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ADDITIVITIES IN FUZZY COALITION GAMES WITH SIDE-PAYMENTS[1]

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The fuzzy coalition game theory brings a more realistic tools for the mathematical modelling of the negotiation process and its results. In this paper we limit our attention to the fuzzy extension of the simple model of coalition games with side-payments, and in the frame of this model we study one of the elementary concepts of the coalition game theory, namely its "additivities", i. e., superadditivity, subadditivity and additivity in the strict sense. In the deterministic game theory these additivites indicate the structure of eventual cooperation, namely the extent of finally formed coalitions, if the cooperation is possible. The additivities in fuzzy coalition games play an analogous role. But the vagueness of the input data about the expected coalitional incomes leads to consequently vague validity of the superadditivity, subadditivity and additivity. In this paper we formulate the model of this vagueness depending on the fuzzy quantities describing the expected coalitional pay-offs, and we introduce some elementary results mostly determining the links between additivities in a deterministic coalition game and its fuzzy extensions.

1. INTRODUCTION

The fuzzy coalition game theory is a natural field of applications of the fuzzy set theory to an originally deterministic model of essentially deeply vague phenomenon. There are several models of coalition games differing in the degree of freedom in the distributions of coalition pay-offs among its members. The simplest type of coalition game is the game with side-payments which accepts the assumption of existence of a universal and linear representative of utility. This representative is used as a medium for the re-distribution of the total profit of a coalition among the players without any loss or other deformation of utility values transferred inside the coalition. In the practical situations, this assumption appears non-realistic, and it can be fulfilled only approximately. Anyhow, the games with side-payments are the first type of coalition games which were thoroughly investigated, and the results derived for them belong to the fundaments of the coalition game theory.

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In this paper we deal with a fuzzification of the mentioned type of game. The previous attempt to this problem was done in [9], and some elementary results were derived. Anyhow, the width of the considered game model justifies the endeavour to investigate its particular components separately and in more details. The concepts of superadditivity, subadditivity and additivity belong to the most significant. The fulfilment of these properties indicates the form of the negotiated coalition structure. If the game is superadditive then the negotiated stable distributions of profit (the "core" of the game), if such stable distribution exists, can be realized by the coalition of all players (cf. [12, 13]). On the other hand, if the game is subadditive then no cooperation can be achieved, and also in the additive game the pay-offs of players are their pay-offs in one-element coalitions without respect to the extent of the coalition in which they are distributed.

The fuzzification of the coalition game models of various types is natural (see [1, 3, 4]). It can include the possibility of a set of players to play more games parallelly with different "intensity" of participation (see [1], and certain pre-attempt was done also in [8]), or it can reflect the fact that, in the time of the negotiation and coalition forming, the terminal coalitional profits can be known only vaguelly (cf., [3, 4, 9]). The latter approach was chosen also in this paper. We suppose that the knowledge of the expected results of cooperation can be only vague, and the terminal result depends on numerous external factors. The vagueness of the input knowledge implies the vagueness in the validity of some important properties of the game like the existence of solution (see [4, 9]) or the validity of the superadditivity property (see [3]). Mainly, some of the first results presented in [3] are extended, completed and discussed in the following sections.

2. FUZZY QUANTITY

The concepts of fuzzy quantity and fuzzy number were delt in several works (see, for example, [2, 5, 6]) and they are well known as fuzzy set theoretical models of vague numbers. As we use them in the following sections to describe the vague pay-offs it is useful to remember briefly some concepts and notations related to them.

We denote by R the set of real numbers. Fuzzy subset a of R, with membership function $\mu_a : R \to [0,1]$ such that:

there exists $x_a \in R$ such that $\mu_a(x_a) = 1$, (1)

there exist $x_1,\ x_2 \in R,\ x_1 < x_a < x_2$, such that $\mu_a(x) = 0$ for all $x \notin [x_1, x_2]$, (2)

is called a *fuzzy quantity* and x_a is its *modal value*. The sets of fuzzy quantities will be denoted by $\mathbb{R}$.

If $a,\ b \in \mathbb{R}$ then their *sum* $a \oplus b$ is also fuzzy quantity with membership function

$$\mu_{a\oplus b}(x) = \sup_{y \in R} \left[\min(\mu_a(y),\ \mu_b(x-y))\right], \quad x \in R. \tag{3}$$

It is also possible to substract the fuzzy quantities by means of the formula $a \oplus (-b)$ where $\mu_{-b}(x) = \mu_b(-x)$ for all $x \in R$ (cf. [5] and [6]).

Fuzzy quantities can be compared and there exist numerous conceptions of ordering-like relations over $\mathbb{R}$. In this paper we use one of them which stresses the fuzzy charracter of the ordering of fuzzy numbers. If $a, b \in \mathbb{R}$ then the possibility of $a \succsim b$, i.e. of "*a is greater or equivalent to b*" is defined as a number

$$\nu(a \succsim b) = \sup_{\substack{x \in R,\, y \in R \\ x \geq y}} [\min(\mu_a(x),\, \mu_b(y))]\,. \tag{4}$$

It is also possible to define fuzzy "equality" between fuzzy quantities (see [5, 6]) as a fuzzy relation $a \sim b,\ a, b \in \mathbb{R}$ which is valid with possibility

$$\nu^*(a \sim b) = \sup_{x \in R} [\min(\mu_a(x),\, \mu_b(x))]\,. \tag{5}$$

Its relation to fuzzy "inequality" $a \succsim b$ is discussed in [6], e.g. Here, we remember that the logical conjunction $a \succsim b$ and $b \succsim a$ is not equivalent to $a \sim b$, what means a significant difference between fuzzy and crisp quantities. In general

$$\nu^*(a \sim b) = \nu^*(b \sim a) \leq \min\Big(\nu(a \succsim b),\, \nu(b \succsim a)\Big)\,. \tag{6}$$

If fuzzy quantities are partwise monotonous, i.e., their membership functions μ_a, μ_b are continuous and increasing for $x < x_a$ or $x < x_b$, $\mu_a(x) = 1$ for $x \in [x_a, x'_a]$ or $\mu_b(x) = 1$ for $x \in [x_b, x'_b]$ and they are decreasing (and continuous) for $x > x'_a$ or $x > x'_b$, respectively (where $x_a \leq x'_a$, $x_b \leq x'_b$) then the inequality in (6) turns into equality. The properties of the operation of summation $a \oplus b$ and of the ordering relation $a \succsim b$ are introduced in [5] and briefly summarized in [6].

3. DETERMINISTIC COALITION GAME WITH SIDE–PAYMENTS

The type of game whose fuzzy extension will be investigated in the following sections is in its deterministic form defined as a pair (I, v) where I is a (non-empty and finite) *set of players* and v is a mapping, $v : 2^I \to R$ such that for every *coalition* $K \subset I$, value $v(K) \in R$ represents the total pay-off of the coalition K. The mapping v is called the *characteristic function* of (I, v), and we suppose $v(\emptyset) = 0$.

We say that the game (I, v) is *superadditive* if for all pairs of disjoint coalitions $K, L \subset I,\ K \cap L = \emptyset$,

$$v(K \cup L) \geq v(K) + v(L), \tag{7}$$

we say that the game is *subadditive* iff for any pair $K, L \subset I,\ K \cap L = \emptyset$, the inequality

$$v(K \cup L) \leq v(K) + v(L) \tag{8}$$

holds, and we say that it is *additive* iff it is both, superadditive and subadditive. Moreover, we say that (I, v) is *convex* iff for any pair of coalitions $K, L \subset I$

$$v(K \cup L) + v(K \cap L) \geq v(K) + v(L). \tag{9}$$

It is evident that convexity implies superadditivity.

Finally, let us denote for every coalition $K \subset T$ by $\mathcal{V}(K) \subset R^I$ the set

$$\mathcal{V}(K) = \left\{ \boldsymbol{x} = (x_i)_{i \in I} \in R^I : \sum_{i \in K} x_i \le v(K) \right\}. \tag{10}$$

Then it can be easily seen (cf. [9]) that the game (I, v) is superadditive if and only if for any $K,\ L \subset I,\ K \cap L = \emptyset$, the inclusion

$$\mathcal{V}(K \cup L) \supset \mathcal{V}(K) \cap \mathcal{V}(L) \tag{11}$$

holds. Each vector $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$ or, more generally, $\boldsymbol{x}_K = (x_i)_{i \in K} \in R^K$ is called an *imputation* in the game (I, v).

4. FUZZY-QUANTITIES-BASED APPROACH

Let us consider a coalition game with side-payments (I, v), and let us suppose that the crisp numbers $v(K)$, $K \subset I$, are not exactly known. It means that we can substitute them by vague (i. e. fuzzy) quantities. Then the pair (i, w) where $w : 2^I \to I\!R$, i. e., $w(K)$ is a fuzzy quantity with membership function $\mu_K : R \to [0, 1]$, such that

$$\mu_K(v(K)) = 1, \tag{12}$$

and for $K = \emptyset$, $\mu_\emptyset(0) = 1$, $\mu_\emptyset(x) = 0$ for $x \neq 0$, is called a *fuzzy extension of the game* (I, v). As follows from (12), $v(K)$ is (not necessarily single) modal value of $w(K)$. The mapping w will be called *fuzzy characteristic function of* (I, w).

4.1. Fuzzy superadditivity

Let (I, v) be a coalition game with side-payments and (I, w) be its fuzzy extension. Then (I, w) is *fuzzy superadditive* iff for any pair of disjoint coalitions $K,\ L \subset I$, $K \cap L = \emptyset$,

$$w(K \cup L) \succsim w(K) \oplus w(L). \tag{13}$$

The possibility of (13) for a given pair $K,\ L \subset I$ is, due to (4), equal to the number

$$\nu(w(K \cup L) \succsim w(K) \oplus w(L)) = \sup_{\substack{x,\, y \in R \\ x \ge y}} \left[\min(\mu_{w(K \cup L)}(x),\ \mu_{w(K) \oplus w(L)}(y)) \right], \tag{14}$$

which we briefly denote $\overline{\nu}(K, L)$, and the possibility that the fuzzy coalition game (I, w) is superadditive is

$$\nu_{\text{super}}(I, w) = \min \left(\overline{\nu}(K, L) : K,\ L \subset I,\ K \cap L = \emptyset \right). \tag{15}$$

The following elementary statements can be formulated.

Remark 1. If $K,\ L \subset I,\ K \cap L = \emptyset$, then evidently $\overline{\nu}(K, L) = \overline{\nu}(L, K)$ as $K \cup L = L \cup K$ and (3) implies $w(K) \oplus w(L) = w(L) \oplus w(K)$ (see also [5, 6]).

Remark 2. If (I,w) is a fuzzy extension of (I,v) and $K,\ L \subset I,\ K \cap L = \emptyset$, then $v(K \cup L) \geq v(K) + v(L)$ implies $\overline{\nu}(K,L) = 1$ as follows from (12) and (4).

Lemma 1. If (I,w) is a fuzzy extension of a game (I,v) and if (I,v) is superadditive then $\nu_{\text{super}}(I,w) = 1$.

Proof. Lemma follows from Remark 2 and (15), immediately. □

Lemma 2. Let (I,w) and (I,w') be fuzzy extensions of a game (I,v), with membership functions $\mu_K,\ \mu'_K$ for $w(K),\ w'(K)$, respectively, where $K \subset I$. Let for a pair of disjoint coalitions $K,\ L \subset I, \mu_{K\cup L}(x) \geq \mu'_{K\cup L}(x),\ \mu_K(x) \geq \mu'_K(x)$ and $\mu_L(x) \geq \mu'_L(x)$ for all $x \in R$, and let us denote by $\overline{\nu}(K,L)$ and $\overline{\nu}'(K,L)$ values (14) for the games (I,w) and (I,w'), respectively. Then $\overline{\nu}(K,L) \geq \overline{\nu}'(K,L)$.

Proof. The statement follows from (14) and from the assumptions, immediately. □

The previous lemma immediately implies the following statement.

Theorem 1. Let (I,w) and (I,w') fulfil the assumptions of Lemma 2, then

$$\nu_{\text{super}}(I,w) \geq \nu_{\text{super}}(I,w'),$$

if $\mu_K(x) \geq \mu'_K(x)$ for all $K \subset I,\ x \in R$.

The methodological principle used in the previous statements can lead to a conclusion that increasing fuzziness of a coalition game can increase the possibility of its superadditivity which idea can be extended ad extremum.

Theorem 2. Let (I,v) be a deterministic coalition game. Then there exists its fuzzy extension (I,w) which is superadditive with possibility $\nu_{\text{super}}(I,w) = 1$.

Proof. Let (I,v) be coalition game. If it is superadditive then any its extension (I,w) is also superadditive with possibility $\nu_{\text{super}}(I,w) = 1$, as follows from Lemma 1. Let (I,v) be not superadditive and let us construct, for any coalition $K \subset I$ the number $y_K \in R$ in the following way. If K is one-element coalition, $K = \{i\},\ i \in I$, then $y_{\{i\}} = v(\{i\})$.

If $K = \{i,j\},\ i,\ j \in I,\ i \neq j$, then

$$y_K = \max\left(v(K),\ y_{\{i\}} + y_{\{j\}}\right),$$

and we continue iteratively. For any $K \subset I$ having at least three elements and any $i \in K$ we denote $K = (K - \{i\}) \cup \{i\}$. If the numbers $y_{K-\{i\}},\ y_{\{i\}}$ are constructed then we put

$$y_K = \max_{i\in K}\left(\max\left(v(K),\ y_{\{i\}} + y_{K-\{i\}}\right)\right). \tag{16}$$

Then $y_K \geq v(K)$ for every $K \subset I$ and we construct the fuzzy extension (I, w) of (I, v) so that for any $K \subset I$,

$$\mu_K(x) = 1 \quad \text{for} \quad x \in [v(K), y_K]$$

and fulfils the demands of (2). Then for every pair of disjoint coalitions K, L the inequality

$$y_{K \cup L} \gtrsim y_K + y_L$$

is fulfilled. Definitions (4) and (14), immediately imply that $\overline{\nu}(K, L) = 1$ and the statement is valid. □

The previous two theorems reflect a relatively natural principle. Increasing vagueness means that more properties become possible. On the other hand, the extremal possibility – certainty – in the case of superadditivity being represented by the value $\nu_{\text{super}}(I, w) = 1$, is achievable only if we extend the set of certain profits of coalitions, i.e. the set

$$\{x \in R : \mu_K(x) = 1\}\,. \tag{17}$$

If (I, w) extends the game (I, v) in rather more realistic way where the sets (17) contain only one number $x = v(K)$ then the certainty of superadditivity is limited to the fuzzy extensions of superadditive games as shown in the following statement.

Theorem 3. Let (I, w) be a fuzzy extension of a coalition game (I, v) and let for every coalition $K \subset I$ $\mu_K(x) = 1$ iff $x = v(K)$. Then $\nu_{\text{super}}(I, w) = 1$ if and only if (I, v) is superadditive.

Proof. One implication follows from Theorem 1. Let us prove the remaining one. Let K, $L \subset I$ be disjoint coalitions and let

$$v(K \cup L) < v(K) + v(L),$$

so that (I, v) is not superadditive. Then $\overline{\nu}(K, L) < 1$ as for any pair x, $y \in R$, $x \geq y$, either $x \neq v(K \cup L)$ or $y \neq v(K) + v(L)$ and, hence, $\mu_{w(K) \oplus w(L)}(y) < 1$ as follows from (3) (cf. also [5]). It means that also $\nu_{\text{super}}(I, w) < 1$. □

4.2. Fuzzy subadditivity

The concept of fuzzy subadditivity is in certain sense a counterpart to the superadditivity, and the definitions and statements which can be formulated in this subsection have their analogies in the previous one.

If (I, w) is a fuzzy extension of coalition game with side-payments (I, v) then it is *fuzzy subadditive* iff for any pair of disjoint coalitions K, $L \subset I$

$$w(K) \oplus w(L) \succsim w(K \cup L). \tag{18}$$

The possibility of (18) for a given pair $K,\ L \subset I$ is, due to (4), equal to the number

$$\nu\Big((w(K) \oplus w(L)) \succsim w(K \cup L)\Big) = \sup_{\substack{x,\, y \in R \\ x \geq y}} \left[\min(\mu_{w(K)\oplus w(L)}(x),\ \mu_{w(K\cup L)}(y))\right] \tag{19}$$

which we briefly denote $\underline{\nu}(K, L)$, and the possibility that the fuzzy coalition game (I, w) is subadditive is

$$\nu_{\mathsf{sub}}(I, w) = \min\left(\underline{\nu}(K, L) : K,\ L \subset I,\ K \cap L = \emptyset\right). \tag{20}$$

The direct analogies of the statements on superadditivity are the following ones.

Remark 3. If $K,\ L \subset I,\ \ K \cap L = \emptyset$ then $\underline{\nu}(K, L) = \underline{\nu}(L, K)$.

Remark 4. If (I, w) is a fuzzy extension of (I, v) and $K,\ L \subset I,\ \ K \cap L = \emptyset$, then $v(K) + v(L) \geq v(K \cup L)$ implies $\underline{\nu}(K, L) = 1$.

Lemma 3. If (I, w) is a fuzzy extension of a game (I, v) and if (I, v) is subadditive then $\nu_{\mathsf{sub}}(I, w) = 1$.

Proof. The theorem follows from Remark 4 and (20), immediately. □

Lemma 4. Let (I, w) and (I, w') be fuzzy extensions of a game (I, v) with membership functions μ_K, μ'_K for $w(K)$, $w'(K)$, respectively. Let for a pair of disjoint coalitions $K,\ L \subset I,\ \ \mu_{K\cup L}(x) \geq \mu'_{K\cup L}(x),\ \ \mu_K(x) \geq \mu'_K(x)$ and $\mu_L(x) \geq \mu'_L(x)$ for all $x \in R$, and let us denote by $\underline{\nu}(K, L)$ and $\underline{\nu}'(K, L)$ values (16) for the respective games. Then $\underline{\nu}(K, L) \geq \underline{\nu}'(K, L)$.

Proof. The statement follows from (19), immediately. □

Theorem 4. Let $(I, w),\ (I, w')$ be fuzzy extensions of (I, v), then

$$\nu_{\mathsf{sub}}(I, w) \geq \nu_{\mathsf{sub}}(I, w'),$$

if $\mu_K(x) \geq \mu'_K(x)$ for all $K \subset I$ and $x \in R$.

Proof. The statement follows from Lemma 4 and from (20), immediately. □

Also in this case the previous result can be driven to the extremal case.

Theorem 5. Let (I, v) be a deterministic coalition game. Then there exists its fuzzy extension (I, w) which is subadditive with possibility $\nu_{\text{sub}}(I, w) = 1$.

Proof. The proof is similar to the one of Theorem 2. We briefly remember only the steps which are rather modified. If (I, v) is subadditive then, due to Lemma 3, any of its fuzzy extensions is subadditive. If (I, v) is not subadditive then we construct numbers $z_K \in R$ for each $K \subset I$ such that for $K = \{i\}$, $z_{\{i\}} = v(\{i\})$. For any K having at least two elements, $K = (K - \{i\}) \cup \{i\}$ for each $i \in K$ and we put

$$z_K = \min \left(v(K), \min_{i \in K} (z_{K-\{i\}}, z_{\{i\}}) \right). \tag{21}$$

Then for any $K \subset I$, $z_K \leq v(K)$ and for any K, $L \subset I$, $K \cap L = \emptyset$, $z_{K \cup L} \leq z_K + z_L$. We construct the fuzzy extension (I, w) of (I, v) such that for each $K \subset I$

$$\mu_K(x) = 1 \quad \text{for} \quad x \in [z_K, v(K)].$$

Then it can be easily shown, using (4) and (19), that for any disjoint K, $L \subset I$ the equality $\underline{\nu}(K, L) = 1$ holds and $\nu_{\text{sub}}(I, w) = 1$. □

Even in the case of subadditivity it is easy to show that its extension to any game by means of its fuzzification is possible only if we admit also other certain values than $v(K)$. Otherwise, the equivalence between certain subadditivity of a fuzzy game and the subadditivity of its crisp pattern is easily provable.

Theorem 6. Let (I, w) be a fuzzy extension of a coalition game (I, v) and let for every $K \subset I$, $\mu_K(x) = 1$ iff $x = v(K)$. Then $\nu_{\text{sub}}(I, w) = 1$ if and only if (I, v) is subadditive.

Proof. The proof is analogous to the proof of Theorem 3. □

4.3. Fuzzy additivity

The additivity of a deterministic coalition game is a conjunction of its super- and subadditivity, as formulated in Section 3. Its generalization can be defined as follows:

We say that a fuzzy game (I, w) if *fuzzy additive* if it is both, fuzzy superadditive and fuzzy subadditive. Then the possibility of additivity is equal to the number

$$\nu_{\text{addit}}(I, w) = \min \left(\nu_{\text{super}}(I, w), \nu_{\text{sub}}(I, w) \right). \tag{22}$$

Remark 5. If we denote for any K, $L \subset I$, $K \cap L = \emptyset$ the number

$$\nu(K, L) = \min \left(\overline{\nu}(K, L), \underline{\nu}(K, L) \right) \tag{23}$$

then, evidently,

$$\nu_{\text{addit}}(I, w) = \min \left(\nu(K, L) : K, L \subset I, K \cap L = \emptyset \right). \tag{24}$$

Remark 6. It is evident that $\nu(K, L) = \nu(L, K)$ for any disjoint K, $L \subset I$, and for $v(K \cup L) = v(K) + v(L)$ also $\nu(K, L) = 1$.

It means that we can formulate an analogy of Lemmas 1 and 3.

Lemma 5. If (I, w) is a fuzzy extension of a game (I, v) and if (I, v) is additive then $\nu_{\text{addit}}(I, w) = 1$.

Proof. The statement follows from Remark 6 and (22), immediately. □

Theorem 7. Let (I, w) and (I, w') be fuzzy extensions of a game (I, v) with membership functions μ_K, μ'_K for $w(K)$, $w'(K)$, respectively. Let for any coalition $K \subset I$ and for any $x \in R$, $\mu_K(x) \geq \mu'_K(x)$. Then

$$\nu_{\text{addit}}(I, w) \geq \nu_{\text{addit}}(I, w').$$

Proof. The theorem follows from Theorems 1 and 4 and from (22). □

Theorem 8. Let (I, v) be a deterministic coalition game. Then there exists its fuzzy extension (I, w) which is additive with possibility $\nu_{\text{addit}}(I, w) = 1$.

Proof. It is sufficient to combine the proofs of Theorems 2 and 5 and to construct for every $K \subset I$ numbers y_K, $z_K \in R$ such that for one-element coalitions $\{i\}$, $i \in I$, $y_{\{i\}} = z_{\{i\}} = v\{i\}$ and for other coalitions the iterative formulas (16) and (21) are used. Evidently $z_K \leq v(K) \leq y_K$ for all $K \subset I$, and it is correct to define $w(K)$ by $\mu_K(x) = 1$ for all $x \in [z_K, y_K]$. Then, analogously to the proofs of Theorems 2 and 5, $\nu(K, L) = 1$ for all disjoint K, $L \subset I$, and, consequently, $\nu_{\text{addit}}(I, w) = 1$. □

Theorem 9. Let (I, w) be a fuzzy extension of a game (I, v) such that for all $K \subset I$, $\mu_K(x) = 1$ iff $x = v(K)$. Then $\nu_{\text{addit}}(I, w) = 1$ if and only if (I, v) is additive.

Proof. The statement follows from Theorems 3 and 6 and from (24), immediately. □

The deterministic coalition games with side-payments offer two equivalent approaches to the definition of additivity.

By the first approach, (I, v) is additive iff it is both, superadditive and subadditive.

By the second approach, (I, v) is additive iff for any pair of disjoint coalitions $K, L \subset I$

$$v(K \cup L) = v(K) + v(L). \tag{25}$$

The equivalence of both approaches, which is evident in the deterministic case, is generally not preserved for the fuzzy extensions of the coalition games. In the previous paragraphs, we have generalized the first one, and defined the possibility $\nu_{\text{addit}}(I,w)$ by (22) which can be supported, via (24), by analogous relation (23) related to particular pairs of disjoint coalitions.

The fuzzification of the second approach demands using fuzzy "equality" (5), to define for any pair of disjoint coalitions $K,\ L \subset I$ the possibility

$$\nu^* \left(w(K \cup L) \sim w(K) \oplus w(L)\right) = \sup_{x \in R} \left[\min\left(\mu_{K \cup L}(x),\ \mu_{w(K) \oplus w(L)}(x)\right)\right] \tag{26}$$

which we briefly denote $\nu^*(K,L)$. Using (26) we may define the possibility that (I,w) is additive under the second approach as a number

$$\nu^*_{\text{addit}}(I,w) = \min\left(\nu^*(K,L) : K,\ L \subset I,\ K \cap L = \emptyset\right). \tag{27}$$

Relation (6) implies that

$$\nu^*(K,L) \le \nu(K,L) \quad \text{and} \quad \nu^*_{\text{addit}}(I,w) \le \nu_{\text{addit}}(I,w) \tag{28}$$

for disjoint $K,\ L \subset I$. The result mentioned in the conclusive paragraph of Section 2 shows one of the conditions under which inequalities (28) turn into equalities.

The possibilities $\nu^*(K,L)$ and $\nu^*_{\text{addit}}(I,w)$ fulfil properties, analogous to the ones of $\nu(K,L)$ and $\nu_{\text{addit}}(I,w)$. Namely, it is easy to see the validity of the following statements.

Remark 7. If $K,\ L$ are disjoint coalitions then $\nu^*(K,L) = \nu^*(L,K)$ and $\nu^*(K,L) = 1$ if $v(K) + v(L) = v(K \cup L)$.

Lemma 6. If (I,w) is a fuzzy extension of (I,v) and if (I,v) is additive then $\nu^*_{\text{addit}}(I,w) = \nu_{\text{addit}}(I,w) = 1$.

Proof. The statement follows from Remark 7 and (27). □

Theorem 10. Let (I,w) and (I,w') be fuzzy extensions of a game (I,v) fulfilling assumptions of Theorem 7. Then $\nu^*_{\text{addit}}(I,w) \ge \nu^*_{\text{addit}}(I,w')$.

Proof. The theorem follows from (26) and (27), immediately. □

Theorem 11. Let (I,v) be a deterministic coalition game. Then there exists its fuzzy extension (I,w) which is additive with possibility $\nu^*_{\text{addit}}(I,w) = 1$.

Proof. The proof is completely analogous to the proof of Theorem 8. □

Theorem 12. Let (I,w) be a fuzzy extension of a game (I,v) such that for all $K \subset I$, $\mu_K(x) = 1$ iff $x = v(K)$. Then $\nu^*_{\text{addit}}(I,w) = 1$ if and only if (I,v) is additive.

Proof. The statement follows from (26) and (27), immediately. □

4.4. Discussion of fuzzy convexity

The fuzzification of the convexity concept (9) is connected with certain ambiguity. In the deterministic coalition game (I, v), two equivalent inequalities, namely

$$v(K \cup L) + v(K \cap L) \geq v(K) + v(L) \tag{29}$$

and

$$v(K \cup L) \geq v(K) + v(L) - v(K \cap L)$$

for $K,\ L \subset I$, are arbitrarily used to verify the convexity property. In the case of fuzzy extension (I, w) of (I, v) this arbitrarity cannot be accepted as the fuzzy inequality relations

$$w(K \cup L) \oplus w(K \cap L) \succsim w(K) \oplus w(L) \tag{30}$$

and

$$w(K \cup L) \succsim w(K) \oplus w(L) \oplus (-w(K \cap L)),$$

where $\mu_{-w(K\cap L)}(x) = \mu_{K\cap L}(-x)$ (remember Section 2) for all $x \in R$, are not equivalent. Some more details about their mutual relations can be derived from the results summarized in [5] and [6]. They reach beyond the subjects of this paper. For our purpose we accept the convention that the fuzzy game (I, w) is convex iff it fulfills (30) as a fuzzification of (29). (In fact, the approach based on the inequality $w(K \cup L) \succsim w(K) \oplus w(L) \oplus (-w(K \cap L))$ leads to quantitatively different values of possibilities but the qualitative relations, namely between the convexity of (I, v) and (I, w) are equivalent.)

If we accept this approach then the procedure can be very analogous to the one used in the previous subsections. For every pair of coalitions $K,\ L \subset I$ we denote the possibility of (30) as a real number $\pi(K, L)$ by means of (4)

$$\begin{aligned} \pi(K, L) &= \nu\Big(w(K \cup L) \oplus w(K \cap L) \succsim w(K) \oplus w(L)\Big) \\ &= \sup_{\substack{x,\, y \in R \\ x \geq y}} \left[\min\left(\mu_{w(K\cup L)\oplus w(K\cap L)}(x),\ \mu_{w(K)\oplus w(L)}(y)\right)\right]. \end{aligned} \tag{31}$$

Then the possibility that (I, w) is convex is

$$\pi_{\text{conv}}(I, w) = \min\left(\pi(K, L) : K,\ L \subset I\right). \tag{32}$$

Analogously to the previous cases, it is easy to verify the following statements.

Remark 8. Evidently, $\pi(K, L) = \pi(L, K)$ for $K,\ L \subset I$.

Remark 9. If (I, w) is a fuzzy extension of (I, v) and if, for some $K,\ L \subset I$, $v(K \cup L) + v(K \cap L) \geq v(K) + v(L)$ then $\pi(K, L) = 1$.

Lemma 7. If the K, $L \subset I$, $K \cap L = \emptyset$ then $\pi(K,L) = \overline{\nu}(K,L)$.

Proof. Using (3), (12) and the assumption that, for empty coalition $\emptyset$, $\mu_\emptyset(0) = 1$, $\mu_\emptyset(x) = 0$ if $x \neq 0$ (cf. introduction of Section 4) it is easy to prove (see [5]) that $w(K \cup L) \oplus w(K \cap L) = w(K \cup L)$. Then the statement follows from (30) and (13), or from (31) and (14), immediately. □

Lemma 8. If (I,w) is a fuzzy extension of a game (I,v) and if (I,v) is convex then $\pi_{\text{conv}}(I,w) = 1$.

Proof. The statement follows from Remark 9, immediately. □

Lemma 9. Let (I,w) and (I,w') be fuzzy extensions of a game (I,v) with membership functions μ_K, μ'_K of $w(K)$, $w'(K)$, respectively, where $K \subset I$. Let for a pair of coalitions K, $L \subset I$, $\mu_K(x) \geq \mu'_K(x)$, $\mu_L(x) \geq \mu'_L(x)$, $\mu_{K\cup L}(x) \geq \mu'_{K\cup L}(x)$, $\mu_{K\cap L}(x) \geq \mu'_{K\cap L}(x)$, for all $x \in R$, and let us denote by $\pi(K,L)$ and $\pi'(K,L)$ values (31) for (I,w), (I,w'), respectively. Then $\pi(K,L) \geq \pi'(K,L)$.

Proof. The statement follows from (31), and from the assumptions, immediately. □

Theorem 11. Let (I,w), (I,w') be fuzzy extensions of (I,v), let the notations of Lemma 9 be preserved and let $\mu_K(x) \geq \mu'_K(x)$ for all $x \in R$ and $K \subset I$. Then

$$\pi_{\text{conv}}(I,w) \geq \pi_{\text{conv}}(I,w').$$

Proof. The theorem follows from Lemma 9, and (32), immediately. □

Theorem 12. Let (I,v) be a deterministic coalition game. Then there exists its fuzzy extension (I,w) which is convex with possibility $\pi_{\text{conv}}(I,w) = 1$.

Proof. The proof of this statement uses procedure analogous to the one of Theorems 2, 5 and 8.

For every $K \subset I$ we define number $t_K \in R$ in the following way. For one-element coalitions $K = \{i\}$ we put $t_{\{i\}} = v(\{i\})$ and then, iteratively, for any $M \subset I$ we put

$$t_M = \max\left\{v(M), \max\{t_K + t_L - v(K \cap L) : K,\ L \subset I,\ K \cup L = M\}\right\}. \tag{33}$$

Then for every K, $L \subset I$ $t_K \geq v(K)$ and

$$t_{K\cup L} + t_{K\cap L} \geq t_{K\cup L} + v(K \cap L) \geq t_K + t_L$$

and we have constructed new deterministic game (I,v') where $v'(K) = t_K$, $K \subset I$. This game is convex. Let us define its extension (I,w) such that for every $K \subset I$,

$$t_K(x) = 1 \quad \text{for} \quad x \in [v(K),\ v'(K)].$$

The fuzzy game (I, w) is a fuzzy extension of both (I, v) and (I, v') and the convexity of (I, v') means, due to Theorem 11, that (I, w) is fuzzy convex with possibility $\pi_{\text{conv}}(I, w) = 1$. □

Even in the case of convexity, the fuzzy extension which respects the condition $\mu_K(x) < 1$ for $x \neq v(K)$ cannot be certainly fuzzy convex.

Theorem 13. Let (I, w) be a fuzzy extension of a coalition game (I, v) and let for every $K \subset I$, $\mu_K(x) = 1$ iff $x = v(K)$. Then $\pi_{\text{conv}}(I, w) = 1$ if and only if (I, v) is convex.

Proof. The proof is analogous to the one of Theorem 3. If (I, v) is convex then $\pi_{\text{conv}}(I, w) = 1$ due to Theorem 11. If it is not convex then there exists a pair of coalitions K, $L \subset I$ such that

$$v(K \cup L) + v(K \cap L) < v(K) + v(L)$$

and then, due to the assumption, and due to (4) and (31), $\pi(K, L) < 1$. Hence, also $\pi_{\text{conv}}(I, w) < 1$. □

5. FUZZY ADMISSIBLE IMPUTATIONS

Another approach to the possibilities of superadditivity and similar concepts in the fuzzy coalition game is generally possible, as well. It is based on the sets of achievable and admissible imputations and for the deterministic case it was briefly mentioned in Section 3 in connection with formulas (10) and (11). Here, we formulate this alternative model for the superadditivity case, only. The treatment of subadditivity and additivity analogous to (11) demands rather wider aparatus (see [7]) and the analogous formulation of the convexity concept differs from (9) but also from (11) in a significant degree (as shown in [10]).

Let us consider a deterministic coalition game (I, v) and its fuzzy extension (I, w), and let us define for every coalition $K \subset I$ a fuzzy subset $\mathcal{W}(K)$ of R^I with membership function $\lambda_K : R^I \to [0, 1]$, where for every $\boldsymbol{x} = (x_i)_{i \in I}$,

$$\lambda_K(\boldsymbol{x}) = \sup \left[\mu_K(y) : y \in R, \sum\nolimits_K x_i \leq y\right]. \tag{34}$$

For empty coalition $K = \emptyset$, $\lambda_\emptyset(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in R^I$, which is correct with respect to (34).

Remark 10. If (I, v) is a coalition game and (I, w) is its fuzzy extension then for every $\boldsymbol{x} \in \mathcal{V}(K)$, where $\mathcal{V}(K)$ is defined by (10), the equality $\lambda_K(\boldsymbol{x}) = 1$ follows from (10), (34) and (12).

It is possible to proceed completely analogously to (11) and to say that the fuzzy game (I, w) is *imputationally superadditive* iff for any K, $L \subset I$, $K \cap L = \emptyset$,

$$\mathcal{W}(K \cup L) \supset \mathcal{W}(K) \cap \mathcal{W}(L) \tag{35}$$

in the fuzzy set theoretical sense, i.e.

$$\lambda_{K\cup L}(\boldsymbol{x}) \geq \min(\lambda_K(\boldsymbol{x}), \lambda_L(\boldsymbol{x})) \quad \text{for all } \boldsymbol{x} \in R^I. \tag{36}$$

This procedure copies the deterministic approach (11) (see also [3]) but it excludes the fuzziness from the superadditivity of fuzzy games. Inclusion (35) is either valid or not and there is no space for any vagueness. It is possible to base the possibility of the superadditivity on the method of α-cuts, suggested in [3]. If $\alpha \in R$, $\alpha \in (0,1]$, and if $K \subset I$ then we denote by $\mathcal{W}^{(\alpha)}$ a fuzzy subset of R^I with the membership function $\lambda^{(\alpha)} : R^I \to [0, \alpha]$

$$\lambda_K^{(\alpha)}(\boldsymbol{x}) = \min\{\alpha, \lambda_K(\boldsymbol{x})\}, \quad \boldsymbol{x} \in R^I. \tag{37}$$

Then we say that the game (I, w) is *imputationally α-superadditive* iff

$$\alpha = \min\Big[\sup(\beta \in (0,1] : \mathcal{W}^{(\beta)}(K \cup L) \supset \mathcal{W}^{(\beta)}(K) \cap \mathcal{W}^{(\beta)}(L)) : K,\, L \subset I,\, K \cap L = \emptyset\Big], \tag{38}$$

it means

$$\alpha = \min\Big[\sup(\beta \in (0,1] : \forall \boldsymbol{x} \in R^I,\, \lambda_{K\cup L}^{(\beta)}(\boldsymbol{x}) \geq \min(\lambda_K^{(\beta)}(\boldsymbol{x}),\, \lambda_L^{(\beta)}(\boldsymbol{x}))) : K,\, L \subset I,\, K \cap L = \emptyset\Big]. \tag{39}$$

The imputational superadditivity (35), (36) is an imputational 1-superadditivity in the terms of (38), (39). The value $\alpha \in (0,1]$ for which the fuzzy game (I, w) is imputationally α-superadditive can be also considered for an alternative possibility of the fuzzy superadditivity. Unfortunately, mutual relations between both fuzzy superadditivities, the imputational α-superadditivity and $\nu_{\text{super}}(I, w)$, are very weak. Namely, they reverse the directions of implication between the deterministic and fuzzy superadditivity.

Theorem 14. If (I, w) is a fuzzy extension of a game (I, v) and if (I, w) is imputationally 1-superadditive then (I, v) is superadditive.

P r o o f. The statement follows from (38), (7), from (11) and from Remark 10. If (I, w) is imputationally 1-superadditive then $\mathcal{W}(K \cup L) \supset \mathcal{W}(K) \cap \mathcal{W}(L)$ for any disjoint pair of coalitions. It means, due to Remark 10, that also $\mathcal{V}(K \cup L) \supset \mathcal{V}(K) \cap \mathcal{V}(L)$ and the equivalence between the superadditivity and (11), mentioned in Section 3 (see also [7]) proves the statement. □

Corollary. If (I, w) is a fuzzy extension of (I, v) and if (I, w) is imputationally 1-superadditive then $\nu_{\text{super}}(I, w) = 1$ as follows from Lemma 1 and Theorem 14.

The previous theorem and its corollary can be for a special but important class of fuzzy games extended into the following statement.

Theorem 15. Let (I, w) be a fuzzy extension of a coalition game (I, v) and let for every coalition $K \subset I$, $\mu_K(x)$ be increasing for $x < v(K)$ and decreasing for $x > v(K)$. If (I, w) is imputationally α-superadditive then

$$\alpha \leq \nu_{\text{super}}(I, w).$$

Proof. If (I, v) is superadditive then $\nu_{\text{super}}(I, w) = 1$, due to Lemma 1, and the inequality is guaranteed as $\alpha \in (0, 1]$. If $\nu_{\text{super}}(I, w) < 1$ then (I, v) is not superadditive as follows from Lemma 1, and there exists a pair of disjoint coalitions K, $L \subset I$ such that $\nu(K, L) < 1$,

$$v(K \cup L) < v(K) + v(L)$$

and, moreover,

$$s_1 = \sup\left(x \in R : \mu_{K \cup L}(x) = 1\right) < \inf\left(x \in R : \mu_{w(K) \oplus w(L)}(x) = 1\right) = s_2$$

as follows from (4) and (14). Then, due to the assumptions of this theorem, with respect to (4) (or (14)),

$$\nu(K, L) = \mu_{K \cup L}(x_0) = \mu_{w(K) \oplus w(L)}(x_0)$$

for some $x_0 \in R$ such that $s_1 < x_0 < s_2$. It means that for all $x > x_0$,

$$\mu_{K \cup L}(x) \leq \mu_{K \cup L}(x_0) < 1$$

and, as follows from the monotonicity of $\mu_{K \cup L}$, also $\lambda_{K \cup L}(x) \leq \lambda_{K \cup L}(x_0) < 1$ for $x > x_0$. It means that also for $x_1 = v(K) + v(L) > x_0$, this inequality holds. On the other hand, $\lambda_K(v(K)) = \lambda_L(v(L)) = 1$ and, consequently,

$$\lambda_{K \cup L}(x_1) < \min\left(\lambda_K(v(K)),\ \lambda_L(v(L))\right) = 1,$$

and, certainly, $\mathcal{W}(K \cup L) \supset \mathcal{W}(K) \cap \mathcal{W}(L)$ is not true. If there exists $\alpha \in (0, 1]$ such that

$$\mathcal{W}^{(\alpha)}(K \cup L) \supset \mathcal{W}^{(\alpha)}(K) \cap \mathcal{W}^{(\alpha)}(L),$$

it means

$$\lambda^{(\alpha)}_{K \cup L}(x) \geq \min\left(\lambda^{(\alpha)}_K(x),\ \lambda^{(\alpha)}_L(x)\right),$$

then the α-cut must be such that for all $x \in R$

$$\lambda^{(\alpha)}_{K \cup L} \leq \lambda_{K \cup L}(x_1) \leq \lambda_{K \cup L}(x_0) = \mu_{K \cup L}(x_0) = \nu(K, L),$$

and this must be valid also for $\lambda^{(\alpha)}_{K \cup L}(x) = \alpha$. Hence, $\alpha \leq \nu(K, L)$ also for the pair $K, L \subset I$, $K \cap L = \emptyset$, which minimizes the right-hand-side of formula (15) and, consequently

$$\alpha \leq \nu_{\text{super}}(I, w). \qquad \square$$

The approach to the superadditivity, which is based on the concept of the sets $\mathcal{V}(K)$ and $\mathcal{W}(K)$ of achievable (possibly achievable) imputations is rather more adequate to the model of coalition games without side-payments. In the deterministic case, it was shown in [7]. For the case of coalition games with side-payments, which is investigated in this paper, it is limited to the lineary bounded achievable sets of imputations. The imputational superadditivity or α-superadditivity has weaker bounds to the deterministic superadditivity of the original deterministic game which was extended to its fuzzy counterpart than the superadditivity based on processing of fuzzy quantities dealt in Subsection 4.1. It is caused by the close similarity between both, deterministic and fuzzy, processing of the coalitional pay-offs, represented by (7) and (14), (15). On the other hand, the imputational superadditivity more evidently reflects the proper sense of the superadditivity, namely the fact that in a superadditive game everything what can be realized by a group of (disjoint) coalitions can be realized also by their union, including the set of all players. In more formal presentation, if $\mathcal{K} = \{K_1, \ldots, K_m\}$, $K_i \cap K_j = \emptyset$, $i \neq j$, $K_1 \cup \cdots \cup K_m = I$ is a *coalition structure* and if the vector of imputations $\boldsymbol{x} = (x_i)_{i \in I}$ can be realized by coalitions from $\mathcal{K}$, i. e.

$$x_{K_j} \leq v(K_j), \quad j = 1, \ldots, m, \quad \text{where} \quad x_{K_j} = \sum\nolimits_{K_j} x_i$$

or if $\boldsymbol{x} \in \mathcal{W}(K_j)$, $j = 1, \ldots, m$, with possibility $\lambda_{K_j}(\boldsymbol{x})$, then $\boldsymbol{x} \in \mathcal{W}(I)$ with possibility

$$\lambda_I(\boldsymbol{x}) \geq \lambda_{K_j}(\boldsymbol{x}) \quad \text{for all} \quad K_j \in \mathcal{K}.$$

However, this relation is more evident for the imputational superadditivity, it is in its essential sense valid also for the superadditivity based on the concept of $\nu_{\text{super}}(I, w)$. Namely, if $\mathcal{K} = \{K_1, \ldots, K_m\}$ is a coalition structure, if $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$, where we denote for any $K_j \in \mathcal{K}$

$$x_{K_j} = \sum\nolimits_{K_j} x_i, \quad x_I = \sum\nolimits_I x_i = \sum\nolimits_{j=1}^m x_{K_j},$$

if we denote by $w(\mathcal{K})$ the fuzzy quantity

$$w(\mathcal{K}) = w(K_1) \oplus w(K_2) \oplus \cdots \oplus w(K_m)$$

and if $\mu_{\mathcal{K}}$ is the membership function of $w(\mathcal{K})$ then (3) easily implies that

$$\mu_{\mathcal{K}}(x) \geq \mu_{K_j}(x) \quad \text{for all } j = 1, \ldots, m.$$

6. CONCLUSIVE REMARKS

The superadditivity and related concepts like subadditivity, additivity and in certain degree also convexity belong to the elementary ones in the deterministic coalition game theory. They represent the first criteria for the admissibility of large (or small) coalitions and the first indicators of the possible forms of eventual cooperation.

Their elementarity and simplicity causes that they are only briefly mentioned in the works subjected to the classical problems of the coalition game theory in its deterministic form (see, for example, [12, 13]) and they were treated analogously briefly in the previous papers oriented to the fuzzy extensions of those games (like [3, 4, 9, 11]). Even the transition from the games with side-payments to those ones without side-payments demands more attention to superadditivity and related concepts even in the deterministic case, as follows from [7] or [10]. The attention paid to these concepts in the fuzzified coalition game theory would be proportional to the increased complexity of the model and the flexibility of its elements. The previous sections are devoted to the fuzzification of the superadditivity and similar concepts for the simpler case of the games with side payments and to a brief note on the topics connected with the transition to games without side payments, which is submitted in Section 5.

It would be mentioned that the transition from the games with side payments to those without them is relatively simple in the case of superadditivity which was treated above. It is analogously simple in the deterministic case but rather complicated in the fuzzified games if subadditivity and additivity are considered. This follows from the rather complicated fuzzification of the domination and superoptimum concepts (see, e. g., [4]). The convexity concept becomes quite complicated even if it is translated to the deterministic games without side-payments (see [10]) and its fuzzification for those games was not approached, yet.

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