Kybernetika

Jiřina Vejnarová

# A note on the interval-valued marginal problem and its maximum entropy solution

*Kybernetika*, Vol. 34 (1998), No. 1, [17]--26

Persistent URL: http://dml.cz/dmlcz/135182

KYBERNETIKA — VOLUME 34 (1998), NUMBER 1, PAGES 17-26

# A NOTE ON THE INTERVAL-VALUED MARGINAL PROBLEM AND ITS MAXIMUM ENTROPY SOLUTION[1]

JIŘINA VEJNAROVÁ

This contribution introduces the marginal problem, where marginals are not given precisely, but belong to some convex sets given by systems of intervals. Conditions, under which the maximum entropy solution of this problem can be obtained via classical methods using maximum entropy representatives of these convex sets, are presented. Two counterexamples illustrate the fact, that this property is not generally satisfied. Some ideas of an alternative approach are presented at the end of the paper.

## 1. INTRODUCTION

The marginal problem, the question of the existence of an extension – a measure with given marginals, belongs to the most challenging aspects of probability theory. The reason is not only its applicability in various questions of statistics, but also a great deal of relevant theoretical problems. One of them is that problem of finding a sufficient condition for the existence of a solution of the marginal problem. The necessary condition – the projectivity (or weak compatibility) condition – is very natural; it means, loosely speaking, that the given marginal measures have common "lower-dimensional" marginals. In some specific situations it becomes a necessary and sufficient condition ([4, 6]).

If an extension exists, it is usually not unique, but the problem has an infinite number of solutions. Therefore the problem of the existence of an extension is usually solved together with the problem of the choice of an – in some sense – optimum representative of the set of these solutions. One of the possibilities is to choose a measure $P^*$ with maximum entropy, which corresponds to the well-known (and widely used) maximum entropy principle ([1]). This choice is justified by the fact that the maximum entropy measure utilizes all information contained in the family of marginal measures. If we choose another measure with lower entropy, we would add some new information to the family.

[1]The work was partially supported by the grant VS96008 of the Ministry of Education of the Czech Republic.

In this paper the marginal problem is modified in such a way, that the marginals are not given precisely, but are supposed to belong to some convex sets. In our case it means that any of these sets is given by a system of intervals. This results in a nonlinear programming problem (a nonlinear function has to be maximized under linear constraints), but it is a problem of very high complexity even in low dimensional case.

It will be shown that in some specific situations maximum entropy solution of this problem can be obtained as the solution of a "classical" marginal problem, where the marginals are maximum entropy measures from the given convex sets.

## 2. "CLASSICAL" MARGINAL PROBLEM AND ITS REFORMULATION

The marginal problem is usually understood as follows: Let us suppose $(X_i, \mathcal{X}_i)$, $i \in N$, $\emptyset \neq N < \infty$ be measurable spaces, $\mathcal{T}$ be a system of nonempty subsets of $N$ and

$$\mathcal{S} = \{P_T, T \in \mathcal{T}\}$$

a family of probability measures, where any $P_T$ is a measure on a product space $(X_T, \mathcal{X}_T) = \times_{i \in T}(X_i, \mathcal{X}_i)$. The problem is the existence of an *extension*, i.e. a measure $P$ on $(X_N, \mathcal{X}_N)$, whose marginals are measures from $\mathcal{S}$.

Denoting $P^S$ the marginal of $P$ on $(X_S, \mathcal{X}_S)$, the set

$$\mathcal{P} = \{P : P^T = P_T, T \in \mathcal{T}\}$$

is of interest. The existence of an extension is equivalent to the nonemptiness of the set $\mathcal{P}$.

The maximum entropy measure can be found using Lagrange multipliers ([1]) or by the IPFP algorithm ([2]).

The *projectivity condition*, as mentioned in the introduction, a necessary condition for the existence of an extension, means that the equality

$$P_S^{S \cap T} = P_T^{S \cap T}$$

is satisfied for any pair $S, T \in \mathcal{T}$, such that $S \cap T \neq \emptyset$.

Let us suppose $X_i, i \in N$, with finite ranges. Now, the probabilities $P_T$ belong to some convex sets $\mathcal{P}_T$, i.e. the family $\mathcal{S}$ is now

$$\mathcal{S} = \{\mathcal{P}_T, T \in \mathcal{T}\}. \tag{1}$$

The existence of an extension – in this situation – means that the set

$$\mathcal{P} = \{P : P^T \in \mathcal{P}_T, T \in \mathcal{T}\} \tag{2}$$

is nonempty. We shall consider situation, when the set $\mathcal{P}_T$ is given by a system of inequalities, i.e.

$$\mathcal{P}_T = \{P_T : a(x_T) \leq P_T(x_T) \leq b(x_T), x_T \in X_T\},$$

where $0 \leq a(x_T) \leq b(x_T) \leq 1$ for all $x_T \in X_T$.

From the fact that the equality

$$b(x_T) = 1 - a(x_T^C)$$

(where $x^C$ denotes the complement of $x$) must be satisfied for any $x_T \in X_T, T \in \mathcal{T}$ (see e. g. [7]), it follows that

$$a(x_T) + \sum_{y_T \in X_T : y_T \neq x_T} b(y_T) \geq 1$$

and

$$b(x_T) + \sum_{y_T \in X_T : y_T \neq x_T} a(y_T) \leq 1$$

must be again satisfied for any $T \in \mathcal{T}$ and any choice of $x_T \in X_T$.

If $a(x_T) = b(x_T)$, for all $x_T \in X_T, T \in \mathcal{T}$, then we obtain "classical" marginal problem. On the other hand, $a(x_T) = 0$ and $b(x_T) = 1$ for all $x_T \in X_T, T \in \mathcal{T}$, means that we have no constraints, and therefore this is not a marginal problem. The cases of interest, from the viewpoint of this paper, are all those between these two extreme ones.

The *projectivity condition* is in this situation weakened to

$$\mathcal{P}_S^{S \cap T} \cap \mathcal{P}_T^{S \cap T} \neq \emptyset$$

for any pair $S, T \in \mathcal{T}$, such that $S \cap T \neq \emptyset$. It means that there exists at least one measure in $\mathcal{P}_S$ and at least one in $\mathcal{P}_T$, whose marginals on $S \cap T$ coincide.

## 3. MAXIMUM ENTROPY SOLUTION

As mentioned above, the maximum entropy solution can be obtained, in special cases, as a solution of "classical" marginal problem, where the marginals are maximum entropy measures $P_T^*$ from $\mathcal{P}_T$. The simplest situation is described by the following proposition.

**3.1. Proposition.** Let $\mathcal{T}$ be a system of pairwise disjoint subsets of $N$. Then the maximum entropy measure $P^*$ of $\mathcal{P}$ is given by the equality

$$P^*(x_N) = \prod_{T \in \mathcal{T}} P_T^*(x_T) \tag{3}$$

for any $x_N \in X_N$, $P_T^*$ being maximum entropy measures from $\mathcal{P}_T, T \in \mathcal{T}$.

Proof. The proof is an immediate consequence of the inequality

$$H(P) \leq \sum_{T \in \mathcal{T}} H(P_T), \tag{4}$$

holding for any distribution $P$ with marginals $P_T$, $T \in \mathcal{T}$, where the equality holds if and only if $P$ is a product of $P_T$, $T \in \mathcal{T}$. The maximum value of the right side of (4) is obtained for $P_T^*$, $T \in \mathcal{T}$, and the equality for $P^*$ defined by (3). □

But if the system $\mathcal{T}$ does not consist of pairwise disjoint sets, $P^*$ is not, in general, derivable from $P_T^*$, $T \in \mathcal{T}$. It is demonstrated by the following example.

**3.2. Example.** Let $X_1 = X_2 = X_3 = \{0,1\}$ and $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ be defined as follows:

$$\mathcal{P}_{12} = \{P_{12} : \begin{array}{l} \frac{1}{6} \le P_{12}(0,0) \le \frac{1}{3}, \\ \frac{1}{6} \le P_{12}(0,1) \le \frac{1}{3}, \\ \frac{1}{6} \le P_{12}(1,0) \le \frac{1}{3}, \\ \frac{1}{6} \le P_{12}(1,1) \le \frac{1}{3}\} \end{array}$$

and

$$\mathcal{P}_{13} = \{P_{13} : \begin{array}{l} 0 \le P_{13}(0,0) \le \frac{1}{3}, \\ \frac{1}{3} \le P_{13}(0,1) \le \frac{2}{3}, \\ 0 \le P_{13}(1,0) \le \frac{1}{3}, \\ 0 \le P_{13}(1,1) \le \frac{1}{3}\}. \end{array}$$

It can be easily verified that the sets are projective, i. e.

$$\mathcal{P}_{12}^1 = \{P_{12}^1 : \ \tfrac{1}{3} \le P_{12}^1(0) \le \tfrac{2}{3}, \ \tfrac{1}{3} \le P_{12}^1(1) \le \tfrac{2}{3}\},$$

and

$$\mathcal{P}_{13}^1 = \{P_{13}^1 : \ \tfrac{1}{3} \le P_{13}^1(0) \le 1, \ 0 \le P_{13}^1(1) \le \tfrac{2}{3}\}$$

have nonempty intersection. On the other hand

$$P_{12}^*(i,j) = \tfrac{1}{4},$$

for $i$, $j = 0,1$, is the maximum entropy measure from $\mathcal{P}_{12}$ and

$$P_{13}^*(0,0) = \tfrac{2}{9}, \ P_{13}^*(0,1) = \tfrac{1}{3}, \ P_{13}^*(1,i) = \tfrac{2}{9},$$

for $i = 0,1$, is the one from $\mathcal{P}_{13}$. Therefore $P_{12}^*$ and $P_{13}^*$ are not projective (since $P_{12}^{*1} \neq P_{13}^{*1}$).

This is caused by the obvious fact, that the projectivity of the families of measures from $\mathcal{S}$ defined by (1) does not imply the projectivity of measures from

$$\mathcal{S}^* = \{P_T^*, T \in \mathcal{T}\}$$

of their maximum entropy representatives.

It is obvious that the projectivity condition of maximum entropy measures is the necessary condition for getting the solution of our marginal problem via the solution of the "classical" one. It is, in general, not the sufficient one, since the projectivity condition in "classical" marginal problem does not imply the existence of an extension. But we can get the following result. For this purpose, let us remind the notion of *running intersection property.* A system $\mathcal{T}$ possesses the running intersection property, if there exists an ordering of subsets of $\mathcal{T}$ such that for each $i$ exists $j < i$ such that

$$(T_1 \cup \cdots \cup T_{i-1}) \cap T_i = T_j \cap T_i.$$

**3.3. Proposition.** Let $\mathcal{T}$ be a system of subsets of $N$ possessing the running intersection property and $\mathcal{S}$ be a family of measures possessing the projectivity condition. If the family $\mathcal{S}^*$ satisfies this condition as well, then the measure $P^*$ defined by the equality

$$P^*(x_N) = \frac{\prod_{T\in\mathcal{T}} P_T^*(x_T)}{\prod_{U\in\mathcal{U}} P_U^*(x_U)}, \tag{5}$$

for all $x_N \in X_N$, is the maximum entropy solution of the interval-valued marginal problem (where $P_T^*$ and $P_U^*$ are measures from $\mathcal{S}^*$ and their restrictions to the elements from $\mathcal{U} = \{U : U = S \cap T, S, T \in \mathcal{T}\}$, respectively).

Proof. The proof is an analogy of the one of Proposition 3.1. The maximum entropy measure has maximum entropy marginals and the maximum entropy extension of them (the solution of the "classical" marginal problem) is given by the equality (5) (see [6]). □

Propositions 3.1 and 3.3 describe the only known situations, in which we were able to express the maximum entropy solution of interval-valued marginal problem in explicit form. The following existence theorem comprises wider class of problems.

**3.4. Proposition.** Let $\mathcal{S}$ be a family of projective measures and $\mathcal{S}^*$ the family of their maximum entropy representatives. If $P^*$ is the maximum entropy extension of measures from $\mathcal{S}^*$, then it is also the maximum entropy extension of measures from $\mathcal{S}$.

Proof. The proof is obvious. Let $P^*$ be the maximum entropy extension of measures from $\mathcal{S}^*$ and $Q \in \mathcal{P}$ such that $H(Q) > H(P^*)$. Then either $H(Q_T) > H(P_T^*)$ for at least one $T \in \mathcal{T}$, which violates the maximality of $H(P_T^*)$ or $H(Q_T) = H(P_T^*)$ for all $T \in \mathcal{T}$ and $H(Q) > H(P^*)$, which violates the maximality of the entropy of the extension. □

This existence proposition allows us to obtain the maximum entropy solution even if the running intersection property is not fulfilled. We can use some iterative methods to obtain the maximum entropy extension of the maximum entropy representatives. Such an extension is the optimal solution of the interval-valued marginal problem.

It should be stressed the meaning of the implication in Proposition 3.4. If the extension of the measures from $\mathcal{S}^*$ does not exist, it does not mean that there exists no extension of measures from $\mathcal{S}$, which is shown in the next example.

**3.5. Example.** Let $X_1 = X_2 = X_3 = \{0,1\}$ and $\mathcal{P}_{12}$, $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$ be defined as follows:

$$\begin{aligned}\mathcal{P}_{12} = \{P_{12} : \quad & \tfrac{1}{6} \le P_{12}(0,0) \le \tfrac{1}{3},\\ & \tfrac{1}{6} \le P_{12}(0,1) \le \tfrac{1}{3},\\ & \tfrac{1}{6} \le P_{12}(1,0) \le \tfrac{1}{3},\\ & \tfrac{1}{6} \le P_{12}(1,1) \le \tfrac{1}{3}\},\end{aligned}$$

$$\mathcal{P}_{13} = \{P_{13} : \begin{array}{l} 0 \le P_{13}(0,0) \le \frac{1}{12}, \\ \frac{5}{12} \le P_{13}(0,1) \le \frac{1}{2}, \\ \frac{5}{12} \le P_{13}(1,0) \le \frac{1}{2}, \\ 0 \le P_{13}(1,1) \le \frac{1}{12}\} \end{array}$$

and

$$\mathcal{P}_{23} = \{P_{23} : \begin{array}{l} 0 \le P_{13}(0,0) \le \frac{1}{12}, \\ \frac{5}{12} \le P_{13}(0,1) \le \frac{1}{2}, \\ \frac{5}{12} \le P_{13}(1,0) \le \frac{1}{2}, \\ 0 \le P_{13}(1,1) \le \frac{1}{12}\}\,. \end{array}$$

Their maximum entropy representatives are

$$P^*_{12}(i,j) \quad = \quad \tfrac{1}{4},$$

for $i$, $j = 0, 1$,

$$P^*_{13}(0,0) = \tfrac{1}{12}, \; P^*_{13}(0,1) = \tfrac{5}{12}, \; P^*_{13}(1,0) = \tfrac{5}{12}, \; P^*_{13}(1,1) = \tfrac{1}{12},$$

and

$$P^*_{23}(0,0) = \tfrac{1}{12}, \; P^*_{23}(0,1) = \tfrac{5}{12}, \; P^*_{23}(1,0) = \tfrac{5}{12}, \; P^*_{23}(1,1) = \tfrac{1}{12}.$$

It can be easily verified that the one-dimensional marginals coincide, which means that the projectivity condition is satisfied. Nevertheless, denoting

$$P(0,0,0) = \alpha$$

(this $\alpha$ will be later specified), the following equalities are obtained from $P^*_{12}$, $P^*_{13}$ and $P^*_{23}$:

$$\begin{array}{lll} P(0,0,1) = \frac{1}{4} - \alpha, & P(0,1,0) = \frac{1}{12} - \alpha, & P(0,1,1) = \frac{1}{6} + \alpha, \\ P(1,0,0) = \frac{1}{12} - \alpha, & P(1,0,1) = \frac{1}{6} + \alpha, & P(1,1,0) = \frac{1}{3} + \alpha, \\ P(1,1,1) = -\frac{1}{12} - \alpha. & & \end{array}$$

It is evident that for any $\alpha \ge 0$ the last expression is negative, which implies the emptiness of the set $\mathcal{P}^* = \{P : P^{12} = P^*_{12}, P^{13} = P^*_{13}, P^{23} = P^*_{23}\}$. On the other hand, the set $\mathcal{P} = \{P : P^{12} \in \mathcal{P}_{12}, P^{13} \in \mathcal{P}_{13}, P^{23} \in \mathcal{P}_{23}\}$ is nonempty, e.g. the measure

$$\begin{array}{llll} P(0,0,0) = 0, & P(0,0,1) = \frac{1}{3}, & P(0,1,0) = \frac{1}{12}, & P(0,1,1) = \frac{1}{12}, \\ P(1,0,0) = \frac{1}{12}, & P(1,0,1) = \frac{1}{12}, & P(1,1,0) = \frac{1}{3}, & P(1,1,1) = 0 \end{array}$$

belongs to $\mathcal{P}$.

## 4. ANOTHER APPROACH

Situations, in which we are able to obtain the maximum entropy solution via classical methods using maximum entropy representatives of $\mathcal{P}_T, T \in \mathcal{T}$, are very specific. Either $\mathcal{T}$ must be a system of pairwise disjoint subsets of $N$ (which is trivial problem) or $\mathcal{S}^*$ must be a family of projective measures (which is very strong assumption). The rest of situations is unsolved. Therefore we want to present some ideas which seem to help us to solve at least some of them.

If $\mathcal{T}$ is not a system of pairwise disjoint subsets, the projectivity condition of $\mathcal{P}_T, T \in \mathcal{T}$, is a necessary one. It has been shown in Example 3.2 that the projectivity of the families of measures from $\mathcal{S}$ does not imply the projectivity of the measures from $\mathcal{S}^*$. In other words: if $\mathcal{P}_S$ and $\mathcal{P}_T, S \cap T \neq \emptyset$, are families of projective measures, it does not mean that any $P_T \in \mathcal{P}_T$ and any $P_S \in \mathcal{P}_S, S \cap T \neq \emptyset$, are projective. Moreover, there may exist a measure $Q_T \in \mathcal{P}_T$ and a family $\mathcal{Q}_S$ $(S \in \mathcal{T}, S \neq T)$ such that

$$Q_T^{S \cap T} \neq Q_S^{S \cap T}$$

for all $Q_S \in \mathcal{Q}_S$.

This is the reason why to introduce the projectivity of a family $\mathcal{S}$. The family $\mathcal{S}$ defined by (1) is *projective* if for any $P_T \in \mathcal{P}_T, T \in \mathcal{T}$, there exists at least one $P_S \in \mathcal{P}_S (S \in \mathcal{T}, S \cap T \neq \emptyset)$ such that

$$P_S^{S \cap T} = P_T^{S \cap T}.$$

**4.1. Proposition.** For any $\mathcal{P}_T \in \mathcal{S}$ let us define

$$\mathcal{P}_T^W = \mathcal{P}_T \cap \left( \bigcap_{S: S \cap T \neq \emptyset} \mathcal{P}_{T \cap S} \right), \tag{6}$$

where

$$\mathcal{P}_{T \cap S} = \left\{ P_T : a(x_{S \cap T}) \leq P_T^{T \cap S}(x_{T \cap S}) \leq b(x_{S \cap T}), x_{T \cap S} \in X_{T \cap S} \right\},$$
$$\left( a(x_{S \cap T}) = \sum_{x_S : \Pi_{T \cap S}(x_S) = x_{T \cap S}} a(x_S) \right)$$

then

$$\mathcal{S}^W = \{\mathcal{P}_T^W, T \in \mathcal{T}\} \tag{7}$$

is projective family of sets of measures.

Proof. The proof is straightforward. Let $P_T$ be arbitrary measure from $\mathcal{P}_T^W$ and $S \in \mathcal{T}$ such that $S \cap T \neq \emptyset$, then

$$\sum_{x_S : \Pi_{T \cap S}(x_S) = x_{T \cap S}} a(x_S) \leq P_T^{T \cap S}(x_{T \cap S}) \leq \sum_{x_S : \Pi_{T \cap S}(x_S) = x_{T \cap S}} b(x_S), \tag{8}$$

for all $x_{T \cap S} \in X_{T \cap S}$ and since

$$a(x_S) \leq P_S(x_S) \leq b(x_S),$$

for all $x_S \in X_S$ the inequality (8) is satisfied for $P_S^{S\cap T}(x_{T\cap S})$ as well, and therefore we can find such $P_S$ which satisfies

$$P_S^{S\cap T} = P_T^{S\cap T}. \qquad \square$$

This fact seems to be very useful. If we use $\mathcal{S}^W$ instead of $\mathcal{S}$, the maximum entropy representatives of $\mathcal{P}_T^W$ may satisfy projectivity condition (due to more restrictive character of $\mathcal{S}^W$) or – at least – we will get the following existence proposition.

**4.2. Proposition.** Let $\mathcal{T}$ be a system of subsets of $N$ possessing running intersection property. If $\mathcal{P}_T^W$ defined by (6) is nonempty for all $T \in \mathcal{T}$, then $\mathcal{P}$ defined by (2) is nonempty, i.e. the solution of the marginal problem exists.

Proof. The proof is an immediate consequence of the fact that $\mathcal{T}$ satisfies running intertsection property and for any $P_T \in \mathcal{P}_T$ there exists some $P_S \in \mathcal{P}_S, S \in \mathcal{T}$, which are projective. $\square$

Let us note that Proposition 4.2 assures the existence of the solution of the problem presented in Example 3.2.

It is evident that this existence proposition does not give us the answer to the question: *How to get the maximum entropy solution?* It seems to us, at this stage, that the most promising is the following approach.

Let us suppose that $\mathcal{T} = \{T, S\}$. We will start from the extreme points $P^e_{T\cap S}, e \in E$ of $\mathcal{P}_T \cap \mathcal{P}_S = \mathcal{P}_T^W \cap \mathcal{P}_S^W$ (which follows from the definition of $\mathcal{P}_T^W$). The set of all marginals $\mathcal{P}_{S\cap T}$ is then

$$\mathcal{P}_{S\cap T} = \left\{ P^\alpha_{S\cap T} = \sum\nolimits_{e\in E} \alpha_e P^e_{S\cap T}(x_{S\cap T}), \alpha_e \geq 0, \sum\nolimits_{e\in E} \alpha_e = 1 \right\}. \tag{9}$$

Since we are interested in maximum entropy measure from (2) we can confine ourselves on the maximum entropy measures from $\mathcal{P}_S$ and $\mathcal{P}_T$, respectively, satisfying the projectivity condition given by (9) for any $\alpha = (\alpha_e \geq 0, e \in E, \sum_{e\in E} \alpha_e = 1)$. Let us denote them $P_S^\alpha$ and $P_T^\alpha$. Their maximum entropy extensions are (according to (5))

$$P^\alpha(x) = \frac{P_S^\alpha(x_S) \cdot P_T^\alpha(x_T)}{P^\alpha_{S\cap T}(x_{S\cap T})},$$

and any $P^\alpha$ defined this way is solution of marginal problem (1). Let us note that it is not the whole set $\mathcal{P}$, but these measures are, let us say, quasi-optimal solutions of (1). Among them we will find the maximum entropy solution. It is again nonlinear programming problem, but the number of constraints was substantially decreased in comparison with the original problem. It is demonstrated in the following example.

**4.3. Example.** Let $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ be as in Example 3.2. We will find that $\mathcal{P}_{12}^W = \mathcal{P}_{12}$ and

$$\mathcal{P}_{13}^W = \left\{ P_{13} \in \mathcal{P}_{13} : \ \tfrac{1}{3} \leq P_{13}^1(0) \leq \tfrac{2}{3}, \quad \tfrac{1}{3} \leq P_{13}^1(0) \leq \tfrac{2}{3} \right\}.$$

From the extreme points of $\mathcal{P}_1 = \mathcal{P}^1_{12} \cap \mathcal{P}^1_{13}$

$$P_1^1(0) = \tfrac{1}{3}, \quad \text{and} \quad P_1^2(0) = \tfrac{2}{3},$$
$$P_1^1(1) = \tfrac{2}{3}, \qquad P_1^2(1) = \tfrac{1}{3},$$

we obtain

$$P_1^\alpha(0) = \tfrac{1}{3} + \tfrac{1}{3}\alpha, \quad P_1^\alpha(1) = \tfrac{2}{3} - \tfrac{1}{3}\alpha,$$

$\alpha \in [0,1]$. The maximum entropy measures from $\mathcal{P}_{12}$ and $\mathcal{P}_{13}$ satisfying this marginal condition for $\alpha \in [0,1]$ are

$$P_{12}^\alpha(0,i) = \tfrac{1}{6} + \tfrac{1}{6}\alpha, \quad P_{12}^\alpha(1,i) = \tfrac{1}{3} - \tfrac{1}{6}\alpha,$$

$i = 0, 1$ and

$$P_{13}^\alpha(0,0) = \tfrac{1}{3}\alpha, \quad P_{13}^\alpha(0,1) = \tfrac{1}{3}, \quad P_{13}^\alpha(1,i) = \tfrac{1}{3} - \tfrac{1}{6}\alpha,$$

$i = 0, 1$. Their maximum entropy extension for $\alpha \in [0,1]$ is

$$P^\alpha(0,i,0) = \tfrac{1}{6}\alpha, \quad P^\alpha(0,i,1) = \tfrac{1}{6}, \quad P^\alpha(1,i,j) = \tfrac{1}{6} - \tfrac{1}{12}\alpha,$$

$i, j = 0, 1$. The maximum entropy within this class of measures is obtained for $\alpha = \frac{2}{3}$, i.e. the maximum entropy solution of the marginal problem is

$$P^*(0,i,0) = \tfrac{1}{9}, \quad P^*(0,i,1) = \tfrac{1}{6}, \quad P^*(1,i,j) = \tfrac{1}{9},$$

$i, j = 0, 1$.

The original problem was to maximize a function of seven variables under sixteen inequalities. It was successively transformed to the problem of maximization of a function of one variable on the interval $[0,1]$ (i.e. under two inequalities), which was easy to solve.

It is obvious that the complexity of this problem depends on the size of $S \cap T$ and the one of any $X_i$, $i \in S \cap T$, nevertheless, the decrease of the complexity (in comparison with the original problem) is substantial.

## 5. CONCLUSIONS

Interval-valued marginal problem seems to be quite a natural modification of marginal problem. In practical situations, the marginals need not be known precisely, but can be obtained from statistical data in the form of confidence intervals or from expert as his/her subjective upper and/or lower probabilities.

The solution of this problem is of a very high complexity, which was demonstrated even in our simple examples. The explicit solution using maximum entropy representatives of given sets of measures can be obtained only in very specific situations (cf. Propositions 3.1 and 3.3), the existence proposition comprises the foregoing ones and points out how iterative methods can be used in this kind of problems.

Since the class of situations covered by preceding propositions is very small, another approach has been suggested in fourth section. The existence theorem (Proposition 4.2) states the sufficient condition for the existence of the solution of the interval-valued marginal problem (but the class of solvable problems is wider, cf. Example 3.5). The method suggesting how to find the maximum entropy solution of marginal problem (1) seems to be promising, but the main ideas must be yet formalized (for richer systems $\mathcal{T}$) and the optimality of the proposed solution should be proven. It will be one topic of our future work together with the study of the possible enlargement of solvable problems, i.e. finding weaker sufficient condition (or that defining another class of solvable problems).

(Received November 7, 1997.)

## REFERENCES

[1] P. Cheeseman: A method of computing generalized Bayesian probability values for expert systems. In: Proc. 6th Internat. Joint Conference on Artificial Intelligence (IJCAI-83), Karlsruhe 1983, pp. 198–202.

[2] W. E. Deming and F. F. Stephan: On a least square adjustment of a sampled frequency table when marginal totals are known. Ann. Math. Statist. *11* (1940), 427–444.

[3] R. R. Gallager: Information Theory and Reliable Communication. J. Wiley, New York 1968.

[4] R. Jiroušek and A. Perez: A partial solution of the marginal problem. In: Trans. of the 10th Prague Conference on Inform. Theory, vol. B, Academia, Prague 1987, pp. 11–19.

[5] H. G. Kellerer: Verteilungsfunktionen mit gegebenem Marginalverteilungen. Z. Wahrsch. verw. Gebiete *3* (1964), 247–270.

[6] F. M. Malvestuto: Existence of extensions and product extensions for discrete probability distributions. Discrete Math. *69* (1988), 61–77.

[7] P. Walley: Measures of uncertainty in expert systems. Artificial Intelligence *83* (1996), 1–58.

*RNDr. Jiřina Vejnarová, CSc., Laboratory of Intelligent Systems, Faculty of Informatics and Statistics, University of Economics, Ekonomická 957, 148 00 Praha 4*
*and*
*Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.*
*e-mail: vejnar@vse.cz*