: http://dml.cz/dmlcz/135289

KYBERNETIKA — VOLUME 35 (1999), NUMBER 3, PAGES 281–308

CONTIGUITY AND LAN-PROPERTY OF SEQUENCES OF POISSON PROCESSES

FRIEDRICH LIESE AND UDO LORZ

Using the concept of Hellinger integrals, necessary and sufficient conditions are established for the contiguity of two sequences of distributions of Poisson point processes with an arbitrary state space.

The distribution of logarithm of the likelihood ratio is shown to be infinitely divisible. The canonical measure is expressed in terms of the intensity measures. Necessary and sufficient conditions for the LAN-property are formulated in terms of the corresponding intensity measures.

1. INTRODUCTION

The concept of local asymptotic normality (LAN) of families of distributions $Q_{n,h}$, $h \in H_n \subseteq \mathbb{R}_k$ goes back to LeCam [7] and proved to be fruitful in asymptotic statistics. The LAN-condition means that $\ln \frac{dQ_{n,h}}{dQ_{n,0}}$ admits an approximate linearization $\langle Z_n, h\rangle - \frac{1}{2}\|h\|^2$ so that the central sequence Z_n is asymptotically sufficient and asymptotic inference may be based on Z_n. In this paper we study distributions $P_{\Lambda_{n,\vartheta}}$, $\vartheta \in \Theta \subseteq \mathbb{R}_k$ of Poisson processes Φ_n with intensity measures $\Lambda_{n,\vartheta}$. We introduce a local parameter h by setting $\mu_{n,h} = \Lambda_{n,\vartheta_0+A_n h}$ where $A_n \to 0$ is a sequence of $k \times k$ matrices and denote by $Q_{n,h}$ the distribution of a Poisson process with intensity measure $\mu_{n,h}$.

A first step for proving the LAN-condition is to study the problem under which conditions P_{Λ_1} is absolutely continuous w.r.t. P_{Λ_2} ($P_{\Lambda_1} \ll P_{\Lambda_2}$). This problem was solved in Brown [1] and in Liese [8].

The concept of contiguity is a natural generalization of the absolute continuity to sequences and, in view of first Lemma of LeCam, is automatically fulfilled if the LAN-condition holds. In this paper we give necessary and sufficient conditions for the contiguity of sequences of distributions of Poisson processes and ask for further conditions which imply the LAN-property. For this purpose we use the fact that the distribution of $\ln \frac{dP_{\Lambda_1}}{dP_{\Lambda_2}}$ is infinitely divisible and calculate the canonical measure. The application of convergence criteria for infinitely divisible distributions leads to limit theorems for $\left(\ln \frac{dP_{\Lambda_{1,n}}}{P_{\Lambda_{2,n}}} - a_n\right) b_n^{-1}$. The representation of $\ln \frac{dP_{\Lambda_1}}{dP_{\Lambda_2}}$ in Karr [4] works only for finite Λ_i. Therefore in this paper we assume contiguity and approximate $Q_{n,h}$ by an accompanying sequence $Q^N_{n,h}$ so that $\ln \frac{dQ^N_{n,h}}{dQ_{n,0}}$ has a representation as an L_2-integral w. r. t. $\Phi_n - \mu_{n,0}$ and it holds $\left\|Q^{N(n)}_{n,h} - Q_{n,h}\right\| \xrightarrow{n\to\infty} 0$ for every sequence $N(n) \to \infty$.

A next step is a suitable linearization of $\ln \frac{dQ^N_{n,h}}{dQ_{n,0}}$ using the concept of L_2-differentiability with respect to a sequence $\mu_{n,0}$. The derivative is then a sequence $\dot{l}_n$ and the local sequence Z_n is a stochastic integral of $\dot{l}_n$ w. r. t. $\Phi_n - \mu_{n,0}$. Using this approach in conjunction with the limit theorems for the logarithm of the likelihood ratios we obtain necessary and sufficient conditions for the LAN-property if the L_2-differentiability is fulfilled. Conditions for the LAN-property for Poisson processes were already established in Kutoyants [5] for Poisson processes in the real line and in Lorz [12] for Poisson processes with arbitrary state space. But the systematic use of the concept of contiguity, which is necessary for LAN, and the application of L_2-differentiabillity simplify the situation considerably and lead to necessary and sufficient conditions for the LAN-property. We will show that an i. i. d. sequence of Poisson point processes satisfies the LAN-condition if the family of corresponding intensity measures is L_2-differentiable. Examples for models which have the LAN-property can be found in Kutoyants [5] and [6], where the theory of statistics of Poisson point processes was systematically developed. The applicability of this theory and of the LAN-concept is demonstrated in Kutoyants [5] and [6] on many concrete problems.

2. DISTRIBUTION OF THE LOGARITHM OF THE LIKELIHOOD RATIO OF POISSON PROCESSES

To prepare for the main result of this chapter, we first use the concept of Hellinger integral to formulate and prove conditions for the absolute continuity of distributions of Poisson processes.

Let μ_1, μ_2 be σ-finite measures on the measurable space $(\Omega, \mathcal{F})$. Let μ be any σ-finite dominating measure and denote by p_1 and p_2 the respective densities of μ_1 and μ_2 w. r. t. μ. Set for $0 < s < 1$

$$
\begin{aligned}
H_s(\mu_1, \mu_2) &= \int p_1^s p_2^{1-s}\, d\mu \qquad \text{and} \\
J_s(\mu_1, \mu_2) &= \int \left(s p_1 + (1-s) p_2 - p_1^s p_2^{1-s}\right) d\mu.
\end{aligned}
$$

If μ_1 and μ_2 are probability measures then

$$
\begin{aligned}
\left(2 J_{\frac{1}{2}}(\mu_1, \mu_2)\right)^{\frac{1}{2}} &= \left(2\left(1 - H_{\frac{1}{2}}(\mu_1, \mu_2)\right)\right)^{\frac{1}{2}} \\
&= \left(\int \left(\sqrt{p_1} - \sqrt{p_2}\right)^2 d\mu\right)^{\frac{1}{2}}
\end{aligned}
$$

is the well-known Hellinger distance of μ_1 and μ_2. Let $\mu_{i,j}$, $j = 1, \ldots, n$, $i = 1, 2$ be σ-finite measures on $(\Omega_j, \mathcal{F}_j)$ and denote by $\mu_{1,1} \times \cdots \times \mu_{1,n}$ and $\mu_{2,1} \times \cdots \times \mu_{2,n}$

the corresponding product measures. Then it is easily seen that

$$H_s(\mu_{1,1} \times \cdots \times \mu_{1,n}, \mu_{2,1} \times \cdots \times \mu_{2,n}) = \prod_{j=1}^{n} H_s(\mu_{1,j}, \mu_{2,j}). \tag{2.1}$$

Introduce the families of convex functions f_s and g_s by

$$\begin{aligned} f_s(x) &= -x^s, \quad 0 < s < 1,\ x \geq 0 \\ g_s(x) &= sx + (1-s) - x^s, \quad 0 < s < 1,\ x \geq 0. \end{aligned}$$

Using the convention $f_s\left(\frac{a}{0}\right) 0 = g_s\left(\frac{a}{0}\right) 0 = 0$ we see that

$$\begin{aligned} -H_s(\mu_1, \mu_2) &= \int f_s\left(\frac{p_1}{p_2}\right) p_2 \,\mathrm{d}\mu \qquad \text{and} \\ J_s(\mu_1, \mu_2) &= \int g_s\left(\frac{p_1}{p_2}\right) p_2 \,\mathrm{d}\mu . \end{aligned}$$

Consequently, both $-H_s$ and J_s are special f-divergences in the sense of Csiszár [2]. As we will show later the behaviour of H_s and J_s as $s \uparrow 1$ and $s \downarrow 0$ is closely related to the question whether $P \ll Q$ and $Q \ll P$, respectively. We set

$$\begin{aligned} \mathring{g}_0(x) &= \lim_{s \downarrow 0} \frac{g_s(x)}{s} = x - 1 - \ln x \quad \text{and} \\ \mathring{g}_1(x) &= \lim_{s \uparrow 1} \frac{g_s(x)}{1-s} = x \ln x - x + 1 \end{aligned} \tag{2.2}$$

and use the conventions $\mathring{g}_i\left(\frac{a}{0}\right) 0 = \lim_{t \downarrow 0} \mathring{g}_i\left(\frac{a}{t}\right) t$ if $a > 0$ and $\mathring{g}_i\left(\frac{0}{0}\right) 0 = 0$. Note that $\mathring{g}_i(x) \geq 0$. The function

$$f(x) = s(1-s)(x \ln x - x + 1) - (sx + (1-s) - x^s)$$

has the following properties:

$$f(1) = f'(1) = 0, \quad f''(x) = s(1-s)\left(-x^{s-2} + \frac{1}{x}\right).$$

Hence

$$\frac{g_s(x)}{1-s} \leq x \ln x - x + 1 \qquad \text{if} \quad 1 \leq x < \infty. \tag{2.3}$$

Now we list further properties of the family g_s used in the sequel. For $\frac{1}{2} < s < 1$ put $\alpha = \frac{1}{2s}$ and note that $x^{\frac{1}{2}} = (x^s)^\alpha 1^{1-\alpha} \leq \alpha x^s + 1 - s$. Hence $g_{\frac{1}{2}}(x) \geq \frac{1}{2s} g_s(x)$ and similar $2(1-s)\, g_{\frac{1}{2}}(x) \leq g_s(x)$. Consequently

$$2(1-s)\, g_{\frac{1}{2}}(x) \leq g_s(x) \leq 2s g_{\frac{1}{2}}(x), \quad \frac{1}{2} \leq s < 1,\ x \geq 0. \tag{2.4}$$

Set $h(x) = 4s(1-s)\, g_{\frac{1}{2}}(x) - g_s(x)$. Then $h(1) = h'(1) = 0$ and

$$h''(x) = s(1-s)\left(x^{-\frac{3}{2}} - x^{s-2}\right) \geq 0$$

for $0 < x \leq 1$. Hence $h(x) \geq 0$ and

$$g_s(x) \leq 4s(1-s)\, g_{\frac{1}{2}}(x), \qquad \frac{1}{2} \leq s < 1,\ 0 \leq x \leq 1. \tag{2.5}$$

Set

$$I_i(\mu_1, \mu_2) = \int \overset{\circ}{g}_i \left(\frac{p_1}{p_2}\right) p_2 \,\mathrm{d}\mu.$$

Note that $I_1(\mu_1, \mu_2) = I_0(\mu_2, \mu_1)$. If $\mu_i = P_i$, $i = 1, 2$ are probability measures then

$$I_1(P_1, P_2) = \int \left(\frac{p_1}{p_2} \ln \frac{p_1}{p_2}\right) \mathrm{d}P_2$$

is the well-known Kullback–Leibler information. As the convergence in (2.2) is uniform on $\frac{1}{N} \leq x \leq N$ for every fixed N and $g_s \geq 0$ we get

$$\liminf_{s \uparrow 1} \frac{J_s(\mu_1, \mu_2)}{1-s} \geq \int_{\left\{\frac{1}{N} \leq \frac{p_1}{p_2} \leq N\right\}} \overset{\circ}{g}_1 \left(\frac{p_1}{p_2}\right) p_2 \,\mathrm{d}\mu.$$

The monotone convergence Theorem and $\overset{\circ}{g}_1 \geq 0$ yield

$$\liminf_{s \uparrow 1} \frac{J_s(\mu_1, \mu_2)}{1-s} \geq \int \overset{\circ}{g}_1 \left(\frac{p_1}{p_2}\right) p_2 \,\mathrm{d}\mu = I_1(\mu_1, \mu_2). \tag{2.6}$$

The converse inequality is trivially fulfilled if $I(\mu_1, \mu_2) = \infty$. If $I(\mu_1, \mu_2) < \infty$ and $J_{\frac{1}{2}}(\mu_1, \mu_2) < \infty$ then by (2.3), (2.5) and the Lebesgue Theorem

$$\lim_{s \uparrow 1} \frac{J_s(\mu_1, \mu_2)}{1-s} = I_1(\mu_1, \mu_2). \tag{2.7}$$

We have for probability measures P_1, P_2

$$J_s(P_1, P_2) = 1 - H_s(P_1, P_2) < \infty.$$

Consequently,

$$\lim_{s \uparrow 1} \frac{1 - H_s(P_1, P_2)}{1-s} = I_1(P_1, P_2) \tag{2.8}$$

independent whether $I_1(P_1, P_2) < \infty$ or $I_1(P_1, P_2) = \infty$. Assume $I_1(P_1, P_2) < \infty$ then

$$I_1(P_1, P_2) = \int \left(\ln \frac{\mathrm{d}P_1}{\mathrm{d}P_2}\right) \mathrm{d}P_1$$

and by Jensen's inequality

$$\begin{aligned} H_s(P_1, P_2) &= \int \left(\frac{\mathrm{d}P_1}{\mathrm{d}P_2}\right)^s \mathrm{d}P_2 = \int \left(\frac{\mathrm{d}P_1}{\mathrm{d}P_2}\right)^{s-1} \mathrm{d}P_1 \\ &= \int \exp\left\{-(1-s) \ln \frac{\mathrm{d}P_1}{\mathrm{d}P_2}\right\} \mathrm{d}P_1 \geq \exp\left\{-(1-s) I_1(P_1, P_2)\right\} \end{aligned}$$

or

$$H_s(P_1, P_2) \geq \exp\{-(1-s)I_1(P_1, P_2)\}\,. \tag{2.9}$$

We now summarize the properties of f-divergences which will be systematically used in the sequel. For proofs we refer to Liese, Vajda [10] and Vajda [17].

Denote by $\mathcal{I}$ the family of all sub-σ-algebras $\mathcal{A}$ of $\mathcal{F}$ and note that $\mathcal{I}$ is a directed set by the inclusion, i. e. $\mathcal{A}_1 \leq \mathcal{A}_2$ iff $\mathcal{A}_1$ is a sub-σ-algebra $\mathcal{A}_2$. Denote by $\mu_{i,\mathcal{A}_j}$ the restriction of μ_i to $\mathcal{A}_j$. Then by the monotonicity of f-divergences (Theorem 1.24 in Liese, Vajda [10])

$$H_s(\mu_{1,\mathcal{A}_1}, \mu_{2,\mathcal{A}_1}) \;\geq\; H_s(\mu_{1,\mathcal{A}_2}, \mu_{2,\mathcal{A}_2}) \quad \text{and} \tag{2.10}$$

$$J_s(\mu_{1,\mathcal{A}_1}, \mu_{2,\mathcal{A}_1}) \;\leq\; J_s(\mu_{1,\mathcal{A}_2}, \mu_{2,\mathcal{A}_2})\,. \tag{2.11}$$

Every function $\varphi : \mathcal{I} \rightarrow [-\infty, \infty]$ is a Moor–Smith sequence (net). We denote by $\lim_{\mathcal{A}\in\mathcal{I}} \varphi(\mathcal{A})$ its limit, provided the limit exists. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be any directed subset of $\mathcal{I}$ and denote by $\sigma(\mathcal{I}_0)$ the σ-algebra generated by all $\mathcal{A} \in \mathcal{I}_0$. Suppose $\mu_{i,\mathcal{A}}$ is σ-finite for every $\mathcal{A} \in \mathcal{I}_0$. Then by Theorem 9.15 in Vajda [17]

$$\lim_{\mathcal{A}\in\mathcal{I}_0} J_s(\mu_{1,\mathcal{A}}, \mu_{2,\mathcal{A}}) = J_s\left(\mu_{1,\sigma(\mathcal{I}_0)}, \mu_{2,\sigma(\mathcal{I}_0)}\right)\,. \tag{2.12}$$

If for some $\mathcal{A}_0 \in \mathcal{I}_0$, $H_s(\mu_{1,\mathcal{A}_0}, \mu_{2,\mathcal{A}_0}) < \infty$ then

$$\lim_{\mathcal{A}\in\mathcal{I}_0} H_s(\mu_{1,\mathcal{A}}, \mu_{2,\mathcal{A}}) = H_s\left(\mu_{1,\sigma(\mathcal{I}_0)}, \mu_{2,\sigma(\mathcal{I}_0)}\right)\,. \tag{2.13}$$

The statements (2.12) and (2.13) were shown in Vajda [17] only for probability measures. But the generalizations in (2.12) and (2.13), respectively, are straightforward. Denote by $\mu_1 = \mu_{1,a} + \mu_{1,si}$ the Lebesgue decomposition of μ_1 into the part $\mu_{1,a}$ being absolutely continuous w. r. t. μ_2 and $\mu_{1,si}$ which is singular w. r. t. μ_2. Note that μ-a. e.

$$\begin{aligned}\frac{\mathrm{d}\mu_{1,a}}{\mathrm{d}\mu} &= p_1 I_{\{p_2>0\}}\\ \frac{\mathrm{d}\mu_{1,s}}{\mathrm{d}\mu} &= p_1 I_{\{p_2=0\}},\end{aligned}$$

where I_A denotes the indicator function of the set A. Inequality (2.4) yields

$$sx + (1-s)y - x^s y^{1-s} \leq \left(\sqrt{x} - \sqrt{y}\right)^2 \tag{2.14}$$

for every $0 \leq x, y < \infty$. Suppose $J_{\frac{1}{2}}(\mu_1, \mu_2) < \infty$. Then by the Lebesgue Theorem

$$\begin{aligned}\lim_{s\uparrow 1} J_s(\mu_1, \mu_2) &= \int I_{\{p_2=0\}} p_1 \,\mathrm{d}\mu \\ &= \mu_{1,si}(\Omega).\end{aligned} \tag{2.15}$$

If both μ_1 and μ_2 are finite then

$$\begin{aligned}\int (\sqrt{p_1} - \sqrt{p_2})^2 \,\mathrm{d}\mu &\leq \int |\sqrt{p_1} - \sqrt{p_2}|\,|\sqrt{p_1} + \sqrt{p_2}| \,\mathrm{d}\mu \\ &= \int |p_1 - p_2| \,\mathrm{d}\mu < \infty\end{aligned} \tag{2.16}$$

and

$$\lim_{s\uparrow 1} H_s(\mu_1,\mu_2) = \mu_1(\Omega) - \lim_{s\uparrow 1} J_s(\mu_1,\mu_2) = \mu_{1,a}(\Omega). \tag{2.17}$$

Moreover, if μ_1, μ_2 are any σ-finite measures then the definition of H_s yields that μ_1 and μ_2 are mutually singular ($\mu_1 \perp \mu_2$) iff

$$H_{\frac{1}{2}}(\mu_1,\mu_2) = 0$$

which is equivalent to

$$H_s(\mu_1,\mu_2) = 0 \qquad \text{for every } 0 < s < 1. \tag{2.18}$$

Suppose now that $(\mathcal{X},\mathcal{A})$ is a measurable space which will serve as the state space of Poisson processes. Denote by M the set of all measures φ on $(\mathcal{X},\mathcal{A})$ taking values in $\{0,1,\ldots,\infty\}$. For every $B \in \mathcal{A}$ we introduce the mapping $Z_B : M \to \{0,\ldots,\infty\}$ by $Z_B(\varphi) = \varphi(B)$. Given $\boldsymbol{z} \subseteq \mathcal{A}$ we denote by $\mathcal{M}_{\boldsymbol{z}}$ the σ-algebra of subsets generated by the mappings $Z_B : M \to \{0,1,\ldots,\infty\}$, $B \in \boldsymbol{z}$. Instead of $\mathcal{M}_{\mathcal{A}}$ we shortly write $\mathcal{M}$. By a point process we shall mean a random variable Φ defined on some probability space which takes values in $(M,\mathcal{M})$.

Let Λ be a σ-finite measure on $(\mathcal{X},\mathcal{A})$. A point process Φ is called a Poisson process with intensity measure Λ if for every $\boldsymbol{z} = \{B_1,\ldots,B_n\}$ with $B_i \cap B_j = \emptyset$, $i \neq j$, $B_i \in \mathcal{A}$ the random variables $\Phi(B_1),\ldots,\Phi(B_n)$ are independent and for every $B \in \mathcal{A}$ with $\Lambda(B) < \infty$ the random variable $\Phi(B)$ has a Poisson distribution with parameter $\Lambda(B)$, i.e.

$$P(\Phi(B) = k) = \pi_{\Lambda(B)}(k) = \frac{(\Lambda(B))^k}{k!} e^{-\Lambda(B)}.$$

The existence of Poisson process with arbitrary state space and σ-finite intensity measure was shown in Mecke [14].

Suppose now Λ_1, Λ_2 are σ-finite measures on $(\mathcal{X},\mathcal{A})$. Denote by ν a σ-finite dominating measure. Let $\mathcal{R} \subseteq \mathcal{A}$ be the ring of all sets $B \in \mathcal{A}$ for which

$$\Lambda_i(B) < \infty, \quad i = 1,2, \ \nu(B) < \infty.$$

Let $\mathcal{Z}$ be the collection of all finite selections of disjoint subsets from $\mathcal{R}$. For $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathcal{Z}$ we write $\boldsymbol{z}_1 \leq \boldsymbol{z}_2$ if every $B \in \boldsymbol{z}_1$ is the union of some sets from $\boldsymbol{z}_2$. Note that $\boldsymbol{z} \in \mathcal{Z}$ is not necessarily a decomposition of $\mathcal{X}$, i.e. it may happen that $\bigcup_{B\in\boldsymbol{z}} B$ is a strict subset of $\mathcal{X}$. We note that $(\mathcal{Z},\leq)$ is a directed set. Define

$$J_{s,\boldsymbol{z}}(\Lambda_1,\Lambda_2) = \sum_{B\in\boldsymbol{z}} \left[s\frac{\Lambda_1(B)}{\nu(B)} + (1-s)\frac{\Lambda_2(B)}{\nu(B)} - \left(\frac{\Lambda_1(B)}{\nu(B)}\right)^s \left(\frac{\Lambda_2(B)}{\nu(B)}\right)^{1-s} \right].$$

A simple calculation shows that

$$H_s(\pi_{\lambda_1},\pi_{\lambda_2}) = \exp\left\{-\left(s\lambda_1 + (1-s)\lambda_2 - \lambda_1^s\lambda_2^{1-s}\right)\right\}.$$

Hence by (2.1)

$$H_s\left(P_{\Lambda_1,\mathcal{M}_{\boldsymbol{z}}}, P_{\Lambda_2,\mathcal{M}_{\boldsymbol{z}}}\right) = J_{s,\boldsymbol{z}}(\Lambda_1,\Lambda_2).$$

Set $\mathcal{I}_0 = \{\mathcal{M}_{\boldsymbol{z}},\ \boldsymbol{z} \in \mathcal{Z}\}$ and apply (2.12), (2.13) to get

$$H_s\left(P_{\Lambda_1}, P_{\Lambda_2}\right) = \exp\left\{-J_s(\Lambda_1,\Lambda_2)\right\}. \tag{2.19}$$

Put $\lambda_i = \frac{\mathrm{d}\Lambda_i}{\mathrm{d}\nu}$. By (2.18) we see that $P_{\Lambda_1} \perp P_{\Lambda_2}$ iff for the Hellinger distance

$$\left(2J_{\frac{1}{2}}(\Lambda_1,\Lambda_2)\right)^{\frac{1}{2}} = \left(\int\left(\sqrt{\lambda_1}-\sqrt{\lambda_2}\right)^2 \mathrm{d}\nu\right)^{\frac{1}{2}} = \infty.$$

Conversely, if $J_{\frac{1}{2}}(\Lambda_1,\Lambda_2) < \infty$ then by (2.15)

$$\lim_{s\uparrow 1} H_s\left(P_{\Lambda_1}, P_{\Lambda_2}\right) = \exp\left\{-\Lambda_{1,si}(\mathcal{X})\right\}. \tag{2.20}$$

Otherwise, (2.17) implies

$$\lim_{s\uparrow 1} H_s\left(P_{\Lambda_1}, P_{\Lambda_2}\right) = P_{\Lambda_1,a}(M)\ .$$

Hence $P_{\Lambda_1} \ll P_{\Lambda_2}$ iff $\Lambda_1 \ll \Lambda_2$ and $J_{\frac{1}{2}}(\Lambda_1,\Lambda_2) < \infty$.

Summarizing the above results we get the following statement which can already be found for special cases in Brown [1] and Liese [8].

Proposition 1. Let $P_{\Lambda_1}, P_{\Lambda_2}$ be distributions of Poisson processes. It holds

$$J_{\frac{1}{2}}(\Lambda_1,\Lambda_2) = \infty$$

iff $P_{\Lambda_1} \perp P_{\Lambda_2}$. It holds $P_{\Lambda_1} \ll P_{\Lambda_2}$ iff $\Lambda_1 \ll \Lambda_2$ and $J_{\frac{1}{2}}(\Lambda_1,\Lambda_2) < \infty$.

The concept of contiguous sequences plays a key role in the asymptotic decision theory. It is in some sense a generalization of the concept of absolute continuity. To be more precise we suppose that $\{P_n\}, \{Q_n\}$ are sequences of distributions on $(\Omega_n, \mathcal{F}_n)$. Then $\{P_n\}$ is called contiguous w. r. t. $\{Q_n\}$ ($\{P_n\} \lhd \{Q_n\}$) if $Q_n(A_n) \overset{n\to\infty}{\longrightarrow} 0$ implies $P_n(A_n) \overset{n\to\infty}{\longrightarrow} 0$, $\quad A_n \in \mathcal{F}_n$. If $\{P_n\}\lhd\{Q_n\}$ and $\{Q_n\}\lhd\{P_n\}$ then we write $\{P_n\} \lhd\rhd \{Q_n\}$. Assume now $P_{\Lambda_{1,n}}, P_{\Lambda_{2,n}}$ are distributions of Poisson point processes with state space $(\mathcal{X}_n, \mathcal{A}_n)$.

Theorem 1. The following statements are equivalent

$$\left\{P_{\Lambda_{1,n}}\right\} \lhd \left\{P_{\Lambda_{2,n}}\right\} \tag{2.21}$$

$$\liminf_{s\uparrow 1}\liminf_{n\to\infty} H_s\left(P_{\Lambda_{1,n}}, P_{\Lambda_{2,n}}\right) = 1 \tag{2.22}$$

$$\limsup_{s\uparrow 1}\limsup_{n\to\infty} J_s(\Lambda_{1,n},\Lambda_{2,n}) = 0 \tag{2.23}$$

$$\begin{aligned} &\limsup_{n\to\infty} \quad J_{\frac{1}{2}}(\Lambda_{1,n},\Lambda_{2,n}) < \infty \\ &\lim_{n\to\infty} \quad \Lambda_{1,n,si}(\mathcal{X}_n) = 0 \qquad \text{and} \\ &\limsup_{N\to\infty} \quad \limsup_{n\to\infty} \int_{\{\lambda_n > N\}} \left(\sqrt{\lambda_n} - 1\right)^2 \mathrm{d}\Lambda_{2,n} = 0. \end{aligned} \tag{2.24}$$

where λ_n is the density of $\Lambda_{1,n,a}$ w. r. t. $\Lambda_{2,n}$.

Corollary 1. If

$$\limsup_{n\to\infty} I_1(\Lambda_{1,n},\Lambda_{2,n}) < \infty$$

then

$$\{P_{\Lambda_{1,n}}\} \lhd \{P_{\Lambda_{2,n}}\}.$$

Proof. In view of Liese [9] and Jacod, Shiryaev [3] it holds for any sequences of distributions

$$\{P_n\} \lhd \{Q_n\} \iff \liminf_{s\uparrow 1} \liminf_{n\to\infty} H_s(P_n,Q_n) = 1$$

which implies the equivalence of (2.21) and (2.22). The equivalence of (2.22) and (2.23) follows from the representation of $H_s(P_{\Lambda_1},P_{\Lambda_2})$ in (2.19). Hence it remains to prove the equivalence of (2.23) and (2.24). Note that $g_s(x)\, g_{\frac{1}{2}}^{-1}(x)$ is a continuous function of (s,x) which is uniformly continuous in $\frac{1}{2} \le s \le 1,\ 0 \le x \le N$. Hence

$$C_N(s) = \max_{0\le x\le N} \frac{g_s(x)}{g_{\frac{1}{2}}(x)}$$

is also continuous. Therefore

$$g_s(x) \le C_N(s)\, g_{\frac{1}{2}}(x) \tag{2.25}$$

and

$$\lim_{s\uparrow 1} C_N(s) = C_N(1) = 0.$$

Note that

$$J_s(\Lambda_{1,n},\Lambda_{2,n}) = \int g_s(\lambda_n)\,\mathrm{d}\Lambda_{2,n} + s\Lambda_{1,n,si}(\mathcal{X}_n).$$

Applying the inequality (2.25) we get

$$J_s(\Lambda_{1,n},\Lambda_{2,n}) \le \int_{\{\lambda_n\le N\}} C_N(s) g_{\frac{1}{2}}(\lambda_n)\,\mathrm{d}\Lambda_{2,n} + \int_{\{\lambda_n > N\}} g_{\frac{1}{2}}(\lambda_n)\,\mathrm{d}\Lambda_{2,n} + s\Lambda_{1,n,si}(\mathcal{X}_n).$$

Assume the conditions in (2.24) are fulfilled. Taking at first $n\to\infty$ then $s\uparrow 1$ and finally $N\to\infty$ we get (2.23). To prove (2.23) $\Rightarrow$ (2.24) we note by (2.4)

$$\limsup_{n\to\infty} J_{\frac{1}{2}}(\Lambda_{1,n},\Lambda_{2,n}) = \infty$$

iff

$$\limsup_{n\to\infty} J_s(\Lambda_{1,n}, \Lambda_{2,n}) = \infty$$

for every $\frac{1}{2} < s < 1$. Therefore (2.23) implies the first condition in (2.24). The second condition follows from $J_s(\Lambda_{1,n}, \Lambda_{2,n}) \geq s\Lambda_{1,n,si}(\mathcal{X}_n)$ which is a direct consequence of the definition of J_s. It holds $\lim_{s\uparrow 1} g_s(x) = 0$ for every fixed x. But since

$$\lim_{x\to\infty} \frac{g_{\frac{1}{2}}(x)}{g_s(x)} = \frac{1}{2s},$$

we find for every $\frac{1}{2} < s < 1$ some $N(s) \overset{s\uparrow 1}{\longrightarrow} \infty$ such that

$$g_{\frac{1}{2}}(x) \leq 4g_s(x) \qquad \text{for } N(s) \leq x < \infty.$$

Put

$$A(N) = \limsup_{n\to\infty} \int_{\{\lambda_n > N\}} \left(\sqrt{\lambda_n} - 1\right)^2 \mathrm{d}\Lambda_{2,n}$$

and notice that $A(N)$ is nonincreasing. Hence

$$\begin{aligned}\lim_{N\to\infty} A(N) &= \limsup_{s\uparrow 1} A(N(s)) \\ &\leq 4 \limsup_{s\uparrow 1} \limsup_{n\to\infty} \int_{\{\lambda_n > N(s)\}} g_s(\lambda_n)\, \mathrm{d}\Lambda_{2,n} \\ &\leq 4 \limsup_{s\uparrow 1} \limsup_{n\to\infty} J_s(\Lambda_{1,n}, \Lambda_{2,n}) = 0\end{aligned}$$

which proves the third statement in (2.24). □

The proof of the Corollary follows from (2.23) and inequality (2.9).

Now we will study the distribution of the logarithm of the likelihood ratio $\ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$. In contrary to other papers we do not assume that $\Lambda_i(\mathcal{X}) < \infty$. But if the last condition is violated then the representation of $\ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ in Karr [4] is not applicable. Therefore we employ the representation (2.19) of the Hellinger integral which is the moment generating function of $\ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ to get the characteristic function. To derive the characteristic function of the logarithm of likelihood ratio we suppose $P_{\Lambda_1} \sim P_{\Lambda_2}$ ($P_{\Lambda_1} \ll P_{\Lambda_2}$ and $P_{\Lambda_2} \ll P_{\Lambda_1}$) which is equivalent to $\Lambda_1 \sim \Lambda_2$ and $J_{\frac{1}{2}}(\Lambda_1, \Lambda_2) < \infty$ by Theorem 1. Note that in this case both

$$\varphi(z) = \int \exp\left\{z \ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\} \mathrm{d}P_{\Lambda_2} \tag{2.26}$$

and

$$\psi(z) = \int \left(z\lambda_1 + (1-z)\lambda_2 - \lambda_1^z \lambda_2^{1-z}\right) \mathrm{d}\nu$$

are well defined for $0 < \mathrm{Re}(z) < 1$, which follows from the inequalities

$$\left|\exp\left\{(s+it)\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\}\right| \leq \left(\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right)^s \tag{2.27}$$

and

$$|z\lambda + (1-z) - \lambda^z| \leq C(t)\left(\sqrt{\lambda}-1\right)^2 \tag{2.28}$$

where $\lambda = \frac{\lambda_1}{\lambda_2}$, $C(t)$ is some constant and $z = s + it$. Note that both φ and ψ are analytic functions in the stripe $0 < \mathrm{Re}(z) < 1$. The uniqueness theorem for analytic functions and $\varphi(s) = \exp\{-\psi(s)\}$ (see (2.19)) imply that

$$\varphi(z) = \exp\{-\psi(z)\} \tag{2.29}$$

for every $0 < \mathrm{Re}(z) < 1$.

Given two real or complex valued functions f, g on the real line we write $|f| \preceq |g|$ if there is some constant c such that

$$|f(x)| \leq c|g(x)|$$

for every $x \in \mathbb{R}_1$. Using this notation we remark that for every fixed $t \in \mathbb{R}_1$

$$\begin{aligned}\left|e^{itx} - 1 - it\frac{x}{1+x^2}\right| &\preceq x^2 I_{[-1,1]}(x) + |x| I_{\mathbb{R}_1\setminus[-1,1]}(x) \\ &\preceq \left(e^{\frac{1}{2}x} - 1\right)^2\end{aligned} \tag{2.30}$$

and similarly

$$\left|e^x - 1 - \frac{x}{1+x^2}\right| \preceq \left(e^{\frac{1}{2}x} - 1\right)^2. \tag{2.31}$$

Note that

$$J_{\frac{1}{2}}(\Lambda_1, \Lambda_2) = \frac{1}{2}\int \left(e^{\frac{1}{2}\ln\lambda} - 1\right)^2 \mathrm{d}\Lambda_2 .$$

Consequently, if $J_{\frac{1}{2}}(\Lambda_1, \Lambda_2) < \infty$ then the integrals

$$a = \int \left(\frac{\ln\lambda}{1+(\ln\lambda)^2} + 1 - \lambda\right) \mathrm{d}\Lambda_2 \tag{2.32}$$

and

$$\int \left(e^{it\ln\lambda} - 1 - \frac{it\ln\lambda}{1+(\ln\lambda)^2}\right) \mathrm{d}\Lambda_2$$

are well defined.

Let $K(t,x)$ be the kernel

$$\begin{aligned}K(t,x) &= \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}, \quad x \neq 0 \\ K(t,x) &= -\frac{t^2}{2}, \qquad x = 0.\end{aligned}$$

Introduce a measure κ by

$$\kappa(B) = \int \frac{(\ln\lambda)^2}{1+(\ln\lambda)^2} I_{B\setminus\{0\}}(\ln\lambda)\,\mathrm{d}\Lambda_2. \tag{2.33}$$

Note that $J_{\frac{1}{2}}(\Lambda_1,\Lambda_2) = \frac{1}{2}\int(\sqrt{\lambda}-1)^2\,\mathrm{d}\Lambda_2 < \infty$ and

$$\frac{(\ln\lambda)^2}{1+(\ln\lambda)^2} \preceq \left(\sqrt{\lambda}-1\right)^2$$

imply that κ is finite. Assume $0 < s < 1$ and $\alpha > 1$ such that $\alpha s \le 1$ then

$$\int \left(\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right)^{s\alpha} \mathrm{d}P_{\Lambda_2} \le 1.$$

Hence $\left(\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right)^s$, $0 < s < \frac{1}{\alpha}$, is uniformly integrable and (2.27) implies

$$\lim_{s\downarrow 0}\varphi(s+it) = \int \exp\left\{it\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\}\mathrm{d}P_{\Lambda_2}.$$

Moreover, (2.28) and the Lebesgue Theorem yield

$$\lim_{s\downarrow 0}\psi(s+it) = \int \left(it\lambda + (1-it) - \lambda^{it}\right)\mathrm{d}\Lambda_2.$$

Consequently by (2.29)

$$E_{P_{\Lambda_2}}\exp\left\{it\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\} = \exp\left\{-\int \left(it\lambda + (1-it) - e^{it\ln\lambda}\right)\mathrm{d}\Lambda_2\right\}. \tag{2.34}$$

The representation of the characteristic function of $\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ in (2.34) yields

$$E_{P_{\Lambda_2}}\exp\left\{it\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\} = \exp\left\{ita + \int K(t,x)\,\kappa(\mathrm{d}x)\right\}.$$

Recall that the characteristic function $\varphi(t)$ of every infinitely divisible distribution Q on the real line $\mathbb{R}_1$ has the representation

$$\varphi(t) = \exp\left\{iat + \int K(t,x)\,\kappa(\mathrm{d}x)\right\}$$

where κ is a finite measure and the characteristic pair (a,κ) is uniquely determined. Thus we get the following statement.

Proposition 2. If $P_{\Lambda_1} \sim P_{\Lambda_2}$ then the distribution of $\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ (w.r.t. P_{Λ_2}) is infinitely divisible with the characteristic pair (a,κ) where a is defined in (2.32) and κ is given by (2.33).

We recall to the following well-known criteria for the weak convergence, denoted by $\Rightarrow$, of infinitely divisible distributions (see Petrov [15]).

Proposition 3. If Q, Q_1, $Q_2, \ldots$ are infinitely divisible distributions with characteristic pairs $(\alpha, \kappa), (\alpha_1, \kappa_1), (\alpha_2, \kappa_2), \ldots$ then

$$Q_n \Longrightarrow Q, \quad \text{as } n \to \infty$$

iff

$$\alpha_n \longrightarrow \alpha, \kappa_n \Longrightarrow \kappa, \quad \text{as } n \to \infty.$$

Denote by $N(\mu, \sigma^2)$ the normal distribution with mean μ and variance $\sigma^2 \geq 0$, where $N(\mu, 0) = \delta_\mu$ is the δ-distribution concentrated at the point μ. Note that $N(\mu, \sigma^2)$ has the characteristic pair $\alpha = \mu$, $\kappa = \sigma^2 \delta_0$.

Furthermore, if the infinitely divisible r.v. X has the characteristic pair (α, κ) then obviously $Y = \frac{X-a}{b}$, $b \neq 0$ has the characteristic pair $(\widetilde{\alpha}, \widetilde{\kappa})$ where

$$\widetilde{\alpha} = \alpha - \frac{a}{b} - \int \left(K\left(\frac{t}{b}, x\right) - K\left(t, \frac{x}{b}\right) \right) \kappa(\mathrm{d}x) \tag{2.35}$$

$$\widetilde{\kappa}(B) = \int I_B\left(\frac{t}{b}\right) \kappa(\mathrm{d}t). \tag{2.36}$$

Suppose now $P_{\Lambda_{i,n}}$, $i = 1, 2$ are distributions of Poisson processes with state spaces $(\mathcal{X}_n, \mathcal{A}_n)$ and assume $P_{\Lambda_{1,n}} \sim P_{\Lambda_{2,n}}$ for every n. Set

$$L_n = \ln \frac{\mathrm{d}P_{\Lambda_{1,n}}}{\mathrm{d}P_{\Lambda_{2,n}}}, \; \lambda_n = \frac{\mathrm{d}\Lambda_{1,n}}{\mathrm{d}\Lambda_{2,n}}$$

and

$$Q_n = \mathcal{L}\left(\frac{L_n - a_n}{b_n} \middle| P_{\Lambda_{2,n}} \right) := P_{\Lambda_{2,n}} \circ \left(\frac{L_n - a_n}{b_n} \right)^{-1}.$$

Theorem 2. If a_n and b_n are real numbers with $b_n \neq 0$ and the condition $P_{\Lambda_{1,n}} \sim P_{\Lambda_{2,n}}$ is fulfilled then

$$Q_n \Longrightarrow N(\mu, \sigma^2)$$

as $n \to \infty$, iff

$$\lim_{n\to\infty} \left[\int \frac{\lambda_n - 1 - \ln \lambda_n}{1 + (\ln \lambda_n)^2} \mathrm{d}\Lambda_{2,n} - \frac{a_n}{b_n} - \int \left(K\left(\frac{t}{b_n}, \ln \lambda_n\right) - K\left(t, \frac{\ln \lambda_n}{b_n}\right) \right) \frac{(\ln \lambda_n)^2}{1 + (\ln \lambda_n)^2} \mathrm{d}\Lambda_{2,n} \right] = \mu \tag{2.37}$$

$$\lim_{n\to\infty} \frac{1}{b_n^2} \int \frac{(\ln \lambda_n)^2}{1 + (\ln \lambda_n)^2} \mathrm{d}\Lambda_{2,n} = \sigma^2 \tag{2.38}$$

$$\lim_{n\to\infty} \frac{1}{b_n^2} \int_{\{|\ln \lambda_n| > \varepsilon\}} \frac{(\ln \lambda_n)^2}{1 + (\ln \lambda_n)^2} \mathrm{d}\Lambda_{2,n} = 0 \tag{2.39}$$

for every $\varepsilon > 0$.

Proof. Let (α_n, κ_n) be the characteristic pair of $\frac{L_n - a_n}{b_n}$. By (2.36) the conditions (2.38), (2.39) are equivalent to the weak convergence of the measures κ_n to $\sigma^2\delta_0$. In view of (2.35) the condition (2.37) is nothing else than $\alpha_n \longrightarrow \mu$. □

For infinitely divisible distribution with finite second moments both the representation of the characteristic function and corresponding limit theorem can be simplified. If Q is an infinitely divisible distribution with

$$\int e^{itx}\, Q(\mathrm{d}x) = \exp\left\{i\alpha t + \int K(t,x)\,\kappa(\mathrm{d}t)\right\}$$

then

$$\int x^2\, Q(\mathrm{d}x) < \infty \iff \int x^2\, \kappa(\mathrm{d}x) < \infty. \tag{2.40}$$

Introduce the kernel L by

$$\begin{aligned} L(t,x) &= \frac{e^{itx} - 1 - itx}{x^2} && \text{if } x \neq 0 \\ L(t,x) &= -\frac{1}{2}t^2 && \text{if } x = 0. \end{aligned}$$

Then

$$\int e^{itx}\, Q(\mathrm{d}x) = \exp\left\{iat + \int L(t,x)\,\mu(\mathrm{d}x)\right\} \tag{2.41}$$

where

$$\mu(B) = \int_B (1+x^2)\,\kappa(\mathrm{d}x) \tag{2.42}$$

$$\beta = \alpha + \int x\,\kappa(\mathrm{d}x). \tag{2.43}$$

If $P_{\Lambda_1} \sim P_{\Lambda_2}$ then by the definition of κ in (2.33) and relation (2.40)

$$\begin{aligned} E_{P_{\Lambda_2}}\left(\ln\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right)^2 < \infty &\iff \int x^2\,\kappa(\mathrm{d}x) < \infty \\ &\iff \int (\ln\lambda)^2\,\mathrm{d}\Lambda_2 < \infty. \end{aligned}$$

Under this condition we obtain from (2.34)

$$\begin{aligned} E_{P_{\Lambda_2}} \exp\left\{it\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}\right\} &= \exp\left\{\int \left(e^{it\ln\lambda} - it(\lambda-1) - 1\right)\,\mathrm{d}\Lambda_2\right\} \\ &= \exp\left\{ita + \int L(t,x)\,\mu(\mathrm{d}x)\right\} \end{aligned}$$

where

$$a = \int (\ln\lambda - \lambda + 1)\,\mathrm{d}\Lambda_2 \tag{2.44}$$

$$\mu(B) = \int (\ln\lambda)^2\, I_{B\setminus\{0\}}(\ln\lambda)\,\mathrm{d}\Lambda_2 \tag{2.45}$$

and $\lambda = \frac{\mathrm{d}\Lambda_1}{\mathrm{d}\Lambda_2}$. Now we are ready to formulate a limit theorem for the distribution of the logarithm of the likelihood ratio of distributions of Poisson processes with the second moments finite. Note that the relations (2.44) and (2.45) yield

$$E_{P_{\Lambda_2}} \ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}} = \int (\ln \lambda - \lambda + 1)\, \mathrm{d}\Lambda_2 \tag{2.46}$$

$$V_{P_{\Lambda_2}} \left(\ln \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}} \right) = \int (\ln \lambda)^2\, \mathrm{d}\Lambda_2. \tag{2.47}$$

Theorem 3. Suppose $P_{\Lambda_{1,n}} \sim P_{\Lambda_{2,n}}$ and

$$b_n^2 = \int (\ln \lambda_n)^2\, \mathrm{d}\Lambda_{2,n} < \infty. \tag{2.48}$$

Put

$$a_n = \int (\ln \lambda_n - \lambda_n + 1)\, \mathrm{d}\Lambda_{2,n}.$$

Then

$$\mathcal{L}\left(\frac{L_n - a_n}{b_n} \middle| P_{\Lambda_{2,n}} \right) \Longrightarrow N(0,1)$$

iff

$$\frac{1}{b_n^2} \int_{\{|\ln \lambda_n| > \varepsilon\}} (\ln \lambda_n)^2\, \mathrm{d}\Lambda_{2,n} \overset{n\to\infty}{\longrightarrow} 0 \tag{2.49}$$

for every $\varepsilon > 0$.

Proof. The proof follows from the fact that infinitely divisible distributions with characteristic pairs (β_n, μ_n) and variance 1 converge weakly to a standard normal distribution iff $\beta_n \to 0$ and $\mu_n \Rightarrow \delta_0$ as $n \to \infty$. Let (β_n, μ_n) correspond to $\frac{L_n - a_n}{b_n}$. Then by (2.46) we have $\beta_n = 0$. In view of (2.45) and (2.47), the relation (2.49) is equivalent to $\mu_n \Rightarrow \delta_0$, $n \to \infty$ which completes the proof. □

Remark 1. If F_n denotes the distribution function of $\frac{L_n - a_n}{b_n}$ under $P_{\Lambda_{2,n}}$ and Φ is the standard normal distribution function then Theorem 3 states that $F_n(x) \overset{n\to\infty}{\longrightarrow} \Phi(x)$ for every x. As Φ is continuous it follows that $\Delta_n = \sup_x |F_n(x) - F(x)| \overset{n\to\infty}{\longrightarrow} 0$. Under weak additional assumptions one can establish upper bounds for Δ_n in terms of a_n, b_n and $\Lambda_{i,n}$. These bounds correspond to the Berry–Esseen inequality for normalized sums of i. i. d. random variables. For details we refer to Lorz and Heinrich [13]. Here one can also find an Edgeworth expansion for F_n.

In special situations it may happen that the distribution of the logarithm of likelihood ratio converges in distribution without linear transformations, i. e. $a_n = 0$, $b_n = 1$. Such situations are met in localized models which will be studied in the next chapter. It turns out that in this case the expectation μ in the normal distribution in Theorem 2 must take on a special value. To be more precise we suppose that for $L_n = \ln \frac{\mathrm{d}P_{\Lambda_{1,n}}}{\mathrm{d}P_{\Lambda_{2,n}}}$ holds

$$\mathcal{L}\left(L_n | P_{\Lambda_{2,n}}\right) \implies N(\mu, \sigma^2). \tag{2.50}$$

Then by the first Lemma of LeCam (see Strasser [16]) the sequence $\{P_{\Lambda_{1,n}}\}$ is contiguous w. r. t. $\{P_{\Lambda_{2,n}}\}$ iff

$$\int e^y \, N(\mu, \sigma^2)(\mathrm{d}y) = 1. \tag{2.51}$$

We have

$$\int e^y \, N(\mu, \sigma^2)(\mathrm{d}y) = \int \frac{1}{\sqrt{2\pi}\sigma} e^{y - \frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}} \, \mathrm{d}y = \exp\left\{\mu + \frac{1}{2}\sigma^2\right\}.$$

Hence (2.50) implies $\mu = -\frac{1}{2}\sigma^2$. Now we formulate a limit theorem for L_n for contiguous sequences.

Theorem 4. If $P_{\Lambda_{1,n}} \sim P_{\Lambda_{2,n}}$ for every n then

$$\mathcal{L}\left(L_n | P_{\Lambda_{2,n}}\right) \implies N\left(-\frac{1}{2}\sigma^2, \sigma^2\right) \tag{2.52}$$

iff

$$\{P_{\Lambda_{1,n}}\} \lhd \rhd \{P_{\Lambda_{2,n}}\} \tag{2.53}$$

$$\int \frac{(\ln \lambda_n)^2}{1 + (\ln \lambda_n)^2} \, \mathrm{d}\Lambda_{2,n} \quad \stackrel{n\to\infty}{\longrightarrow} \quad \sigma^2 \qquad \text{and} \tag{2.54}$$

$$\int_{\{|\ln \lambda_n| > \varepsilon\}} \frac{(\ln \lambda_n)^2}{1 + (\ln \lambda_n)^2} \, \mathrm{d}\Lambda_{2,n} \quad \stackrel{n\to\infty}{\longrightarrow} \quad 0 \tag{2.55}$$

for every $\varepsilon > 0$.

P r o o f. Suppose that (2.52) is fulfilled. Then $\{P_{\Lambda_{1,n}}\} \lhd \rhd \{P_{\Lambda_{2,n}}\}$ follows from the first Lemma of LeCam. The conditions (2.54) and (2.55) follow from (2.38) and (2.39), respectively, in Theorem 2. Conversely, assume (2.53), (2.54), (2.55) are fulfilled. Then Theorem 2 yields that

$$\lim_{N\to\infty} \limsup_{n\to\infty} \int_{\{\lambda_n > N\}} \left(\sqrt{\lambda_n} - 1\right)^2 \mathrm{d}\Lambda_{2,n} = 0. \tag{2.56}$$

We have from (2.55) and (2.56)

$$\int_{\{|\ln \lambda_n| \geq \varepsilon\}} \frac{\ln \lambda_n - \lambda_n + 1}{1 + (\ln \lambda_n)^2} \, \mathrm{d}\Lambda_{2,n} \longrightarrow 0 \tag{2.57}$$

for every $\varepsilon > 0$. As $\lim_{x\to 1} \frac{x - 1 - \ln x}{(\ln x)^2} = \frac{1}{2}$ we obtain from (2.54) and (2.57) that

$$\lim_{n\to\infty} \int \frac{\ln \lambda_n - \lambda_n + 1}{1 + (\ln \lambda_n)^2} \, \mathrm{d}\Lambda_{2,n} = -\frac{\sigma^2}{2}$$

Hence (2.37) in Theorem 2 is fulfilled and the proof is complete. □

3. LOCAL ASYMPTOTIC NORMALITY OF DISTRIBUTIONS OF POISSON PROCESSES

Suppose P_Λ is the distribution of a Poisson point process with state space $(\mathcal{X}, \mathcal{A})$ and σ-finite intensity measure Λ. If $f_j = \sum_{i=1}^n a_{i,j} I_{A_{i,j}}$ are step functions with $\Lambda(A_{i,j}) < \infty$ then for

$$Y(f_j) = \int f_j \, \mathrm{d}(\varphi - \Lambda) := \int f_j \, \mathrm{d}\varphi - \int f_j \, \mathrm{d}\Lambda$$

it holds

$$E_{P_\Lambda} Y(f_1) Y(f_2) = \int f_1 f_2 \, \mathrm{d}\Lambda. \tag{3.1}$$

If $f \in L_2(\Lambda)$ is any function then we choose a sequence of step functions $f_n \in L_2(\Lambda)$ with

$$\int (f - f_n)^2 \, \mathrm{d}\Lambda \overset{n\to\infty}{\longrightarrow} 0.$$

The relation (3.1) shows that $Y(f_n)$ is a Cauchy sequence in $L_2(P_\Lambda)$ which converges to some element which will be denoted again by $Y(f)$ or by

$$\int f \, \mathrm{d}(\varphi - \Lambda).$$

Note that by construction (3.1) holds for every f_1, $f_2 \in L_2(\Lambda)$. By approximation of $f \in L_2(\Lambda)$ by step functions one can see that

$$E_{P_\Lambda} \exp\{isY(f)\} = \exp\left\{\int \left(e^{ifs} - 1 - ifs\right) \Lambda(\mathrm{d}s)\right\} \tag{3.2}$$

$$= \exp\left\{\int L(f,s) I_{\{f\neq 0\}}(s) \, f^2(s), \Lambda(\mathrm{d}s)\right\}.$$

Hence $Y(f) = \int f \, \mathrm{d}(\varphi - \Lambda)$ is infinitely divisible with characteristic pair

$$(\beta, \mu) = \left(0, \int I_{\cdot \cap \{f\neq 0\}}(s) \, f^2(s) \, \Lambda(\mathrm{d}s).\right) \tag{3.3}$$

Assume now that Φ_1 and Φ_2 are Poisson point processes with finite intensity measures Λ_1 and Λ_2, respectively, which are assumed to be equivalent. Then by inequality (2.16) and Proposition 1 we have $P_{\Lambda_1} \sim P_{\Lambda_2}$. Set $\lambda = \frac{\mathrm{d}\Lambda_1}{\mathrm{d}\Lambda_2}$. Due to Karr [4] the density $\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ admits the following representation

$$\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}(\varphi) = \exp\left\{\int \ln \lambda \, \mathrm{d}\varphi - \Lambda_1(\mathcal{X}) + \Lambda_2(\mathcal{X})\right\}. \tag{3.4}$$

For any fixed $B \in \mathcal{A}$ we denote by $\mathcal{M}_B$ the σ-algebra of subsets of M generated by the mappings $Z_A(\varphi) = \varphi(A)$, $A \subseteq B$, $A \in \mathcal{A}$. Let P_{Λ_i}, $i = 1, 2$ be equivalent distributions of Poisson point processes with σ-finite intensity measure Λ_i. Denote by P_{B,Λ_i} the restriction of P_{Λ_i} to $\mathcal{M}_B$. Then by (3.4) for every B with $\Lambda_i(B)<\infty$, $i=1,2$

$$\frac{\mathrm{d}P_{B,\Lambda_1}}{\mathrm{d}P_{B,\Lambda_2}}(\varphi) = \exp\left\{\int_B \ln\lambda\,\mathrm{d}\varphi - \Lambda_1(B) + \Lambda_2(B)\right\}. \tag{3.5}$$

Assume now $P_{\Lambda_1} \sim P_{\Lambda_2}$ and choose $B_1 \subseteq B_2 \subseteq \ldots \subseteq \mathcal{X}$, $B_i \in \mathcal{A}$ such that $\bigcup_{i=1}^{\infty} B_i = \mathcal{X}$. Then $\mathcal{M}_{B_1} \subseteq \mathcal{M}_{B_2} \subseteq \ldots$ and $\mathcal{M} = \sigma(\bigcup_{i=1}^{\infty} \mathcal{M}_{B_i})$. Hence $\frac{\mathrm{d}P_{B_i,\Lambda_1}}{\mathrm{d}P_{B_i,\Lambda_2}}$ is an $\{\mathcal{M}_{B_n}\}$-martingale which converges to $\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}$ in $L_1(P_{\Lambda_2})$ and P_{Λ_2}-a. s. Consequently,

$$\lim_{n\to\infty} \int \left| \frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}(\varphi) - \exp\left\{\int_{B_n} \ln\lambda\,\mathrm{d}\varphi - \Lambda_1(B_n) + \Lambda_2(B_n)\right\}\right| P_{\Lambda_2}(\mathrm{d}\varphi) = 0. \tag{3.6}$$

Assume now $\int(\ln\lambda)^2\,\mathrm{d}\Lambda_2 < \infty$ and $\int(\lambda - 1 - \ln\lambda)\,\mathrm{d}\Lambda_2 < \infty$. Then

$$\int_{B_n} \ln\lambda\,\mathrm{d}\varphi - \Lambda_1(B_n) + \Lambda_2(B_n) = \int_{B_n} (\ln\lambda)\,\mathrm{d}(\varphi - \Lambda_2) - \int_{B_n} (\lambda - 1 - \ln\lambda)\,\mathrm{d}\Lambda_2$$

converges in the sense of $L_2(P_{\Lambda_2})$ to $\int(\ln\lambda)\,\mathrm{d}(\varphi - \Lambda_2) - \int(\lambda - 1 - \ln\lambda)\,\mathrm{d}\Lambda_2$. From the represention (3.6) we get

$$\frac{\mathrm{d}P_{\Lambda_1}}{\mathrm{d}P_{\Lambda_2}}(\varphi) = \exp\left\{\int(\ln\lambda)\,\mathrm{d}(\varphi - \Lambda_2) - \int(\lambda - 1 - \ln\lambda)\,\mathrm{d}\Lambda_2\right\}. \tag{3.7}$$

Now we turn to the concept of local asymptotic normality of families of distributions of Poisson processes. To be more precise, we suppose that $(\mathcal{X}_i, \mathcal{A}_i)$, $i = 1,2,\ldots$ are measurable spaces which play the role of state spaces. Introduce $(M_i, \mathcal{M}_i)$ in the same way as $(M, \mathcal{M})$. Suppose $\Theta \subseteq \mathbb{R}_k$ where $\mathbb{R}_k$ is the k-dimensional Euclidean space. Assume that the interior Θ° of Θ is nonempty and $\Lambda_{i,\vartheta}$, $\vartheta \in \Theta$ are σ-finite measures on $(\mathcal{X}_i, \mathcal{A}_i)$. We suppose also that

$$P_{\Lambda_i,\vartheta_1} \sim P_{\Lambda_i,\vartheta_2} \tag{3.8}$$

for every $i = 1,2,\ldots$, $\vartheta_1, \vartheta_2 \in \Theta$. For a $k \times k$-matrix $A = (a_{ij})_{1\le i,j\le k}$ we define the norm

$$\|A\| = \left(\sum_{i,j=1}^{n} a_{i,j}^2\right)^{\frac{1}{2}}.$$

Now fix $\vartheta_0 \in \Theta^\circ$ and a sequence A_n of $k \times k$-matrices with $\|A_n\| \overset{n\to\infty}{\longrightarrow} 0$. Introduce the local parameter h in the following way

$$Q_{n,h} := P_{\mu_{n},h} \tag{3.9}$$

where $\mu_{n,h} = \Lambda_{n,\vartheta_0 + A_n h}$, $h \in H_n = \{h : \vartheta_0 + A_n h \in \Theta\}$. The family $(Q_{n,h})_{h\in H_n}$ is called locally asymptotically normal if there is a sequence of r. v. $Z_n : (M_n, \mathcal{M}_n) \longrightarrow$

$(\mathbb{R}_k, \mathcal{B}_k)$ called central sequence, such that the following expansion of the logarithm of the likelihood ratio holds

$$\ln \frac{\mathrm{d}Q_{n,h}}{\mathrm{d}Q_{n,0}} = \langle Z_n, h\rangle \, ||h||^2 + R_n \tag{3.10}$$

where

$$\mathcal{L}(Z_n|Q_{n,0}) \Rightarrow N(0,I) \quad \text{as } n \to \infty \tag{3.11}$$

and

$$R_n \overset{n\to\infty}{\longrightarrow} 0 \qquad Q_{n,0}\text{-stochastically.} \tag{3.12}$$

I is the k-dimensional unit matrix.

Remark 2. The notion of local asymptotic normality (LAN) goes back to LeCam [7]. We refer to Strasser [16] p. 408 for further historical remarks.

Set

$$\begin{aligned} \lambda_{n,h} &= \frac{\mathrm{d}\mu_{n,h}}{\mathrm{d}\mu_{n,0}} \\ L_{n,h} &= \ln \frac{\mathrm{d}Q_{n,h}}{\mathrm{d}Q_{n,0}}. \end{aligned}$$

First of all we present such necessary conditions for the LAN-property do not using the special structure of the central sequence.

Proposition 4. If $(Q_{n,h})_{h\in H_n}$ from (3.9) has the LAN-property then

$$\{Q_{n,h}\} \lhd \rhd \{Q_{n,0}\} \tag{3.13}$$

$$\int_{\{|\ln\lambda_{n,h}|>\varepsilon\}} \frac{(\ln\lambda_{n,h})^2}{1+(\ln\lambda_{n,h})^2}\, \mathrm{d}\mu_{n,0} \quad \overset{n\to\infty}{\longrightarrow} \quad 0 \tag{3.14}$$

$$\int \frac{(\ln\lambda_{n,h})^2}{1+(\ln\lambda_{n,h})^2}\, \mathrm{d}\mu_{n,0} \quad \overset{n\to\infty}{\longrightarrow} \quad ||h||^2. \tag{3.15}$$

Proof. The LAN-property implies that

$$\mathcal{L}(L_{n,0}|Q_{n,0}) \;\Rightarrow\; N\left(-\frac{1}{2}||h||^2, ||h||^2\right).$$

The application of Theorem 4 yields (3.13), (3.14) and (3.15). □

To prepare the next Theorem we need a suitable approximation of the likelihood ratio. For every $N = 1, 2, \ldots$ we denote by $Q^N_{n,h}$ the distribution of a Poisson process with intensity measure

$$\Lambda^N_{n,h}(B) = \int_B \left(\lambda_{n,h} I_{\{|\ln\lambda_{n,h}|\le N\}} + I_{\{|\ln\lambda_{n,h}|>N\}}\right) \mathrm{d}\mu_{n,0}.$$

Lemma 1. If $\{Q_{n,h}\} \triangleleft \triangleright \{Q_{n,0}\}$ then

$$\int \left| \frac{\mathrm{d}Q_{n,h}}{\mathrm{d}Q_{n,0}} - \frac{\mathrm{d}Q_{n,h}^N}{\mathrm{d}Q_{n,0}} \right| \mathrm{d}Q_{n,0} \leq a(n,N)$$

where $a(n,N(n)) \to 0$ for every sequence $N(n) \to \infty$ as $n \to \infty$.

Proof. We use the well-known inequality (see Strasser [16]) for the variational distance $\|P_1 - P_2\|$

$$\begin{aligned} \|P_1 - P_2\| = \int \left| \frac{\mathrm{d}P_1}{\mathrm{d}Q} - \frac{\mathrm{d}P_2}{\mathrm{d}Q} \right| \mathrm{d}Q &\leq 2\left(1 - H_{\frac{1}{2}}^2(P_1,P_2)\right)^{\frac{1}{2}} \\ &\leq \left[8\left(1 - H_{\frac{1}{2}}(P_1,P_2)\right)\right]^{\frac{1}{2}}. \end{aligned}$$

By the definition of $Q_{n,h}^N$ and (2.19)

$$H_{\frac{1}{2}}(Q_{n,h}, Q_{n,h}^N) = \exp\left\{-\frac{1}{2}\int_{|\ln \lambda_{n,h}|>N} \left(\sqrt{\lambda_{n,h}} - 1\right)^2 \mathrm{d}\mu_{n,0}\right\} \tag{3.16}$$

Set

$$b_1(n,N) = \frac{1}{2}\int_{\{\ln \lambda_{n,h}>N\}} \left(\sqrt{\lambda_{n,h}} - 1\right)^2 \mathrm{d}\mu_{n,0}$$

and note that $b_1(n,N)$ is nonincreasing in N. Hence for every $N(n) \to \infty, n \to \infty$ by Theorem 1 and $\{Q_{n,h}\} \triangleleft \{Q_{n,0}\}$

$$\limsup_{n\to\infty} b_1(n,N(n)) \leq \limsup_{N\to\infty} \limsup_{n\to\infty} b_1(n,N) = 0.$$

Analogously, for

$$\begin{aligned} b_2(n,N) &= \frac{1}{2}\int_{\{-\ln \lambda_{n,N}>N\}} \left(\sqrt{\lambda_{n,h}} - 1\right)^2 \mathrm{d}\mu_{n,0} \\ &= \frac{1}{2}\int_{\{\ln \frac{1}{\lambda_{n,N}}>N\}} \left(\sqrt{\frac{1}{\lambda_{n,h}}} - 1\right)^2 \mathrm{d}\mu_{n,h} \end{aligned}$$

we obtain from $\{Q_{n,0}\} \triangleleft \{Q_{n,h}\}$ and Theorem 1 that

$$\limsup_{n\to\infty} b_2(n,N(n)) = 0.$$

To complete the proof we have only to set

$$a(n,N) = [8\,(1 - \exp\{-(b_1(n,N) + b_2(n,N))\})]^{\frac{1}{2}}. \qquad \square$$

An essential step for proving the LAN-property is the linearization of the logarithm of the likelihood ratio. For this aim we linearize $\lambda_{n,h}$ and introduce a suitable concept of L_2-differentiability for the sequence $\lambda_{n,h}$.

Definition 1. Suppose $\Lambda_{n,\vartheta}$, $\vartheta \in \Theta \subseteq I\!R_k$ are σ-finite measures on $(\mathcal{X}^{(n)}, \mathcal{A}^{(n)})$ with $\Lambda_{n,\vartheta} \ll \Lambda_{n,\vartheta_0}$ for every $\vartheta \in \Theta$. Set

$$\lambda_{n,h} = \frac{\mathrm{d}\Lambda_{n,\vartheta_0+A_n h}}{\mathrm{d}\Lambda_{n,\vartheta_0}}.$$

The sequence $\{\mu_{n,h},\ h \in H_n\} = \{\Lambda_{n,\vartheta_0+A_n h},\ h \in H_n\}$ is called $L_2(\mu_{n,0})$-differentiable with derivative $\dot{l}_n$ if $\dot{l}_n$ are $\mathcal{A}_n - \mathcal{B}_k$-measurable mappings into $I\!R_k$ with $\dot{l}_n \in L_2(\mu_{n,0})$ and

$$\int \left(2\left(\sqrt{\lambda_{n,h}}-1\right) - \left\langle \dot{l}_n, A_n h\right\rangle\right)^2 \mathrm{d}\mu_{n,0} \overset{n\to\infty}{\longrightarrow} 0 \tag{3.17}$$

as $n \to \infty$ for every $h \in H$.

Now we are ready to formulate conditions for the LAN-property under the contiguity condition which is necessary in view of Theorem 4. Furthermore we give an explicit expression for the central sequence. Recall that $\mu_{n,h} = \Lambda_{n,\vartheta_0+A_n h}$, $Q_{n,h} = P_{\Lambda_{n,\vartheta_0+A_n h}}$.

Theorem 5. Suppose $Q_{n,h} \sim Q_{n,0}$ and $\{Q_{n,h}\} \lhd \rhd \{Q_{n,0}\}$ for every $h \in H_n$. Assume $\{\mu_{n,h},\ h \in H_n\}$ is $L_2(\mu_{n,0})$-differentiable with derivative $\dot{l}_n$. If the conditions (3.14) and (3.15) are fulfilled then $Q_{n,h},\ h \in H_n$ has the LAN-property and

$$Z_n(\varphi) = A_n^T \int \dot{l}_n \,\mathrm{d}(\varphi - \mu_{n,0})$$

is a central sequence.

Proof. Set $B_{n,N} = \{|\ln \lambda_{n,h}| \le N\}$. Then

$$\begin{aligned} (\ln x)^2 &\preceq \left(\sqrt{x}-1\right)^2 \\ x - 1 - \ln x &\preceq \left(\sqrt{x}-1\right)^2 \end{aligned}$$

on $\{x :, |\ln x| \le N\}$ and $\int \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} < \infty$ imply that the representation (3.7) may be applied to $L_{n,h}^N = \ln \frac{\mathrm{d}Q_{n,h}^N}{\mathrm{d}Q_{n,0}}$. Hence with $B_{n,N} = \{|\ln \lambda_{n,h}| \le N\}$

$$\begin{aligned} L_{n,h}^N &= \int_{B_{n,N}} (\ln \lambda_{n,h})\, \mathrm{d}(\varphi - \mu_{n,0}) - \int_{B_{n,N}} (\lambda_{n,h} - 1 - \ln \lambda_{n,h})\, \mathrm{d}\mu_{n,0} \\ &= \int_{B_{n,N}} 2\left(\sqrt{\lambda_{n,h}}-1\right) \mathrm{d}(\varphi - \mu_{n,0}) \\ &\quad + \int_{B_{n,N}} \left(\ln \lambda_{n,h} - 2\left(\sqrt{\lambda_{n,h}}-1\right)\right) \mathrm{d}(\varphi - \mu_{n,0}) \\ &\quad + \int_{B_{n,N}} \left(\frac{(\ln \lambda_{n,h})^2}{2(1+(\ln \lambda_{n,h})^2)} - (\lambda_{n,h} - 1 - \ln \lambda_{n,h})\right) \mathrm{d}\mu_{n,0} \\ &\quad - \int_{B_{n,N}} \frac{(\ln \lambda_{n,h})^2}{2(1+(\ln \lambda_{n,h})^2)}\, \mathrm{d}\mu_{n,0} \\ &= T_{1,n} + \ldots + T_{4,n}. \end{aligned}$$

We obtain from (3.1) and the inequality $(a+b)^2 \leq 2(a^2+b^2)$

$$\begin{aligned}
&E_{Q_{n,0}}\left(\langle Z_n, h\rangle - T_{1,n}\right)^2 \\
&\leq 2\int \left(2\left(\sqrt{\lambda_{n,h}}-1\right) - \left\langle A_n^T \dot{l}_n, h\right\rangle\right)^2 \mathrm{d}\mu_{n,0} \qquad (3.18)\\
&\quad + \int_{\{|\ln \lambda_{n,h}|>N\}} 8\left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \\
&\leq 2\int \left(2\left(\sqrt{\lambda_{n,h}}-1\right) - \left\langle A_n^T \dot{l}_n, h\right\rangle\right)^2 \mathrm{d}\mu_{n,0} + 16(b_1(n,N)+b_2(n,N)) \qquad (3.19)
\end{aligned}$$

with $b_i(n,N)$ from the proof of Lemma 1. Set for $t>0$

$$\begin{aligned}
\delta_2(t) &= \sup_{\{x:\,|\ln x|\leq t\}} \left|\ln x - 2\left(\sqrt{x}-1\right)\right|^2 \frac{1+(\ln x)^2}{(\ln x)^2} \\
\text{and}\quad \delta_3(t) &= \sup_{\{x:\,|\ln x|\leq t\}} \left|\frac{(\ln x)^2}{1+(\ln x)^2} - (x-1-\ln x)\right| \frac{1+(\ln x)^2}{(\ln x)^2}
\end{aligned}$$

and note that $\delta_i(t) \stackrel{t\to 0}{\longrightarrow} 0$. To estimate $T_{2,n}$ we apply (3.1) and obtain

$$\begin{aligned}
E_{Q_{n,0}}T_{2,n}^2 &\leq \delta_2(\varepsilon)\int \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0} \qquad (3.20)\\
&\quad +\delta_2(N)\int_{\{|\ln \lambda_{n,h}|>\varepsilon\}} \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0}.
\end{aligned}$$

Furthermore

$$\begin{aligned}
|T_{3,n}| &\leq \delta_3(\varepsilon)\int \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0} \qquad (3.21)\\
&\quad +\delta_3(N)\int_{\{|\ln \lambda_{n,h}|>\varepsilon\}} \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0}.
\end{aligned}$$

By assumption (3.14) we find a sequence $\varepsilon_n \to 0$ such that

$$\int_{\{|\ln \lambda_{n,h}|>\varepsilon_n\}} \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0} \stackrel{n\to\infty}{\longrightarrow} 0. \qquad (3.22)$$

Now we choose a sequence $N(n) \stackrel{n\to\infty}{\longrightarrow} \infty$ such that

$$\delta_i(N(n))\cdot \int_{\{|\ln \lambda_{n,h}|>\varepsilon_n\}} \frac{(\ln \lambda_{n,h})^2}{1+(\ln \lambda_{n,h})^2}\,\mathrm{d}\mu_{n,0} \stackrel{n\to\infty}{\longrightarrow} 0, \quad i=1,2. \qquad (3.23)$$

We have $T_{4,n} \to -\frac{1}{2}\|h\|^2$ by assumptions (3.15) and (3.22). The inequalities (3.18), (3.20) and (3.21) show that

$$\begin{aligned}
&\limsup_{n\to\infty} E_{Q_{n,0}}\left(\langle Z_n, h\rangle - L_{n,h}^{N(n)} + \frac{1}{2}\|h\|^2\right)^2 \\
&\leq \limsup_{n\to\infty} \int \left(\left(2\sqrt{\lambda_{n,h}}-1\right) - \left\langle A_n^T \dot{l}_n, h\right\rangle\right)^2 \mathrm{d}\mu_{n,0} \\
&\quad + \limsup_{n\to\infty} 16(b_1(n,N(n))+b_2(n,N(n))).
\end{aligned}$$

The second term vanishes by the same arguments as in the proof of Lemma 1. The first term on the right hand side is zero by the $L_2(\mu_{n,0})$-differentiability. Thus we arrive at

$$\lim_{n\to\infty} E_{Q_{n,0}} \left(\langle Z_n, h\rangle - L_{n,h}^{N(n)} + \frac{1}{2}||h||^2 \right)^2 = 0$$

which implies

$$\frac{\mathrm{d}Q_{n,h}^{N(n)}}{\mathrm{d}Q_{n,0}} = \exp\left\{ \langle Z_n, h\rangle - \frac{1}{2}||h||^2 + R_n(h) \right\} \tag{3.24}$$

where $R_n(h) \to 0$ $Q_{n,0}$-stochastically. We obtain from Lemma 1 that

$$\frac{\mathrm{d}Q_{n,h}}{\mathrm{d}Q_{n,0}} = \exp\left\{ \langle Z_n, h\rangle - \frac{1}{2}||h||^2 + \widetilde{R}_n(h) \right\} \tag{3.25}$$

with some $\widetilde{R}_n$ which converges $Q_{n,0}$-stochastically to zero. From Theorem 4 we already know that

$$\mathcal{L}\left(\ln \frac{\mathrm{d}Q_{n,h}}{\mathrm{d}Q_{n,0}} | Q_{n,0} \right) \Rightarrow N\left(-\frac{1}{2}||h||^2, ||h||^2 \right)$$

for every $h \in I\!R_k$. Hence by the Cramer–Wold technique

$$\mathcal{L}(Z_n | Q_{n,0}) \Rightarrow N(0, I)$$

which completes the proof. □

We derived the asymptotic normality of Z_n from the asymptotic normality of $L_{n,h}$. To do this one has to verify the Lindeberg condition (3.14), (3.15) for every fixed h. As $\lambda_{n,h}$ depends on h nonlinearly, in general, these conditions are not easy to handle. Therefore we now directly impose conditions on $\dot{l}_n$ to guarantee the asymptotic normality of Z_n.

Consider $\dot{l}_n$ as column vector and introduce the matrix Σ_n by

$$\Sigma_n = \int \dot{l}_n \dot{l}_n^T \, \mathrm{d}\mu_{n,0}.$$

Note that the covariance matrix C_{Z_n} of Z_n from Theorem 5 is given by $A_n^T \Sigma_n A_n$. As the distribution of Z_n is aimed to converge to $N(0, I)$ it is natural to require that $C_{Z_n} = I$. If $\det(\Sigma_n) \neq 0$ the condition $C_{Z_n} = I$ can be fulfilled with $A_n = \Sigma_n^{-\frac{1}{2}}$. Note that in this case

$$\int \left\langle A_n^T \dot{l}_n, h \right\rangle^2 \mathrm{d}\mu_{n,0} = ||h||^2.$$

Furthermore, if $\mu_{n,h}$ is $L_2(\mu_{n,0})$ differentiable we obtain

$$\lim_{n\to\infty} \int 4\left(\sqrt{\lambda_{n,h}} - 1 \right)^2 \mathrm{d}\mu_{n,0} = ||h||^2 \tag{3.26}$$

and

$$\lim_{n\to\infty} \int \left| 4\left(\sqrt{\lambda_{n,h}} - 1 \right)^2 - \left\langle A_n^T \dot{l}_n, h \right\rangle^2 \right| \mathrm{d}\mu_{n,0} = 0. \tag{3.27}$$

Theorem 6. Suppose $Q_{n,h} \sim Q_{n,0}$ and $\{Q_{n,h}\} \lhd \rhd \{Q_{n,0}\}$ for every $h \in \mathbb{R}_k$. Assume $\{\mu_{n,h},\ h \in H_n\}$ is $L_2(\mu_{n,0})$ differentiable with derivative $\dot{l}_n$ and $\det(\Sigma_n) \neq 0$ for every n and $A_n = \Sigma_n^{-\frac{1}{2}}$. If

$$Z_n(\varphi) = \Sigma_n^{-\frac{1}{2}} \int \dot{l}_n \, \mathrm{d}(\varphi - \mu_{n,0})$$

then $\{Q_{n,h}\}$ has the LAN-property with central sequence Z_n iff

$$\lim_{n\to\infty} \int_{\{\|\Sigma_n^{-\frac{1}{2}} \dot{l}_n\| > \varepsilon\}} \|\Sigma_n^{-\frac{1}{2}} \dot{l}_n\|^2 \, \mathrm{d}\mu_{n,0} = 0 \tag{3.28}$$

for every $\varepsilon > 0$.

Proof. The proof is splitted into several steps.

1. We have from (3.2)

$$\begin{aligned} E_{Q_{n,0}} \exp\{i\langle Z_n, h\rangle\} &= \exp\left\{\int \left(e^{i\left\langle \Sigma_n^{-\frac{1}{2}} \dot{l}_n, h\right\rangle} - 1 - i\left\langle \Sigma_n^{-\frac{1}{2}} \dot{l}_n, h\right\rangle\right) \mathrm{d}\mu_{n,0}\right\} \\ &= \exp\left\{\int \left(e^{i\langle x, h\rangle} - 1 - i\langle x, h\rangle\right) \kappa_n(\mathrm{d}x)\right\}. \end{aligned}$$

Consequently, Z_n is an infinitely divisible random vector with $E_{Q_{n,0}} Z_n = 0, C_{Z_n} = I$. The criteria for the weak convergence of the distribution of such vectors to a standard normal distribution yield that

$$\mathcal{L}(Z_n | Q_{n,0}) \Rightarrow N(0, I)$$

iff

$$\int_{\{\|x\| > \varepsilon\}} \|x\|^2 \kappa_n(\mathrm{d}x) \overset{n\to\infty}{\longrightarrow} 0 \tag{3.29}$$

for every $\varepsilon > 0$. But (3.29) is the same as (3.28), which gives the necessity of (3.28).

2. To prove the sufficiency we use a similar splitting of $L_{n,h}^N$ as in the proof of Theorem 5. Set

$$\begin{aligned} L_{n,h}^N &= \int_{B_{n,N}} 2\left(\sqrt{\lambda_{n,h}} - 1\right) \mathrm{d}(\varphi - \mu_{n,0}) \\ &\quad + \int_{B_{n,N}} \left(\ln \lambda_{n,h} - 2\left(\sqrt{\lambda_{n,h}} - 1\right)\right) \mathrm{d}(\varphi - \mu_{n,0}) \\ &\quad + \int_{B_{n,N}} \left(2\left(\sqrt{\lambda_{n,h}} - 1\right)^2 - (\lambda_{n,h} - 1 - \ln \lambda_{n,h})\right) \mathrm{d}\mu_{n,0} \\ &\quad - \int_{B_{n,N}} 2\left(\sqrt{\lambda_{n,h}} - 1\right)^2 \mathrm{d}\mu_{n,0} \\ &= S_{1,n} + \ldots + S_{4,n}. \end{aligned}$$

To proceed as in the proof of Theorem 5 we put

$$\begin{aligned} \Delta_2(\varepsilon) &= \sup_{\{x:,|2(\sqrt{x}-1)|\le\varepsilon\}} \left|\ln x - 2\left(\sqrt{x}-1\right)\right|^2 \left(\sqrt{x}-1\right)^{-2} \\ \text{and}\quad \Delta_3(\varepsilon) &= \sup_{\{x:,|2(\sqrt{x}-1)|\le\varepsilon\}} \left|2\left(\sqrt{x}-1\right)^2 - (x-1-\ln x)\right| \left(\sqrt{x}-1\right)^{-2} \end{aligned}$$

and note that $\Delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$. Furthermore for every x with $|\ln x| \le N$

$$\begin{aligned} &\left|\ln x - 2\left(\sqrt{x}-1\right)\right| \le \Delta_2(e^N)\left(\sqrt{x}-1\right)^2 \\ \text{and}\quad &\left|2\left(\sqrt{x}-1\right)^2 - (x-1-\ln x)\right| \le \Delta_3(e^N)\left(\sqrt{x}-1\right)^2. \end{aligned}$$

By the same arguments as in the proof of Theorem 5 we get for $i = 2, 3$

$$\begin{aligned} E_{Q_{n,0}} S_{2,n}^2 &\le \Delta_2(\varepsilon)\int \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \\ &\quad + \Delta_2(e^N)\cdot \int_{\{|2(\sqrt{\lambda_{n,h}}-1)|>\varepsilon\}\cap B_{n,N}} \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} S_{3,n}^2 &\le \Delta_3(\varepsilon)\int \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \\ &\quad + \Delta_3(e^N)\cdot \int_{\{|2(\sqrt{\lambda_{n,h}}-1)|>\varepsilon\}\cap B_{n,N}} \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0}. \end{aligned}$$

To estimate the second term on the right hand side we set

$$\psi_{1,n} = 2\left(\sqrt{\lambda_{n,h}}-1\right) I_{B_{n,N}},\ \psi_{2,n} = \langle \Sigma_n^{-\frac{1}{2}} \dot{l}_n, h\rangle I_{B_{n,N}}$$

and note that by the definition of $B_{n,N}$

$$\lambda_{n,h} \le e^N$$

on $B_{n,N}$ and consequently $|\psi_{1,n}| \le e^N$. Hence for every $\varepsilon > 0$

$$\begin{aligned} \int_{\{|\psi_{1,n}-\psi_{2,n}|>\frac{\varepsilon}{2}\}} \psi_{1,n}^2 \,\mathrm{d}\mu_{n,0} &\le e^N \mu_{n,0}\left(\left\{|\psi_{1,n}-\psi_{2,n}| > \frac{\varepsilon}{2}\right\}\right) \\ &\le \frac{4e^N}{\varepsilon^2}\int |\psi_{1,n}-\psi_{2,n}|^2 \,\mathrm{d}\mu_{n,0}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_{\{|\psi_{1,n}|>\varepsilon\}} \psi_{1,n}^2 \,\mathrm{d}\mu_{n,0} &\le \int_{\{|\psi_{2,n}|>\frac{\varepsilon}{2}\}} \psi_{1,n}^2 \,\mathrm{d}\mu_{n,0} + \int_{\{|\psi_{1,n}-\psi_{2,n}|>\frac{\varepsilon}{2}\}} \psi_{1,n}^2 \,\mathrm{d}\mu_{n,0} \\ &\le 2\int |\psi_{1,n}-\psi_{2,n}|^2 \,\mathrm{d}\mu_{n,0} + 2\int_{\{|\psi_{2,n}|>\frac{\varepsilon}{2}\}} \psi_{2,n}^2 \,\mathrm{d}\mu_{n,0} \\ &\quad + \frac{4e^N}{\varepsilon^2}\int |\psi_{1,n}-\psi_{2,n}|^2 \,\mathrm{d}\mu_{n,0}. \end{aligned}$$

Hence

$$\int_{\{|\psi_{1,n}|>\varepsilon\}} \psi_{1,n}^2 \,\mathrm{d}\mu_{n,0} \le \left(2+\frac{4e^N}{\varepsilon^2}\right)\int |\psi_{1,n}-\psi_{2,n}|^2 \,\mathrm{d}\mu_{n,0} + 2\int_{\{|\psi_{2,n}|>\frac{\varepsilon}{2}\}} \psi_{2,n}^2 \,\mathrm{d}\mu_{n,0}. \tag{3.31}$$

Note that

$$\psi_{2,n}^2 I_{\{|\psi_{2,n}|>\frac{\varepsilon}{2}\}} \le \|h\|^2 \left\|\Sigma_n^{-\frac{1}{2}} \dot{l}_n\right\|^2 I_{\left\{\left\|\Sigma_n^{-\frac{1}{2}}\dot{l}_n\right\|\cdot\|h\|>\frac{\varepsilon}{2}\right\}}. \tag{3.32}$$

The inequalities (3.31), (3.32) and the assumptions (3.17), (3.28) imply that for every $\varepsilon>0, N>0$

$$\int_{\{|2(\sqrt{\lambda_{n,h}}-1)|>\varepsilon\}\cap B_{n,N}} \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \overset{n\to\infty}{\longrightarrow} 0.$$

Consequently there are $N(n)\to\infty$ and $\varepsilon_n\to 0$ as $n\to\infty$ such that for $i=2,3$

$$\Delta_i\left(e^{N(n)}\right)\int_{\{|2(\sqrt{\lambda_{n,h}}-1)|>\varepsilon_n\}\cap B_{n,N}} \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} \overset{n\to\infty}{\longrightarrow} 0. \tag{3.33}$$

Note that $\limsup_{n\to\infty}\int\left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0}<\infty$ as $\{Q_{n,h}\}\lhd\{Q_{n,0}\}$. Hence (3.30) and (3.33) imply $E_{Q_{n,0}}S_{2,n}^2 \overset{n\to\infty}{\longrightarrow} 0$. Similarly, $E_{Q_{n,0}}S_{3,n} \overset{n\to\infty}{\longrightarrow} 0$. The statements $E_{Q_{n,0}}(S_{1,n}-\langle Z_n,h\rangle)^2 \overset{n\to\infty}{\longrightarrow} 0$ and $S_{4,n} \overset{n\to\infty}{\longrightarrow} -\frac{1}{2}\|h\|$ may be established as in the proof of Theorem 5. Consequently,

$$L_{n,h}^N = \langle Z_n,h\rangle - \frac{1}{2}\|h\|^2 + R_n(h)$$

where $R_n(h)\to 0$ $Q_{n,0}$-stochastically. To complete the proof it remains to apply the same arguments as in the proof of Theorem 5. □

Remark 3. Necessary and sufficient conditions for $\{Q_{n,h}\}\lhd\rhd\{Q_{n,0}\}$ are given in (2.24), in Theorem 1. But sometimes it is more convenient to use the sufficient conditions fromulated in Corollary 1. To be more precise let $Q_{n,h}\sim Q_{n,0}$ for every $n, h\in H_n$. Note that

$$I_1(\mu_{n,h},\mu_{n,0}) = \int (\lambda_{n,h}\ln\lambda_{n,h}-\lambda_{n,h}+1)\,\mathrm{d}\mu_{n,0}$$

$$\text{and}\quad I_1(\mu_{n,0},\mu_{n,h}) = I_0(\mu_{n,h},\mu_{n,0}) = \int (\lambda_{n,h}-1-\ln\lambda_{n,h})\,\mathrm{d}\mu_{n,0}.$$

Hence

$$I_0(\mu_{n,h},\mu_{n,0}) + I_1(\mu_{n,h},\mu_{n,0}) = \int (\lambda_{n,h}-1)\ln\lambda_{n,h}\,\mathrm{d}\mu_{n,0}.$$

Consequently, by the Corollary of Theorem 1,

$$\limsup_{n\to\infty}\int(\lambda_{n,h}-1)\ln\lambda_{n,h}\,\mathrm{d}\mu_{n,0}<\infty$$

implies $\{Q_{n,h}\}\lhd\rhd\{Q_{n,0}\}$.

Now we study a situation in which the assumption of Theorem 6 are fulfilled. Suppose $\Lambda_\vartheta,\ \vartheta\in\Theta\subseteq I\!R_1$ is a family of equivalent σ-finite measures with

$$J_{\frac{1}{2}}(\Lambda_\vartheta,\Lambda_{\vartheta_0})<\infty \tag{3.34}$$

for every $\vartheta\in\Theta$. Assume $\vartheta_0\in\Theta^\circ$ and put

$$\lambda_\vartheta=\frac{\mathrm{d}\Lambda_\vartheta}{\mathrm{d}\Lambda_{\vartheta_0}}.$$

Suppose λ_ϑ is $L_2(\Lambda_{\vartheta_0})$ differentiable in the sense that there is some $\dot{l}_\vartheta\in L_2(\Lambda_{\vartheta_0})$ such that

$$\int\left(2\left(\sqrt{\lambda_\vartheta}-1\right)-\dot{l}_{\vartheta_0}(\vartheta-\vartheta_0)\right)^2\mathrm{d}\Lambda_{\vartheta_0}=o\left(|\vartheta-\vartheta_0|^2\right). \tag{3.35}$$

Suppose we observe i.i.d. Poisson point processes $\Phi_1,\ldots,\Phi_n$ with common distribution P_{Λ_ϑ}. We see from (2.34) that the distribution of $\ln\frac{\mathrm{d}P^n_{\Lambda_\vartheta}}{\mathrm{d}P^n_{\Lambda_0}}$ w.r.t. $P^n_{\Lambda_0}$ is identical with the distribution of $\ln\frac{\mathrm{d}P_{n\Lambda_\vartheta}}{\mathrm{d}P_{n\Lambda_0}}$ w.r.t. $P_{n\Lambda_0}$. Consequently $\left\{\left(P_{\Lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}}\right)^n\right\}$ has the LAN property iff $\left\{P_{n\left(\Lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}\right)}\right\}$ has the LAN-property. To establish the LAN-property for $P_{n\Lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}}$ we apply Theorem 6. First of all we note that $\mu_{n,h}=n\left(\Lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}\right)$, $\lambda_{n,h}=\lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}$. Set $\dot{l}_n=\dot{l}_{\vartheta_0}$. Then

$$\begin{aligned}&\int\left(2\left(\sqrt{\lambda_{n,h}}-1\right)-\langle\dot{l}_n,\frac{1}{\sqrt{n}}h\rangle\right)^2\mathrm{d}\mu_{n,0} \\ =&\int\left(2\sqrt{n}\left(\sqrt{\lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}}-1\right)-\dot{l}_{\vartheta_0}h\right)^2\mathrm{d}\Lambda_{\vartheta_0}\overset{n\to\infty}{\longrightarrow}0.\end{aligned} \tag{3.36}$$

Hence (3.17) is fulfilled. Suppose $I(\vartheta_0)=\int\dot{l}^2_{\vartheta_0}\,\mathrm{d}\Lambda_0>0$. Then $\Sigma_n=nI(\vartheta_0)>0$ and

$$\int_{\left\{\left\|\Sigma_n^{-\frac{1}{2}}\dot{l}_n\right\|>\varepsilon\right\}}\left\|\Sigma_n^{-\frac{1}{2}}\dot{l}_n\right\|^2\mathrm{d}\mu_{n,0}=\frac{1}{I^2(\vartheta_0)}\int_{\{I(\vartheta_0)\sqrt{n}|\dot{l}_{\vartheta_0}|>\varepsilon\}}\dot{l}^2_{\vartheta_0}\,\mathrm{d}\Lambda_{\vartheta_0}\overset{n\to\infty}{\longrightarrow}0.$$

Hence (3.28) is fulfilled. Now we prove the contiguity $\{Q_{n,h}\}\lhd\rhd\{Q_{n,0}\}$. The relation (3.36) implies

$$\limsup_{n\to\infty}\int\left(2\left(\sqrt{\lambda_{n,h}}-1\right)\right)^2\mathrm{d}\mu_{n,0}<\infty. \tag{3.37}$$

For every $\varepsilon > 0$ we get from (3.36)

$$\limsup_{n\to\infty} \int_{\{|\lambda_{n,h}-1|>\varepsilon\}} \left(2\left(\sqrt{\lambda_{n,h}}-1\right)\right)^2 \mathrm{d}\mu_{n,0} \le \limsup_{n\to\infty} \int_{\{|\lambda_{n,h}-1|>\varepsilon\}} h^2 \dot{l}^2_{\vartheta_0}\, \mathrm{d}\Lambda_{\vartheta_0}.$$

Relation (3.37) yields

$$\Lambda_{\vartheta_0}(\{|\lambda_{n,h}-1|>\varepsilon\}) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence by the Lebesgue Theorem

$$\limsup_{n\to\infty} \int_{\{|\lambda_{n,h}-1|>\varepsilon\}} \left(2\left(\sqrt{\lambda_{n,h}}-1\right)\right)^2 \mathrm{d}\mu_{n,0} = 0.$$

Hence

$$\limsup_{N\to\infty}\limsup_{n\to\infty} \int_{\{\lambda_{n,N}>N\}} \left(\sqrt{\lambda_{n,h}}-1\right)^2 \mathrm{d}\mu_{n,0} = 0$$

and

$$\limsup_{N\to\infty}\limsup_{n\to\infty} \int_{\{\frac{1}{\lambda_{n,N}}>N\}} \left(\frac{1}{\sqrt{\lambda_{n,h}}}-1\right)^2 \mathrm{d}\mu_{n,h} = 0.$$

The contiguity $\{Q_{n,h}\} \lhd\rhd \{Q_{n,0}\}$ now follows from Theorem 1. Summarizing the results we get the following Proposition.

Proposition 5. Assume the family λ_ϑ is L_2-differentiable at ϑ_0 in the sense of (3.35), the measures Λ_ϑ, $\vartheta \in \Theta$ are equivalent and (3.34) is fulfilled. If $I(\vartheta_0) = \int \dot{l}^2_{\vartheta_0}\, \mathrm{d}\Lambda_{\vartheta_0} > 0$ then $\left(P_{\Lambda_{\vartheta_0+\frac{1}{\sqrt{n}}h}}\right)^n$ has the LAN-property.

(Received April 16, 1997.),

REFERENCES

[1] M. Brown: Discrimination of Poisson processes. Ann. Math. Statist. *42* (1971), 773–776.

[2] I. Csiszár: Eine informationstheoretische Ungleichung und ihre Anwendung auf den Beweis der Ergodizität Markoffscher Ketten. Publ. Math. Inst. Hungar. Acad. Sci., Ser. A *8* (1963), 85–108.

[3] J. Jacod and A. N. Shiryaev: Limit Theorems for Stochastic Processes. Springer–Verlag, Berlin 1987.

[4] A. F. Karr: Point Processes and their Statistical Inference. Marcel Dekker, New York 1986.

[5] Yu. A. Kutoyants: Parameter Estimation for Stochastic Processes. Helderman, Berlin 1984.

[6] Yu. A. Kutoyants: Statistical inference for spatial Poisson processes. Lab. de Stat. et Proc. Univ. du Maine, Le Mans, manuscript of forthcoming monography (1996).

[7] L. LeCam: Locally asymptotically normal families of distributions. Univ. Calif. Publ. Statist. *3* (1960), 37–98.

[8] F. Liese: Eine informationstheoretische Bedingung für die Äquivalenz unbegrenzt teilbarer Punktprozesse. Math. Nachr. *70* (1975), 183–196.

[9] F. Liese: Hellinger integrals of diffusion processes. Statistics *17* (1986), 63–78.

[10] F. Liese and I. Vajda: Convex Statistical Distances. Teubner, Leipzig 1987.
[11] U. Lorz: Sekundärgrösen Poissonscher Punktprozesse – Grenzwertsätze und Abschätzungen der Konvergenzgeschwindigkeit. Rostock. Math. Kolloq. *29* (1986), 99–111.
[12] U. Lorz: Beiträge zur Statistik unbegrenzt teilbarer Felder mit unabhängigen Zuwächsen. Dissertation, Univ. Rostock 1987.
[13] U. Lorz and L. Heinrich: Normal and Poisson approximation of infinitely divisible distribution function. Statistics *22* (1991), 627– 649.
[14] J. Mecke: Stationäre zufällige Maße auf lokal-kompakten Abelschen Gruppen. Z. Wahrsch. verw. Geb. *9* (1967), 36–58.
[15] V. V. Petrov: Sums of Independent Random Variables. Akademie–Verlag, Berlin 1975.
[16] H. Strasser: Mathematical Theory of Statistics. de Gruyter, Berlin 1985.
[17] I. Vajda: Theory of Statistical Inference and Information. Kluwer, Dordrecht 1989.

Prof. Dr. Friedrich Liese, Universität Rostock, Fachbereich Mathematik, Universitätsplatz 1, D-18055 Rostock. Germany.
e-mail: friedrich.liese@mathematik.uni-rostock.de

Dr. Udo Lorz, Technische Universität – Bergakademie Freiberg, Akademiestraße 6, D-09599 Freiberg. Germany.
e-mail: rektorat@zuv.tu-freiberg.de