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OPTIMAL PLACEMENT OF CONTROLS FOR A ONE-DIMENSIONAL ACTIVE NOISE CONTROL PROBLEM

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In this paper, we investigate the optimal location of secondary sources (controls) to enhance the reduction of the noise field in a one-dimensional acoustic cavity. We first formulate the active control strategy as a linear quadratic tracking (LQT) problem in a Hilbert space, and then formulate the optimization problem as minimizing an appropriate performance criterion based on the LQT cost function with respect to the location of the controls. A numerical scheme based on the Legendre–tau method is used to approximate the control and the optimization problems. Numerical examples are presented to illustrate the effect of location of controls on the reduction of the noise field.

1. INTRODUCTION

The problem of optimal location of controls is of importance in situations where optimum performance of the feedback control is required in order to meet certain design requirements. In this work, we consider optimal reduction of the noise field inside a one-dimensional cavity through introduction of loudspeakers that would generate a secondary acoustic field that interacts destructively with the primary noise field. This active noise control technique is useful in situations where the traditional passive damping methods which involve addition of mass are not practical. In [1, 2] the control problem has been formulated as an abstract linear quadratic tracking problem (LQT) in an infinite dimensional setting. In [3] numerical approximations based on the Legendre–tau method were performed to approximate the infinite dimensional control system and to solve the resulting finite-dimensional ones. The goal of this work is to address the issue of optimizing the performance of the controls with respect to their placement. We will consider optimizing the LQT cost function with respect to the placement of controls, and present necessary optimality conditions for existence of optimal location. We will present numerical examples to illustrate the dependence of the performance of the controls with respect to their location.

2. FORMULATION OF THE PHYSICAL MODEL

For simplicity, we model the primary noise source in the bounded domain $\Omega = [-1,1]$ as a single wave: $p_1(x,t) = \widehat{p}_1(x)\, e^{i\omega t}$. After activation of the controls (speakers) at $t = 0$, it is anticipated that the acoustic pressure field will approach a steady periodic state p_3 in a stable manner. With $\tau = 2\pi/\omega$, and $f = \Sigma_{i=1}^{m}\chi(\widehat{\Omega}_i)F_i(t)$, where $\widehat{\Omega}_i$ is the support of the ith control, the steady state p_3 is expected to be governed by (see [2])

$$\begin{cases} \partial_t^2 p_3 = \gamma^2 \,\Delta\, p_3 + f_c & \text{in } \Omega \times [0,\tau] \\ 0 = \alpha p_3 + \beta\partial_t p_3 + \partial_n p_3 & \text{on } \partial\Omega \times [0,\tau] \\ p_3(0) = p_3(\tau), & \\ \partial_t p_3(0) = \partial_t p_3(\tau) & \text{in } \Omega. \end{cases} \tag{2.1}$$

3. A PERIODIC LINEAR QUADRATIC TRACKING PROBLEM

In order to use the well-established results in control theory, we first write (2.1) in the following first order form:

$$\begin{cases} \partial_t \begin{pmatrix} p_3 \\ \partial_t p_3 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{pmatrix} \begin{pmatrix} p_3 \\ \partial_t p_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \chi(\widehat{\Omega})F(t) \end{pmatrix} & \text{in } \Omega \times (0,\tau) \\ 0 = \alpha p_3 + \beta\partial_t p_3 + \partial_n p_3 & \text{on } \partial\Omega \times [0,\tau) \\ \begin{pmatrix} p_3(0) \\ \partial_t p_3(0) \end{pmatrix} = \begin{pmatrix} p_3(\tau) \\ \partial_t p_3(\tau) \end{pmatrix} & \text{in } \Omega. \end{cases}$$

In the equations above, for simplicity only one control (speaker) is considered. By defining the state variable as

$$P_3(x,t) = \begin{bmatrix} p_3 \\ \partial_t p_3 \end{bmatrix},$$

the state space is taken to be $X = H^1(-1,1) \times L^2(-1,1)$, which is a Hilbert space with the usual product topology inner product $(\cdot,\cdot)_X$. It is also a Hilbert space with the equivalent inner product $\langle\cdot,\cdot\rangle_{\alpha,\gamma}$ given by

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\alpha,\gamma} = \alpha(u_1,u_2)_{L^2(\partial\Omega)} + (\nabla u_1, \nabla u_2)_{L^2(\Omega)} + \gamma^{-2}(v_1,v_2)_{L^2(\Omega)} \tag{3.1}$$

and the associated norm, $|\cdot|_{\alpha,\gamma}^2 = \langle\cdot,\cdot\rangle_{\alpha,\gamma}$.

Define

$$A = \begin{bmatrix} 0 & I \\ \gamma^2\Delta & 0 \end{bmatrix}$$

with

$$\mathcal{D}(A) = \{(u, v) \in X \mid u \in H^2, v \in H^1, \alpha u + \beta v + \partial_n u = 0\}.$$

It can be shown (e.g., see [2]) that the operator A has the following properties:

(i) A is the infinitesimal generator of a contraction semigroup $T(t)$ on X.

(ii) For some μ_0, $M_0 > 0$, we have that

$$|T(t)|_{\alpha,\gamma} \leq M_0 e^{-\mu_0 t} \quad \text{for} \quad t \geq 0. \tag{3.2}$$

The active control strategy is formulated as a periodic linear quadratic tracking (LQT) problem where the goal is to track the primary noise field by the use of a secondary source. The mathematical formulation is to find among functions $F \in L^2(0, \tau : U)$ a function F_{opt} which minimizes

$$J(F) = \int_0^\tau \{(M[P_1 + P_3], [P_1 + P_3])_X + \theta(F, F)_{L^2}\} \, dt$$

subject to

$$\begin{cases} \dot{P}_3 = AP_3 + BF & 0 \leq t \leq \tau \\ P_3(0) = P_3(\tau) \end{cases}$$

where M is a self-adjoint, nonnegative operator, θ is a control design parameter, and $U = \boldsymbol{R}^1$ for one control. Note that for m number of controls, $U = \boldsymbol{R}^m$ and θ is an $m \times m$ matrix.

Here A is the generator defined above and

$$BF(t) = \begin{bmatrix} 0 \\ \chi(\widehat{\Omega})F(t) \end{bmatrix}.$$

In the above, $\chi(\widehat{\Omega})$ can be characterized in terms of a control located at x_c with radius of influence of a as the interval $\chi(\widehat{\Omega}) = [x_c - a, x_c + a]$.

The primary noise vector is

$$P_1(x, t) = \widehat{P}_1(x) \, e^{i\omega t} = \begin{bmatrix} \widehat{p}_1 \\ i\omega\widehat{p}_1 \end{bmatrix} e^{i\omega t}.$$

Under the assumptions of detectability and stabilizibility which in our case follow from the decay estimate in (3.2), it follows that the *optimal control* F_{opt} is given by

$$F_{\text{opt}}(t) = -\theta^{-1} B^* G P_3(t) - \theta^{-1} B^* r(t),$$

where G satisfies the *Algebraic Riccati Equation*

$$GA + A^*G + M - \theta^{-1} GBB^*G = 0,$$

and r is a tracking variable satisfying $r(x, t) = \widehat{r}(x) \, e^{i\omega t}$ where

$$\widehat{r}(x) = -[i\omega + (A^* - \theta^{-1} GBB^*)]^{-1} M \widehat{P}_1.$$

From the observations above, we can conclude that the *optimal state* satisfies:

$$\begin{cases} \dot{P}_3 = (A - \theta^{-1}BB^*G)P_3 - \theta^{-1}BB^*r & 0 \le t \le \tau \\ P_3(0) = P_3(\tau). \end{cases}$$

Moreover, there exists an $\widehat{F}$ such that the optimal control is given by

$$F_{\text{opt}} = \widehat{F}e^{i\omega t},$$

thus the optimal control in our case is sinusoidal [2].

4. OPTIMIZATION PROBLEM

Our goal is to optimize the following minimum LQT cost function with respect to location of the controller, x_c:

$$J_{\min}(F_{\text{opt}}) = \int_0^\tau \{(MP_1, P_1)_X - \theta^{-1}(B^*r, B^*r)_{L^2}\}\, dt.$$

In this expression, only the control operator B and the tracking variable r are dependent on x_c. Since the possible values for the location belong to a compact set, we can prove existence of an optimizing location by proving the cost function is weakly lower semi continuous with respect to x_c. This result can be obtained by showing that B and therefore the Riccati operator G are continuously dependent on x_c. In order to find the equation that gradient of $J_{\min}$ with respect to x_c satisfies, we need to find $\frac{\partial G}{\partial x_c}$ and $\frac{\partial B}{\partial x_c}$. One can show that $\Sigma = \frac{\partial G}{\partial x_c}$ satisfies the following Lyapunov equation:

$$\Sigma(A - \theta^{-1}BB^*G) + (A^* - \theta^{-1}BB^*G)\Sigma = \theta^{-1}G\left(\frac{\partial B}{\partial x_c}B^* + B\frac{\partial B}{\partial x_c}\right)G.$$

This equation has a unique solution as long as $(A - \theta^{-1}BB^*G)$ generates a contraction semigroup that decays exponentially in time which is established for our problem from (3.2). Now that we have the well-posedness of the sensitivity equation, we can proceed with the numerical approximation of the optimization problem.

5. NUMERICAL APPROXIMATIONS AND RESULTS

To carry out the numerical approximation, we employ the Legendre–tau method to cast the infinite dimensional control system in a sequence of finite dimensional spaces of polynomials. In this method, the finite dimensional solution is expanded in terms of the Legendre polynomials. For a rigorous treatment of this method applied to solve the control problem and proof of convergence of the finite-dimensional control systems to the infinite-dimensional one, see [3].

In spectral methods, in particular in the Legendre–tau methods presented here, the approximating spaces X_N (the trial spaces) in which we seek our solutions are taken to be finite-dimensional subspaces of $\mathcal{D}(A)$. Thus the elements in X_N satisfy the boundary conditions. The approximation scheme is defined by projecting the differential equation onto spaces Y_N, (the test spaces), which are appropriately defined finite dimensional spaces spanned by Legendre polynomials.

In our discussions, the following spaces will be used:

$$\begin{aligned} \mathcal{P}_N &= \text{Space of polynomials of degree} \leq N \\ X_N &= \{(u,v) \in \mathcal{P}_N \times \mathcal{P}_N : \alpha u + \beta v + \partial_n u|_{\partial\Omega} = 0\} \\ Y_N &= \mathcal{P}_N \times \mathcal{P}_{N-2} \end{aligned}$$

The orthogonal projection $Q_N : X \to Y_N$ is defined by

$$Q_N = \begin{bmatrix} P^{(1)}_{N,\alpha} & 0 \\ 0 & P^{(0)}_{N-2} \end{bmatrix},$$

where $P^{(1)}_{N,\alpha}$ is the orthogonal projection of the space H^1 onto $\mathcal{P}_N$ with respect to the H^1_α-inner product, defined as

$$\langle u_1, u_2 \rangle_{H^1_\alpha} = \alpha(u_1, u_2)_{L^2(\partial\Omega)} + (\nabla u_1, \nabla u_2)_{L^2(\Omega)},$$

and $P^{(0)}_{N-2}$ is the orthogonal projection of L^2 onto $\mathcal{P}_{N-2}$ with respect to the L^2 norm. Now we define a projection operator Π_N from X to X_N for the Legendre–tau method as follows:

$$\Pi_N \begin{pmatrix} \sum_{n=0}^{\infty} u_n \phi_n s \\ \sum_{n=0}^{\infty} v_n \phi_n \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{N} u_n \phi_n \\ \sum_{n=0}^{N-2} v_n \phi_n + \sum_{m=1}^{2} b_m \phi_{N-2+m} \end{pmatrix} \tag{5.1}$$

where $\phi_n, n = 0, 1, \ldots,$ are the Legendre polynomials of degree n, and $b_m, m = 1, 2,$ are chosen so that projected elements satisfy the boundary constraints

$$\mathcal{B}(\Pi_N \begin{pmatrix} u \\ v \end{pmatrix}) = 0 \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in X;$$

here $\mathcal{B}$ is the linear boundary operator associated with X_N. We see that Π_N as defined maps X to X_N. If we denote the restriction of Π_N to Y_N by Π^0_N, we can easily see that Π_N can be written as

$$\Pi_N \begin{pmatrix} u \\ v \end{pmatrix} = \Pi^0_N Q_N \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{for all } \begin{pmatrix} u \\ v \end{pmatrix} \in X.$$

For a second order hyperbolic system such as the wave equation with two boundary conditions, we seek an approximation to the solution of the form

$$(\widetilde{u}^N(t,x), \widetilde{v}^N(t,x))^T = \sum_{i=0}^{2N-1} w^N_i(t)\, \Phi^N_i(x) \tag{5.2}$$

where Φ_i^N are defined as follows:

$$\Phi_i^N = \begin{cases} (\phi_i, 0)^T & 0 \le i \le N \\ (0, \phi_{i-N-1})^T & N+1 \le i \le 2N-1 \end{cases}$$

It is assumed that $(\tilde{u}^N, \tilde{v}^N)^T \in Y_N$ satisfies an equation of the form

$$\frac{\partial}{\partial t}\begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix} = Q_N \, A\Pi_N^0 \begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix} + Q_N \begin{bmatrix} 0 \\ \chi(\hat{\Omega})\, F(t) \end{bmatrix}. \tag{5.3}$$

The above can be equivalently written in a variational form as

$$\left(\frac{\partial}{\partial t}\begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix}, y\right)_X = \left(A\Pi_N^0 \begin{bmatrix} \tilde{u}^N \\ \tilde{v}^N \end{bmatrix}, y\right)_X + \left(\begin{bmatrix} 0 \\ \chi(\hat{\Omega})\, F(t) \end{bmatrix}, y\right)_X \tag{5.4}$$

for all $y \in Y_N$. Note that $(\cdot,\cdot)_X$ denotes the usual inner product on $X = H^1 \times L^2$. Choosing $y = \Phi_i^N$ and using (5.2), we can, in turn, write (5.4) in matrix form as

$$\underline{D}_N \dot{w}^N = \underline{D}_N \underline{A}_N w^N + \underline{B}_N F(t) \tag{5.5}$$

where

$$(\underline{D}_N) = \begin{bmatrix} \underline{K} & 0 \\ 0 & \underline{L} \end{bmatrix} \quad \text{with}$$

$$(\underline{K})_{i,j} = (\phi_i, \phi_j)_{H^1(-1,1)}, \quad \text{for all} \quad \begin{array}{l} 0 \le i \le N \\ 0 \le j \le N, \end{array}$$

$$\text{and } (\underline{L})_{i,j} = (\phi_i, \phi_j)_{L^2(-1,1)}, \quad \text{for all} \quad \begin{array}{l} 0 \le i \le N-2 \\ 0 \le j \le N-2 \end{array}$$

$$\underline{A}_N = \underline{A}_N \underline{\Pi}_N$$

where

$$\underline{A}_N = \begin{bmatrix} 0_{(N+1)\times(N+1)} & I_{(N+1)\times(N+1)} \\ S^2 & 0_{(N-1)\times(N+1)} \end{bmatrix}.$$

In the above, $I_{(N+1)\times(N+1)}$ = Identity matrix of dimension $(N+1)\times(N+1)$, and S^2 = matrix representation of Δ (second-order differentiation operator) with respect to the Legendre polynomials. It is an $(N-1)$ by $(N+1)$ matrix. Also, $\underline{\Pi}_N$ is the matrix representation for Π_N^0 with dimension $(2N+2)\times 2N$. In (5.5), we also have

$$\underline{B}_N = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} (N+1) \\ (\phi_0, \chi(\widehat{\Omega}))_{L^2(-1,1)} \\ \vdots \\ (\phi_{N-2}, \chi(\widehat{\Omega}))_{L^2(-1,1)} \end{bmatrix}$$

with $w^N =$ col $(w_0^N, \dots, w_{2N-1}^N)$. From equation (5.5), we see that the matrix representations of operators $Q_N A \Pi_N^0$ and $Q_N B$ are the following

$$\mathcal{A}_N = \underline{A}_N \underline{\Pi}_N \quad \text{and} \quad \mathcal{B}_N = (\mathcal{D}_N)^{-1} \underline{B}_N. \tag{5.6}$$

We can define the matrix $\mathcal{M}_N$, the matrix representation of operator $M_N = Q_N M$ by

$$[\mathcal{M}_N]_{i,j} = (\Phi_i, M\Phi_j)_{X=H^1 \times L^2} \tag{5.7}$$

for $0 \le i \le 2N-1, \quad 0 \le j \le 2N-1$. Now let z_d^N be defined as $Q_N z_d$:

$$z_d^N(x,t) = Q_N z_d(x,t) = -\begin{bmatrix} \widehat{p}_1^N(x) \\ i\omega \widehat{p}_1^N(x) \end{bmatrix} e^{i\omega t} = -\widehat{P}_1^N(x)\, e^{i\omega t} = \widehat{z}_d^N(x)\, e^{i\omega t}. \tag{5.8}$$

Expanding $\widehat{z}_d^N$ in terms of the basis elements Φ_i, we obtain $\widehat{z}_d^N = \sum_{i=0}^{2N-1} z_i^N \Phi_i$. From (5.8), we have

$$(\widehat{\underline{z}}_d^N)_i = -(\mathcal{D}_N)_{i,j}^{-1}\, (R[\widehat{P}_1^N(x)])_j \tag{5.9}$$

where $\widehat{\underline{z}}_d^N = (z_0^N, \dots, z_{2N-1}^N)^T$, and $(R[\widehat{P}_1^N(x)])_j = (\Phi_j, \widehat{P}_1^N(x))_X$.

Combining the above, we obtain matrix equations for the optimal control problem in $\boldsymbol{R}^{2N}$. The unique optimal control for the control system in $\boldsymbol{R}^{2N}$ is given in a feedback form by

$$\underline{F}_{\text{opt}}^N(t) = -\theta^{-1} \mathcal{B}_N^T\, (\mathcal{G}_N \underline{P}_3^N + \underline{r}^N(t)) \tag{5.10}$$

where $\underline{F}_{\text{opt}}^N(t) = (F_0^N, \dots, F_{2N-1}^N)^T \in \boldsymbol{R}^{2N}$, and $\underline{P}_3^N = (P_{3,0}^N, P_{3,1}^N \dots, P_{3,2N-1}^N)^T \in \boldsymbol{R}^{2N}$ is the unique solution to the matrix equation

$$\begin{cases} \dot{\underline{P}}_3^N = (\mathcal{A}_N - \theta^{-1} \mathcal{B}_N \mathcal{B}_N^T \mathcal{G}_N) \underline{P}_3^N - \theta^{-1} \mathcal{B}_N \mathcal{B}_N^T \underline{r}^N \\ \underline{P}_3^N(0) = \underline{P}_3^N(\tau), \end{cases} \tag{5.11}$$

and $\mathcal{G}_N$ is the solution to the matrix Algebraic Riccati Equation

$$\mathcal{A}_N^T \mathcal{G}_N + \mathcal{G}_N \mathcal{A}_N - \theta^{-1} \mathcal{G}_N \mathcal{B}_N \mathcal{B}_N^T \mathcal{G}_N + \mathcal{M}_N = 0. \tag{5.12}$$

The tracking variable $\underline{r}^N(x,t)$ is a vector in $\boldsymbol{R}^{2N}$ of the form $\underline{r}^N(x,t) = \widehat{\underline{r}}^N(x)\, e^{i\omega t}$ with $\widehat{\underline{r}}^N(x) \in \boldsymbol{R}^{2N}$ satisfying the following matrix equation:

$$[i\omega + (\underline{\mathcal{A}}_N^T - \theta^{-1}\underline{\mathcal{G}}_N\underline{\mathcal{B}}_N\underline{\mathcal{B}}_N^T)]\,\widehat{\underline{r}}^N(x) = \underline{\mathcal{M}}_N\widehat{\underline{z}}_d^N. \tag{5.13}$$

The cost function can now be represented as

$$J^N = \int_0^T \{(\underline{P}_3^N(t) - \underline{z}_d^N(t))^*\underline{\mathcal{M}}_N(\underline{P}_3^N(t) - \underline{z}_d^N(t)) + \theta(\underline{F}^N)^*(t)\underline{F}^N(t)\}\ \mathrm{d}t \tag{5.14}$$

Equations (5.10) – (5.14) are the basic equations that are used in our computations.

For the numerical optimization, we minimize with respect to x_c the following finite-dimensional cost function evaluated at the optimal finite-dimensional control

$$J^N(\underline{F}_{\text{opt}}^N) = \int_0^T \{(\underline{P}_1^N)^*\mathcal{M}_N\underline{P}_1^N - \theta^{-1}(\underline{r}^N)^*\mathcal{B}_N\mathcal{B}_N^*\underline{r}^N\}\ \mathrm{d}t.$$

For the simple model of the primary noise as $p_1(x,t) = 2e^{i\omega t}$ with the constant amplitude of 2 PA, we consider three different values freq $= 1$, 86.5, 173 Hz, where the last two frequencies correspond to the first two fundamental frequencies of the cavity. In the following simulations we calculate the optimal location x_c of a control with radius $a = 0.1$ for these 3 frequencies. The following graphs are the graphs of the norm of the overall reduced noise field $|\widehat{P}_1^N + \widehat{P}_3^N|$ along the length of the cavity. Since the primary as well as the secondary sound fields are sinusoidal with the same frequency, in these graphs only the spatial norms are demonstrated. All the calculations are performed using the MATLAB Optimization Toolbox and the following parameters: The degree of approximation $= 16$, $\alpha = 2178.4\ 1/s$, $\beta = 0.76185$, $\gamma = 346$ m/s, $\theta = 10^{-5}$.

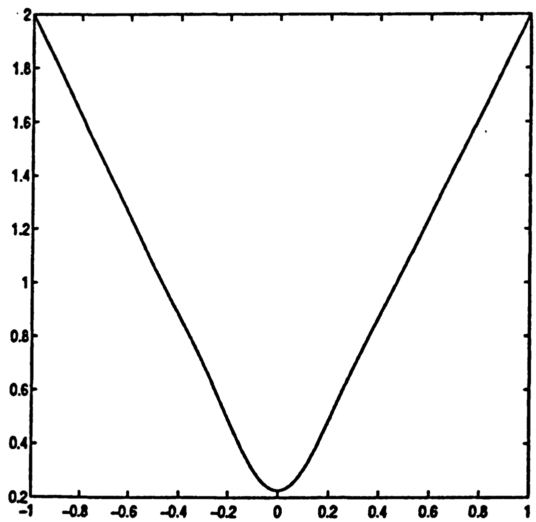

Fig. 1. $|\widehat{P}_1^N + \widehat{P}_3^N|$ vs x, with one control located optimally at $x = 0.0$ for freq $= 1$ Hz Total reduction=4.26 dB.

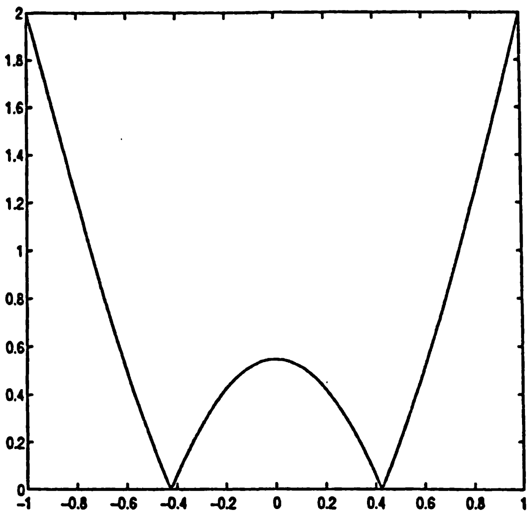

Fig. 2. $|\widehat{P}_1^N + \widehat{P}_3^N|$ vs x, with one control located optimally at $x = 0.0$ for freq $= 86.5$ Hz Total reduction= 7.23 dB.

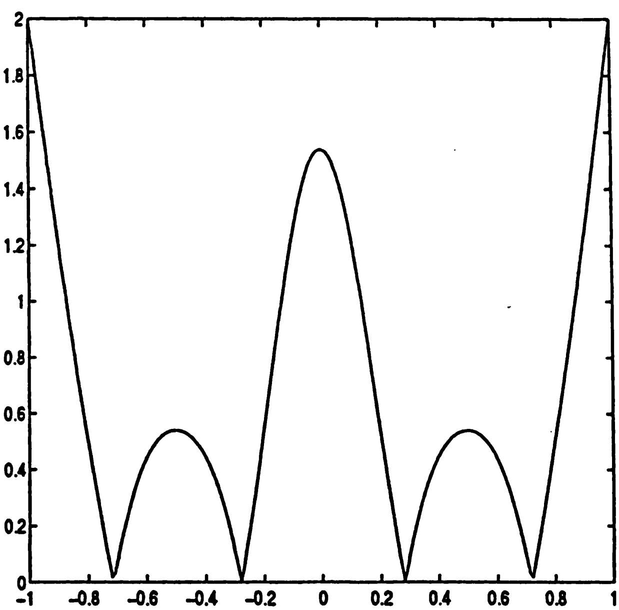

Fig. 3. $|\widehat{P}_1^N + \widehat{P}_3^N|$ vs x, with one control located optimally at $x = 0.0$ for freq $= 173$ Hz Total reduction= 7.5 dB.

From these figures, one can see that one control placed optimally at the center is quite effective in reducing the sound field for these frequencies. To see the effect of location on the effectiveness of the control, we placed the control at a non-optimal point $x = 0.1$, for freq = 173.

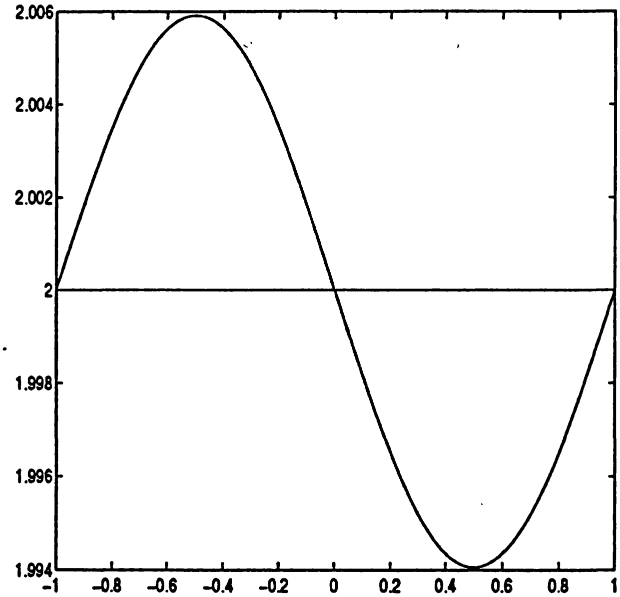

Fig. 4. $|\hat{P}_1^N + \hat{P}_3^N|$ vs x, with one control located at a non-optimal location at $x = 0.1$ for freq = 173 Hz.

As we can see from Figure 4, the non-optimal placement not only reduces the effectiveness of the control, but actually increases the sound field level in some locations $(-1 \leq x \leq 0)$. This figure clearly demonstrates the importance of placing the controls at optimal locations in order to maximize the effectiveness of the controls.

6. CONCLUSIONS

In this paper, the problem of optimal location of actuators for a one-dimensional acoustic cavity was presented as optimizing a cost function based on a linear quadratic tracking problem formulation. In carrying out the approximations of the control and the optimization problems, a numerical approximation based on the Legendre–tau was presented and calculations were carried out with the harmonic primary noise modeled as a sinusoidal single frequency wave. The calculations demonstrated the effectiveness of the control strategy as well as the importance of finding the optimal locations for the controls in order to have the best overall reduction of the noise field.

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