Kybernetika

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*Kybernetika*, Vol. 34 (1998), No. 2, [181]--187

Persistent URL: http://dml.cz/dmlcz/135197

KYBERNETIKA — VOLUME 34 (1998), NUMBER 2, PAGES 181–187

# SECOND ORDER LINEAR DIFFERENCE EQUATIONS OVER DISCRETE HARDY FIELDS

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We shall investigate the properties of solutions of second order linear difference equations defined over a discrete Hardy field via canonical valuations.

## 1. INTRODUCTION

Recently Boshernitzan [3] introduced the notion of discrete hardy field and studied the properties of the sequences satisfying some difference equations. In this paper we shall study the properties of solutions of the second order linear difference equations over a perfect discrete Hardy field via canonical valuations. For related results see [2, 5, 7, 8] and the references contained therein.

The results obtained in this paper have applications in the fields of discrete time systems, numerical analysis, biology, population dynamics, economics, control theory, computer science etc., see [1, 4]. The motivation of the present work stems from [6, 8].

## 2. DEFINITIONS AND NOTATIONS

Denote $B_s$, the class of real valued sequences $\{a_n\}$ where $a_n$ is defined for large values of $n$. The set $B_s$ is a ring with respect to pointwise addition and multiplication and is partially ordered by the relation "$\gg$" defined by $\{a_n\} \gg \{b_n\}$ if and only if $a_n > b_n$ for large $n$. Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be comparable if either $\{a_n\} \gg \{b_n\}$ or $\{a_n\} = \{b_n\}$ or $\{a_n\} \ll \{b_n\}$.

The subrings (subfields) of the ring $B_s$ will be called $B_s$-rings ($B_s$-fields). A $B_s$-ring ($B_s$-field) is said to be ordered if every sequence in it is ultimately of definite sign. An ordered $B_s$-field which is closed under translation is called an ordered $\Delta\, B_s$-field or discrete Hardy field and it is denoted by $K$. A sequence $\{a_n\}$ is said to be $\Delta$ consistent with a discrete Hardy field $K$ if there exists a discrete Hardy field $K'$ containing both $K$ and $\{a_n\}$. The intersection of all minimal discrete Hardy fields is denoted by $E_s$ and it is equal to the set of sequences $\{a_n\} \in B_s$ which are $\Delta$ consistent with every discrete Hardy field [3]. The rational constants belong to $E_s$. It should be noted that every ordered $B_s$-field contains rational constants and so any sequence $\{r_n\}$ in it being comparable with a rational constant must have a limit finite or infinite.

The perfect closure of a discrete Hardy field $K$ is denoted by $E_s(K)$ and is defined as the intersection of all maximal discrete Hardy fields containing $K$. A discrete Hardy field $K$ is said to be perfect if $E_s(K) = K$. The field $E_s$ of sequences is the minimal discrete Hardy field. For further details one can refer to [3].

## 3. THE CANONICAL VALUATIONS OF A DISCRETE HARDY FIELD

Throughout we assume that the discrete Hardy field $K$ is perfect. We now discuss the valuation that is naturally associated with any discrete Hardy field $K$. We begin with the following theorem which is a discrete analogue of Theorem 4 of [6].

**Theorem 1.** Let $K$ be a perfect discrete Hardy field. Then there exists a homomorphism $\nu$ from the set $K^*$ of nonzero elements of $K$ onto an ordered abelian group $\Gamma$ such that

(i) if $a_n, b_n \in K^*$, then $\nu(a_n b_n) = \nu(a_n) + \nu(b_n)$;

(ii) if $a_n \in K^*$ then $\nu(a_n) \geq 0$ if and only if $\lim_{n\to\infty} a_n \in R$ where $R$ denotes the field of real numbers;

(iii) writing symbolically $\nu(0)=\infty$, if $a_n, b_n \in K$ then $\nu(a_n+b_n) \geq \min\{\nu(a_n), \nu(b_n)\}$ with equality if $\nu(a_n) \neq \nu(b_n)$;

(iv) if $a_n, b_n \in K^*$ and $\nu(a_n), \nu(b_n) \neq 0$ then $\nu(\Delta a_n) \geq \nu(\Delta b_n)$ if and only if $\nu(a_n) \geq \nu(b_n)$;

(v) if $a_n \in K^*$ and $\nu(a_n) > \nu(b_n) \neq 0$ then $\nu(\Delta a_n) > \nu(\Delta b_n)$.

Proof. Let $a_n, b_n \in K^*$. Define the relation $\approx$ by $a_n \approx b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a finite nonzero number. Then clearly this relation on $K^*$ is an equivalence relation. Hence the equivalence relation decomposes $K^*$ into union of mutually disjoint equivalent classes. Let $\nu(a_n)$ denote the equivalence class of $a_n \in K^*$. Denote by $\Gamma$, the set of equivalence classes on $K^*$. Thus $\Gamma = \{\nu(a_n) : a_n \in K^*\}$. If $a_n, b_n, e_n, d_n \in K^*$ and $a_n \approx b_n$ and $e_n \approx d_n$ then $a_n e_n \approx b_n d_n$ so that multiplication on $K^*$ induces a composition of elements of $\Gamma$. Thus $\Gamma$ becomes an abelian group with identity element $\nu(1)$ and the map $\nu : K^* \to \Gamma$ is a homomorphism. This group $\Gamma$ is called the value group of $K$.

Now we follow the convention of writing the composition law of $\Gamma$ additively as it is done in the continuous case [7]. If $a_n, b_n \in K^*$, then define $\nu(a_n) > \nu(b_n)$ ($\nu(a_n) < \nu(b_n)$) if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. This definition depends only on the equivalence class $\nu(a_n)$ and $\nu(b_n)$ of $a_n$ and $b_n$ respectively. Further it includes a total ordering on the set $\Gamma$. By the above definition if $a_n \in K^*$ then $\nu(a_n) > 0$ $(= \nu(1))$. This means simply that $\lim_{n\to\infty} a_n = 0$. If $a_n\, b_n \in K^*$ and if $\nu(a_n) > \nu(b_n) > 0$ then $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$ and so $\nu(a_n) + \nu(b_n)$ $(\nu(a_n, b_n)) > 0$. Thus $\Gamma$ is an order abelian group with identity element $\nu(1)$. Also if $a_n, b_n \in K^*$ then $\nu(a_n) > \nu(b_n)$ means $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite. Thus if $a_n \in K^*$ then $\nu(a_n) \geq 0$ if and only if $\lim_{n\to\infty} a_n \in R$ where $R$ is the set of all real numbers.

Hence we have associated with the discrete Hardy field $K$ an ordered abelian group $\Gamma$ and an onto map $\nu : K^* \to \Gamma$ such that (i) if $a_n,\, b_n \in K^*$ then $\nu(a_n, b_n) = \nu(a_n)+\nu(b_n)$, (ii) if $a_n,\, b_n \in K^*$ and $a_n \neq -b_n$ then $\nu(a_n+b_n) \geq \min\{\nu(a_n),\, \nu(b_n)\}$. This map $\nu$ is called the canonical valuation of $K$ with value group $\Gamma$.

Thus we have established the existence of a surjective map $\nu$ from the nonzero elements $K^*$ of $K$ onto an ordered abelian group $\Gamma$ satisfying conditions (i), (ii) and (iii). The order conditions (iv) and (v) can be easily proved by applying discrete L'Hospital rule [2]. This completes the proof of the theorem. □

**Remark.** The kernel of this homomorphism $\nu$ consists of all $f_n \in K^*$ such that $\lim_{n\to\infty} f_n$ is finite and nonzero, while $\nu(a_n) > 0$ if and only if $\lim_{n\to\infty} f_n = 0$ and $\nu(f_n) < 0$ if and only if $\lim_{n\to\infty} f_n = \pm\infty$.

If $a_n,\, b_n \in K^*$ then $\nu(a_n) > \nu(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\nu(a_n) \geq \nu(b_n)$ if and only if $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite.

## 4. APPLICATION TO SECOND ORDER LINEAR DIFFERENCE EQUATIONS

Consider the second order difference equations of the form

$$\Delta^2 y_n - p_n\, y_{n+1} = 0, \tag{E$_1$}$$

over a perfect discrete Hardy field $K$. In the following we establish conditions for the solutions of (E$_1$) to lie in a perfect Hardy field $K$ and study their properties using the canonical valuation $\nu$. A solution (E$_1$) means a nontrivial solution for large values of $n$.

A sequence $\{a_n\} \in B_s$ is said to be nonoscillatory if $a_n\, a_{n+1} > 0$ for all $n$ sufficiently large; otherwise it is called oscillatory. Thus a solution of (E$_1$) is nonoscillatory if it is eventually positive or eventually negative. Moreover, if all solutions of (E$_1$) are nonoscillatory then (E$_1$) is called nonoscillatory otherwise (E$_1$) is called oscillatory.

**Theorem 2.** Let $K$ be any discrete Hardy field and $B_s$ denote the $B_s$-field of real sequences. Assume that $\{y_n\} \in B_s$ satisfies the difference equation (E$_1$) with $p_n \in K$ a rational sequence. If equation (E$_1$) is nonoscillatory, then $\{y_n\} \in E_s[K]$.

Proof. Let $\{y_n\} \in B_s$ be nonoscillatory solution of (E$_1$). Define $Z_n = \frac{y_{n+1}}{y_n}$ then $Z_n \in B_s$ is positive and satisfies the Riccati difference equation

$$Z_{n+1} = p(Z_n, k_n)$$

where $p(Z_n, k_n) = -1/Z_n + p_n + 2$. It is easy to see that $p(Z_n, k_n)$ satisfies all the conditions of Theorem 5.1 [5] and so $Z_n \in E_s[K]$. But $y_{n+1} = y_n\, Z_n$ and so

$$y_n = Z_1,\, Z_2, \ldots, Z_{n-1}\, y_1 \in E_s[K].$$

□

**Remark 2.** Theorem 2 shows that any nonoscillatory sequence satisfying ($E_1$) belongs to $E_s[K]$. This is a partial discrete analogue of Theorem 16.7 of [3] because here the $p_n \in K$ is a rational sequence where as $\phi \in K$ in Theorem 16.7 of [3] need not be so.

**Example 1.** Consider the difference equation

$$\Delta^2 y_n - 4y_{n+1} = 0 \tag{1}$$

over a minimal perfect discrete Hardy field $E_s$. From Proposition 1 [9] equation (1) is nonoscillatory and hence every solution of (1) belongs to $E_s$. In fact $\{(3+2\sqrt{2})^n\}$ and $\{(3-2\sqrt{2})^n\}$ are nonoscillatory solutions of (1) that belong to $E_s$.

**Theorem 3.** Let $K$ be any discrete Hardy field and the difference equation ($E_1$) be nonoscillatory. If $Q_n \in K$, then the solutions of the non-homogeneous equation

$$\Delta^2 y_n - p_n\, y_{n+1} = Q_n \tag{E$_2$}$$

belongs to $E_s[K]$.

Proof. Let $\{y_n^1\}$ and $\{y_n^2\}$ be two linearly independent solutions of equation ($E_1$). By Theorem 2, they belong to $E_s[K]$. The general solution of ($E_2$) is of the form

$$y_n = c_1\, y_n^1 + c_2\, y_n^2 + y_n^p$$

where $c_1$ and $c_2$ are arbitrary constants and $y_n^p$ is the particular integral. To prove the theorem is is enough to show that $y_n^p \in E_s[K]$. From variation of constants method we have

$$y_n^p = y_n^2 \sum_{j=1}^{n-1} Q_j\, y_{j+1}^1 - y_n^1 \sum_{j=1}^{n-1} Q_j\, y_{j+1}^2$$

where we choose the Casoration (the discrete analog of Wronskian) [4, p. 93] is unity. Clearly $y_n^p \in E_s[K]$. This completes the proof of the theorem. □

**Example 2.** Consider the difference equation

$$\Delta^2 y_n - \tfrac{1}{2} y_{n+1} = 4^n + 3n \tag{2}$$

over a discrete hardy field $E_s$. The homogeneous part of nonoscillatory with solution basis $\{2^n,\, 1/2^n\} \in E_s$. Thus all conditions of Theorem 3 are satisfied and so the solution of (3) belongs to $E_s$.

**Theorem 4.** Suppose the equation ($E_1$) has two linearly independent solutions in $K$. Then there are linearly independent solutions $\{y_n^1\}$ and $\{y_n^2\}$ such that $\nu(y_n^1) > \nu(y_n^2)$. If $\{y_n^1\}$ and $\{y_n^2\}$ are chosen positive their Casoration $C[y_n^1,\, y_n^2]$ is a positive constant $d$, $\Delta\left(\frac{y_n^1}{y_n^2}\right) = -\frac{d}{y_n^2\, y_{n+1}^2}$ and $\Delta\left(\frac{y_n^2}{y_n^1}\right) = \frac{d}{y_n^1\, y_{n+1}^1}$. If further $v_i = \frac{\Delta y_n^i}{y_n^i}$, $(i = 1, 2)$ then $v_2 - v_1 = \frac{d}{y_n^1\, y_n^2} > 0$.

Proof. Since any two linearly independent solutions of the difference equation ($E_1$) with the same $\nu$-value have a quotient that approaches a nonzero real limit as $n \to \infty$, they have a nonzero real linear combination with higher $\nu$-value. This shows the existence of $\{y_n^1\}$ and $\{y_n^2\}$ as desired. Since

$$C[y_n^1, y_n^2] = y_n^1 \,\Delta\, y_n^2 - y_n^2 \,\Delta\, y_n^1$$

we have

$$\Delta\left(C[y_n^1, y_n^2]\right) = y_{n+1}^1 \,\Delta^2 y_n^2 - y_{n+1}^2 \,\Delta^2 y_n^1 = 0$$

and hence $C[y_n^1, y_n^2]$ is a constant $d \in R$. This $d \neq 0$, for otherwise $\frac{y_n^1}{y_n^2}$ would be a constant. Since $y_n^1$, $y_n^2 > 0$ and $\nu(y_n^1) > \nu(y_n^2)$ we have $\frac{y_n^1}{y_n^2} \to 0$ as $n \to \infty$, hence decreasing, so $\Delta\left(\frac{y_n^1}{y_n^2}\right) = -\frac{d}{y_n^2\, y_{n+1}^2} < 0$. Since $\{y_n^2\}$ is nonoscillatory, we have $y_n^2\, y_{n+1}^2 > 0$ and therefore $d > 0$. Further

$$\Delta\left(\frac{y_n^2}{y_n^1}\right) = \frac{C[y_n^1, y_n^2]}{y_n^1\, y_{n+1}^1} = \frac{d}{y_n^1\, y_{n+1}^1}$$

and

$$v_2 - v_2 = \frac{\Delta\, y_n^2}{y_n^2} - \frac{\Delta\, y_n^1}{y_n^1} = \frac{C[y_n^1, y_n^2]}{y_n^1\, y_n^2} = \frac{d}{y_n^1\, y_n^2} > 0. \qquad \square$$

**Theorem 5.** Let $\{p_n\}$ and $\{q_n\}$ be elements of a discrete Hardy field $K$ in which each of the difference equations ($E_1$) and $\Delta^2 z_n - q_n\, z_{n+1} = 0$ has two linearly independent solutions. Let $\{y_n^1\}$, $\{y_n^2\}$ and $\{z_n^1\}$, $\{z_n^2\}$ respectively, be linearly independent solutions of the given difference equations with $\nu(y_n^1) > \nu(y_n^2)$ and $\nu(z_n^1) > \nu(z_n^2)$ and suppose that $\nu(y_n^1) > \nu(z_n^1)$. Then

(i) $\nu(y_n^2) < \nu(z_n^2)$,

(ii) $\dfrac{\Delta\, y_n^1}{y_n^1} < \dfrac{\Delta\, z_n^1}{z_n^1}$,

(iii) $\dfrac{\Delta\, y_n^2}{y_n^2} < \dfrac{\Delta\, z_n^2}{z_n^2}$, and

(iv) $p_n > q_n$.

Proof. Since $\nu(y_n^1) > \nu(z_n^1)$ and $\nu(z_n^1) > \nu(z_n^2)$ we have $\nu\left(\frac{y_n^1}{z_n^1}\right) > 0 > \nu\left(\frac{z_n^2}{z_n^1}\right)$. It follows from Theorem 1,

$$\nu\left(\Delta\left(\frac{y_n^1}{z_n^1}\right)\right) > \nu\left(\Delta\left(\frac{z_n^2}{z_n^1}\right)\right)$$

or

$$\frac{\nu\left(z_n^1\,\Delta\, y_n^1 - y_n^1\,\Delta\, z_n^1\right)}{z_n^1\, z_{n+1}^1} > \frac{\nu\left(z_n^1\,\Delta\, z_n^2 - z_n^2\,\Delta\, z_n^1\right)}{z_n^1\, z_{n+1}^1}$$

or

$$\nu\left(z_n^1\,\Delta\,y_n^1 - y_n^1\,\Delta\,z_n^1\right) > \nu\left(z_n^1\,\Delta\,z_n^2 - z_n^2\,\Delta\,z_n^1\right) = 0.$$

Assuming, as we may that $y_n^1$, $z_n^1 > 0$, we have $y_n^1/z_n^1$ is positive and approaching zero as $n \to \infty$, hence $\Delta(y_n^1/z_n^1) < 0$. Thus we have $\nu\left(z_n^1\,\Delta\,y_n^1 - y_n^1\,\Delta\,z_n^1\right) > 0$ and $z_n^1\,\Delta\,y_n^1 - y_n^1\,\Delta\,z_n^1 < 0$ or

$$\frac{\Delta\,y_n^1}{y_n^1} < \frac{\Delta\,z_n^1}{z_n^1}.$$

Since $z_n^1\,\Delta\,y_n^1 - y_n^1\,\Delta\,z_n^1 < 0$ and approaches zero as $n \to \infty$ it follows that $\Delta(z_n^1\,\Delta\,y_n^1 - y_n^1\,\Delta\,z_n^1) > 0$. An easy approximation shows that $(p_n - q_n)\,z_{n+1}^1\,y_{n+1}^1 > 0$ which implies that $p_n > q_n$. To prove (i) it suffices to prove that

$$\nu\left(\frac{1}{y_n^2\,y_{n+1}^2}\right) > \nu\left(\frac{1}{z_n^2\,z_{n+1}^2}\right)$$

and therefore

$$\nu\left(\Delta\left(\frac{y_n^1}{y_n^2}\right)\right) > \nu\left(\Delta\left(\frac{z_n^1}{z_n^2}\right)\right)$$

which implies that

$$\nu\left(\frac{y_n^1}{y_n^2}\right) > \nu\left(\frac{z_n^1}{z_n^2}\right) \quad \text{or} \quad \nu\left(\frac{y_n^2}{y_n^1}\right) < \nu\left(\frac{z_n^1}{z_n^2}\right)$$

$$\text{or} \quad \nu\left(\Delta\left(\frac{y_n^2}{y_n^1}\right)\right) < \nu\left(\Delta\left(\frac{z_n^2}{z_n^1}\right)\right) \quad \text{or} \quad \nu\left(\frac{1}{y_n^2\,y_{n+1}^1}\right) < \nu\left(\frac{1}{z_n^1\,z_{n+1}^1}\right)$$

or $\nu(y_n^1) > \nu(z_n^1)$, which was assumed. Finally, since $\nu(y_n^2) < \nu(z_n^2)$, we have $\nu\left(\frac{z_n^2}{y_n^2}\right) > 0$. Taking $z_n^2$, $y_n^2$ positive, as we may, the positive function $\left(\frac{z_n^2}{y_n^2}\right) \to 0$ as $n \to \infty$, hence $\Delta\left(\frac{z_n^2}{y_n^2}\right) < 0$. Then $y_n^2\,\Delta\,z_n^2 - z_n^2\,\Delta\,y_n^2 < 0$ or $\frac{\Delta\,z_n^2}{z_n^2} < \frac{\Delta\,y_n^2}{y_n^2}$, which proves (iv). The proof of the theorem is complete. □

**Theorem 6.** Let ($E_1$) be nonoscillatory with solution basic $\{y_n^1, y_n^2\}$ ($y_n^i > 0$, $i = 1, 2$). If $\nu(y_n^1) > 0$ or $\nu(y_n^2) > 0$ then $p_n > 0$.

Proof. Assume $\nu(y_n^1) \gg 0 > \nu(n)$. By Theorem 1, it follows that

$$\nu(\Delta\,y_n^1) > \nu(\Delta\,n) = 0. \tag{3}$$

Also from $\nu(y_n^1) > 0$ we have

$$\Delta\,y_n^1 < 0. \tag{4}$$

From (4) and (5) we obtain $\Delta(\Delta\,y_n^1) > 0$, which gives $p_n\,y_{n+1}^1 > 0$ and so $p_n > 0$. Similarly we can prove that $p_n > 0$ if $\nu(y_n^2) > 0$. □

**(Received April 16, 1996.)**

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