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ON CONVERGENCE OF GRADIENT-DEPENDENT INTEGRANDS*

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Dedicated to Jiří V. Outrata on the occasion of his 60th birthday

Abstract. We study convergence properties of $\{v(\nabla u_k)\}_{k\in\mathbb{N}}$ if $v \in C(\mathbb{R}^{m\times n})$, $|v(s)| \leqslant C(1+|s|^p)$, $1 < p < +\infty$, has a finite quasiconvex envelope, $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and for some $g \in C(\Omega)$ it holds that $\int_\Omega g(x)v(\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega g(x)Qv(\nabla u(x))\,\mathrm{d}x$ as $k \to \infty$. In particular, we give necessary and sufficient conditions for L^1-weak convergence of $\{\det \nabla u_k\}_{k\in\mathbb{N}}$ to $\det \nabla u$ if $m = n = p$.

Keywords: bounded sequences of gradients, concentrations, oscillations, quasiconvexity, weak convergence

MSC 2000: 49J45, 35B05

1. Introduction

Oscillations and/or concentrations appear in many problems in the calculus of variations, partial differential equations, or optimal control theory, which admit only L^p but not L^∞ apriori estimates. While Young measures [31] successfully capture oscillatory behavior (see e.g. [17], [23]) of sequences they completely miss concentrations. There are several tools how to deal with concentrations. They can be considered as generalization of Young measures, see for example Alibert's and Bouchitté's approach [1], DiPerna's and Majda's treatment of concentrations [7], or Fonseca's method described in [10]. An overview can be found in [25], [28]. Moreover, in many cases, we are interested in oscillation/concentration effects generated by sequences of gradients. A characterization of Young measures generated by gradients was completely given by Kinderlehrer and Pedregal [14], [16], cf. also [23], [24]. The first attempt to characterize both oscillations and concentrations in sequences

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of gradients is due to Fonseca, Müller, and Pedregal [11]. They dealt with a special situation of $\{gv(\nabla u_k)\}_{k\in\mathbb{N}}$ where v is positively p-homogeneous, $u_k \in W^{1,p}(\Omega;\mathbb{R}^m)$, and g continuous and vanishing on $\partial\Omega$. Later on, a characterization of oscillation/concentration effects in terms of DiPerna's and Majda's generalization of Young measures was given in [13] for arbitrary integrands.

The aim of our paper is to point out a few consequences of this characterization. This leads to a slight generalization of Kinderlehrer's and Pedregal's results on weak convergence of integrands [15]. They proved that if $0 \leqslant v \leqslant C(1+|\cdot|^p)$ is quasiconvex, $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, and $\int_\Omega v(\nabla u_k)\,\mathrm{d}x \to \int_\Omega v(\nabla u)\,\mathrm{d}x$ then possibly for a subsequence $v(\nabla u_k) \to v(\nabla u)$ weakly in $L^1(\Omega)$. Here we show that if we instead assume that $|v| \leqslant C(1+|\cdot|^p)$ and a condition on $\{u_k\}$ which is too involved to be stated here but which is fulfilled e.g. if $u_k = u$ on the boundary we get $v(\nabla u_k) \to v(\nabla u)$ weakly* in measures on $\bar{\Omega}$; cf. Theorem 2.3 and Corollary 2.4. We also give necessary and sufficient conditions under which a nonnegative sequence of $\{\det \nabla u_k\}$ converges weakly to $\{\det \nabla u\}$ in L^1 if $u_k \to u$ in $W^{1,n}(\Omega;\mathbb{R}^n)$ for a smooth bounded domain in $\mathbb{R}^n$. Proposition 2.7 generalizes some results by Müller [22] and Hogan et al. [12]. Finally, we show that while $u \mapsto \int_\Omega v(\nabla u)\,\mathrm{d}x$ does not have to be sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ if $|v| \leqslant C(1+|\cdot|^p)$ is quasiconvex, the weak lower semicontinuity can be recovered by removing an arbitrarily thin "boundary layer" of Ω; cf. Theorem 2.11. The main tool of our analysis is a recently proved characterization of generalized Young measures generated by gradients [13].

1.1. Basic notation

Let us start with a few definitions and with the explanation of our notation. Having a bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $C(\Omega)$ the space of continuous functions: $\Omega \to \mathbb{R}$. Then $C_0(\Omega)$ consists of functions from $C(\Omega)$ whose support is contained in Ω. In what follows "rca(S)" denotes the set of regular countably additive set functions on the Borel σ-algebra on a metrizable set S (cf. [8]), its subset, $\mathrm{rca}_1^+(S)$, denotes regular probability measures on a set S. We write "γ-almost all" or "γ-a.e." if we mean "up to a set with the γ-measure zero". If γ is the n-dimensional Lebesgue measure and $M \subset \mathbb{R}^n$ we omit writing γ in the notation. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \leqslant p < +\infty$ denotes the usual space of measurable mappings which are together with their first (distributional) derivatives integrable in the pth power. The support of a measure $\sigma \in \mathrm{rca}(\Omega)$ is a smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\sigma \in \mathrm{rca}(S)$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition, respectively. Finally, we denote by "w-lim" the weak limit.

If not said otherwise, we will suppose in the sequel that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary. Some generalizations to less regular domains are possible, however they seem to be technically much more involved.

1.2. Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. We say that a function $v\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex [21] if for any $s_0 \in \mathbb{R}^{m\times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega;\mathbb{R}^m)$

$$v(s_0)|\Omega| \leqslant \int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x.$$

If $v\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Qv\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ as

$$Qv = \sup\{h \leqslant v;\ h\colon \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex}\}$$

and if the set on the right-hand side is empty we put $Qv = -\infty$. If v is locally bounded and Borel measurable then for any $s_0 \in \mathbb{R}^{m\times n}$ (see [6])

$$Qv(s_0) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega v(s_0 + \nabla\varphi(x))\,\mathrm{d}x. \tag{1.1}$$

1.3. Young measures

For $p \geqslant 0$ we define the following subspace of the space $C(\mathbb{R}^{m\times n})$ of all continuous functions on $\mathbb{R}^{m\times n}$:

$$C_p(\mathbb{R}^{m\times n}) = \{v \in C(\mathbb{R}^{m\times n});\ v(s) = o(|s|^p) \text{ for } |s| \to \infty\}.$$

The Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x\colon \Omega \to \mathrm{rca}(\mathbb{R}^{m\times n})$ with values in probability measures; and the adjective "weakly* measurable" means that, for any $v \in C_0(\mathbb{R}^{m\times n})$, the mapping $\Omega \to \mathbb{R}\colon\ x \mapsto \langle \nu_x, v\rangle = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\mathrm{rca}(\mathbb{R}^{m\times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{m\times n})^*$, where $C_0(\mathbb{R}^{m\times n})$ stands for the space of all continuous functions $\mathbb{R}^{m\times n} \to \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$. It is known that $\mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega;\mathrm{rca}(\mathbb{R}^{m\times n})) \cong L^1(\Omega;C_0(\mathbb{R}^{m\times n}))^*$, where the subscript "w" indicates the property "weakly* measurable". A classical result [27], [30] is that, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^{m\times n})$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall\, v \in C_0(\mathbb{R}^{m\times n})\colon\ \lim_{k\to\infty} v\circ y_k = v_\nu \ \text{ weakly* in } L^\infty(\Omega), \tag{1.2}$$

where $[v \circ y_k](x) = v(y_k(x))$ and

$$v_\nu(x) = \int_{\mathbb{R}^{m\times n}} v(\lambda)\nu_x(\mathrm{d}\lambda). \tag{1.3}$$

Let us denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created in this way, i.e. by taking all bounded sequences in $L^\infty(\Omega;\mathbb{R}^{m\times n})$. Note that (1.2) actually holds for any $v\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ continuous.

A generalization of this result was formulated by Schonbek [26] (cf. also [2]): if $1 \leqslant p < +\infty$, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^{m\times n})$ such that

$$\forall\, v \in C_p(\mathbb{R}^{m\times n})\colon\ \lim_{k\to\infty} v\circ y_k = v_\nu \ \text{ weakly in } L^1(\Omega). \tag{1.4}$$

We say that $\{y_k\}$ generates ν if (1.4) holds.

Let us denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ the set of all Young measures which are created in this way, i.e. by taking all bounded sequences in $L^p(\Omega;\mathbb{R}^{m\times n})$.

1.4. DiPerna-Majda measures

Let us take a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring $\mathcal{R}$ of continuous bounded functions $\mathbb{R}^{m\times n} \to \mathbb{R}$. It is known [9, Sect. 3.12.21] that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}}\mathbb{R}^{m\times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m\times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$, into which $\mathbb{R}^{m\times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m\times n}$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$. Similarly, we will not distinguish between elements of $\mathcal{R}$ and their unique continuous extensions to $\beta_{\mathcal{R}}\mathbb{R}^{m\times n}$.

Let $\sigma \in \mathrm{rca}(\bar\Omega)$ be a positive Radon measure on a bounded domain $\Omega \subset \mathbb{R}^n$. A mapping $\hat\nu\colon x \mapsto \hat\nu_x$ belongs to the space $L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ if it is weakly* σ-measurable (i.e., for any $v_0 \in C_0(\mathbb{R}^{m\times n})$, the mapping $\bar\Omega \to \mathbb{R}\colon x \mapsto \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)$ is σ-measurable in the usual sense). If additionally $\hat\nu_x \in \mathrm{rca}_1^+(\beta_{\mathcal{R}}\mathbb{R}^{m\times n})$ for σ-a.a. $x \in \bar\Omega$ the collection $\{\hat\nu_x\}_{x\in\bar\Omega}$ is the so-called Young measure on $(\bar\Omega,\sigma)$ [31], see also [2], [25], [27], [29], [30].

DiPerna and Majda [7] showed that having a bounded sequence in $L^p(\Omega;\mathbb{R}^{m\times n})$ with $1 \leqslant p < +\infty$ and Ω an open domain in $\mathbb{R}^n$, there exists its subsequence (denoted by the same indices) a positive Radon measure $\sigma \in \mathrm{rca}(\bar\Omega)$ and a Young measure $\hat\nu\colon x \mapsto \hat\nu_x$ on $(\bar\Omega,\sigma)$ such that $(\sigma,\hat\nu)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}} \subset$

$L^p(\Omega;\mathbb{R}^{m\times n})$ in the sense that $\forall\, g\in C(\bar\Omega)\ \forall\, v_0\in\mathcal{R}$:

$$(1.5)\qquad \lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} g(x)v_0(s)\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x),$$

where

$$v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n}):=\{v_0(1+|\cdot|^p);\ v_0\in\mathcal{R}\}.$$

In particular, putting $v_0=1\in\mathcal{R}$ in (1.5) we can see that

$$(1.6)\qquad \lim_{k\to\infty}(1+|y_k|^p)=\sigma\quad\text{weakly}^*\text{ in }\operatorname{rca}(\bar\Omega).$$

If (1.5) holds, we say that $\{y_k\}_{k\in\mathbb{N}}$ generates $(\sigma,\hat\nu)$. Let us denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ the set of all pairs $(\sigma,\hat\nu)\in\operatorname{rca}(\bar\Omega)\times L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\operatorname{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ attainable by sequences from $L^p(\Omega;\mathbb{R}^{m\times n})$; note that, taking $v_0=1$ in (1.5), one can see that these sequences must inevitably be bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$. The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$, called DiPerna-Majda measures, for unconstrained sequences was done in [19, Theorem 2].

Let us recall that for any $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ there is precisely one $(\sigma^\circ,\hat\nu^\circ)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ such that

$$(1.7)\qquad \int_\Omega\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}} v_0(s)\hat\nu^\circ_x(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x)$$

for any $v_0\in C_0(\mathbb{R}^{m\times n})$ and any $g\in C(\bar\Omega)$ and $(\sigma^\circ,\hat\nu^\circ)$ is attainable by a sequence $\{y_k\}_{k\in\mathbb{N}}$ such that the set $\{|y_k|^p;\ k\in\mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; see [19], [25] for details. We call $(\sigma^\circ,\hat\nu^\circ)$ the nonconcentrating modification of $(\sigma,\hat\nu)$. We call $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ nonconcentrating if

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s)\sigma(\mathrm{d}x)=0.$$

There is a one-to-one correspondence between nonconcentrating DiPerna-Majda measures and Young measures; cf. [25].

We wish to emphasize the following fact: if $\{y_k\}\in L^p(\Omega;\mathbb{R}^{m\times n})$ generates $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ and σ is absolutely continuous with respect to the Lebesgue measure it generally *does not* mean that $\{|y_k|^p\}$ is weakly relatively compact in $L^1(\Omega)$. Simple examples can be found e.g. in [20], [25].

Having a sequence bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ generating a DiPerna-Majda measure $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ it also generates an L^p-Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$. It easily follows from [25, Theorem 3.2.13] that

$$(1.8)\qquad \nu_x(\mathrm{d}s)=d_{\sigma^\circ}(x)\frac{\hat\nu^\circ_x(\mathrm{d}s)}{1+|s|^p}\quad\text{for a.a. } x\in\Omega.$$

Note that (1.8) is well-defined as $\hat\nu_x^\circ$ is supported on $\mathbb{R}^{m\times n}$. As pointed out in [19, Remark 2] for almost all $x\in\Omega$

$$d_\sigma(x)=\Big(\int_{\mathbb{R}^{m\times n}}\frac{\hat\nu_x(\mathrm{d}s)}{1+|s|^p}\Big)^{-1}. \tag{1.9}$$

In fact, that (1.7) can even be improved to

$$\int_\Omega\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x)=\int_{\bar\Omega}\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x^\circ(\mathrm{d}s)g(x)\sigma^\circ(\mathrm{d}x) \tag{1.10}$$

for any $v_0\in\mathcal{R}$ and any $g\in C(\bar\Omega)$. The one-to-one correspondence between Young and DiPerna-Majda measures, in particular (see (1.8) and (1.10))

$$\int_{\mathbb{R}^{m\times n}}v(s)\nu_x(\mathrm{d}s)=d_\sigma(x)\int_{\mathbb{R}^{m\times n}}v_0(s)\hat\nu_x(\mathrm{d}s)$$

whenever $v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$, finally yields that $\forall\, g\in C(\bar\Omega)\ \forall\, v\in\Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$:

$$\begin{aligned}\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\,\mathrm{d}x=&\int_\Omega\int_{\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)d_\sigma(x)g(x)\,\mathrm{d}x\\&+\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}}\frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s)g(x)\sigma(\mathrm{d}x),\end{aligned} \tag{1.11}$$

where $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^{m\times n})$ and $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ are Young and DiPerna-Majda measures generated by $\{y_k\}_{k\in\mathbb{N}}$, respectively. We will denote the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ which are generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$ for some bounded $\{u_k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ by $\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$.

The following proposition from [19] explicitly characterizes elements of $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$.

Proposition 1.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open domain, $\mathcal{R}$ a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m\times n}$ and $(\sigma,\hat\nu)\in\mathrm{rca}(\bar\Omega)\times L^\infty_{\mathrm{w}}(\bar\Omega,\sigma;\mathrm{rca}(\beta_{\mathcal{R}}\mathbb{R}^{m\times n}))$ and $1\leqslant p<+\infty$. Then the following two statements are equivalent to each other:*

(i) *the pair $(\sigma,\hat\nu)$ is a DiPerna-Majda measure, i.e. $(\sigma,\hat\nu)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$,*

(ii) *the following properties are satisfied simultaneously:*

1. *σ is positive,*
2. *$\sigma_{\hat\nu}\in\mathrm{rca}(\bar\Omega)$ defined by $\sigma_{\hat\nu}(\mathrm{d}x)=(\int_{\mathbb{R}^{m\times n}}\hat\nu_x(\mathrm{d}s))\sigma(\mathrm{d}x)$ is absolutely continuous with respect to the Lebesgue measure ($d_{\sigma_{\hat\nu}}$ will denote its density),*

3. *for a.a.* $x \in \Omega$ *it holds*

$$\int_{\mathbb{R}^{m\times n}} \hat{\nu}_x(\mathrm{d}s) > 0, \quad d_{\sigma_{\hat{\nu}}}(x) = \left(\int_{\mathbb{R}^{m\times n}} \frac{\hat{\nu}_x(\mathrm{d}s)}{1+|s|^p}\right)^{-1} \int_{\mathbb{R}^{m\times n}} \hat{\nu}_x(\mathrm{d}s),$$

4. *for* σ*-a.a.* $x \in \bar{\Omega}$ *it holds*

$$\hat{\nu}_x \geqslant 0, \quad \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \hat{\nu}_x(\mathrm{d}s) = 1.$$

The following two theorems were proved in [13].

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with Lipschitz boundary,* $1 < p < +\infty$ *and* $(\sigma, \hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$. *Then then there is* $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ *and a bounded sequence* $\{u_k - u\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega; \mathbb{R}^m)$ *such that* $\{\nabla u_k\}_{k\in\mathbb{N}}$ *generates* $(\sigma, \hat{\nu})$ *if and only if the following three conditions hold:*

$$\text{for a.a. } x \in \Omega: \ \nabla u(x) = d_\sigma(x) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{s}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s), \tag{1.12}$$

for almost all $x \in \Omega$ *and for all* $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ *the following inequality is fulfilled*

$$Qv(\nabla u(x)) \leqslant d_\sigma(x) \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s), \tag{1.13}$$

and for σ*-almost all* $x \in \bar{\Omega}$ *and all* $v \in \Upsilon^p_{\mathcal{R}}(\mathbb{R}^{m\times n})$ *with* $Qv > -\infty$ *it holds that*

$$0 \leqslant \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s). \tag{1.14}$$

The next theorem addresses DiPerna-Majda measures generated by gradients of maps with possibly different traces.

Theorem 1.3. *Let* Ω *be an arbitrary bounded domain,* $1 < p < +\infty$ *and* $(\sigma, \hat{\nu}) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ *be generated by* $\{\nabla u_k\}_{k\in\mathbb{N}}$ *such that* $\mathrm{w}\text{-}\lim_{k\to\infty} u_k = u$ *in* $W^{1,p}(\Omega; \mathbb{R}^m)$. *Then the conditions* (1.12), (1.13) *hold, and* (1.14) *is satisfied for* σ*-a.a.* $x \in \Omega$.

Remark 1.4. (i) It can happen that under the assumptions of Theorem 1.3 the formula (1.14) does not hold on $\partial\Omega$. See an example in [3] showing the violation of weak sequential continuity of $W^{1,2}(\Omega; \mathbb{R}^2) \to L^1(\Omega)\colon\ u \mapsto \det \nabla u$ if $\Omega = (-1,1)^2$.

(ii) On the other hand, having $1 < p < +\infty$ we can ask what condition besides quasiconvexity must $v \in C(\mathbb{R}^{2\times 2})$, $|v| \leqslant C(1+|\cdot|^p)$ satisfy so that $W^{1,p}(\Omega;\mathbb{R}^2) \to \mathbb{R}$: $u \mapsto I(u) := \int_\Omega v(\nabla u(x))\,\mathrm{d}x$ is sequentially weakly lower semicontinuous. Suppose that $v \in C(\mathbb{R}^{2\times 2})$ is positively p-homogeneous, i.e. $v(\lambda s) = \lambda^p v(s)$ for all $s \in \mathbb{R}^{2\times 2}$, $\lambda \geqslant 0$, and take arbitrary $u \in W_0^{1,p}((-1,1)^2;\mathbb{R}^2)$ and extend it by zero to $\mathbb{R}^2$. Define further for all $k \in \mathbb{N}$ $u_k(x) = k^{2/p-1}u(kx)$. Then $\nabla u_k(x) = k^{2/p}\nabla u(kx)$. Clearly, $u_k \to 0$ weakly in $W^{1,p}((-1,1)^2;\mathbb{R}^2)$. Take $\Omega := (-1,0)^2$. A simple calculation shows that if I is weakly lower semicontinuous then for all $u \in W_0^{1,p}((-1,1)^2;\mathbb{R}^2)$ it holds

$$\int_{(-1,0)^2} v(\nabla u(x))\,\mathrm{d}x \geqslant 0. \tag{1.15}$$

2. Convergence of integrands

We start with a remark on (1.14).

Remark 2.1. Condition (1.14) for $x \in \partial\Omega$ is satisfied if $\{u_k - u\} \subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ but the same trace for all terms in the sequence is far from being necessary for (1.14) to hold on $\partial\Omega$. As Ω is supposed to be Lipschitz $\{u_k\}$ can be extended to $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset W^{1,p}(B;\mathbb{R}^m)$ for some ball $\mathbb{R}^n \supset B \supset \Omega$. Moreover, $\{\tilde{u}_k\}$ is uniformly bounded in $W^{1,p}(B;\mathbb{R}^m)$. Then (1.14) holds if $\{|\nabla\tilde{u}_k|^p\}$ is weakly relatively compact in $L^1(B\setminus\Omega)$. If $\{\nabla u_k\}$ satisfies this condition for some ball $B \subset \mathbb{R}^n$ we say that it has a p-equiintegrable extension.

We put

$$V_p = \{v \in C(\mathbb{R}^{m\times n});\ \exists\, C > 0\colon\ |v| \leqslant C(1+|\cdot|^p),\ Qv > -\infty\}. \tag{2.1}$$

As shown in [18] if $v \in V_p$ then $Qv \in V_p$ as well.

Theorem 2.2. *Let $v \in V_p$ and $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$. Suppose that there is $g_0 \in C(\bar{\Omega})$, $g_0(x) = 0$ if $x \in \partial\Omega$ and $g_0(x) > 0$ if $x \in \Omega$ such that for $k \to \infty$*

$$\int_\Omega g_0(x)v(\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega g_0(x)Qv(\nabla u(x))\,\mathrm{d}x. \tag{2.2}$$

Then there is a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ (not relabeled) such that for $k \to \infty$

$$v(\nabla u_k) \to Qv(\nabla u) \quad \text{weakly}^* \text{ in } \mathrm{rca}(\Omega), \tag{2.3}$$

i.e., we can replace g_0 in (2.2) by all $g \in C_0(\Omega)$.

P r o o f. We take a separable subring $\mathcal{R}$ such that $v/(1+|\cdot|^p) \in \mathcal{R}$. As noted in [13], this is always possible. Taking a subsequence of $\{\nabla u_k\}$ generating $(\sigma, \hat{\nu}) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^{m\times n})$ and exploiting Theorem 1.3 we have by (2.2) and (1.11) that for g_0

$$\begin{aligned}\int_\Omega g_0(x) Qv(\nabla u(x))\,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega g_0(x) v(\nabla u_k(x))\,\mathrm{d}x \\ &= \int_\Omega \int_{\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) d_\sigma(x) g_0(x)\,\mathrm{d}x \\ &\quad + \int_\Omega \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) g_0(x)\sigma(\mathrm{d}x) \\ &\geqslant \int_\Omega Qv(\nabla u(x)) g_0(x)\,\mathrm{d}x \\ &\quad + \int_\Omega \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) g_0(x)\sigma(\mathrm{d}x) \\ &\geqslant \int_\Omega g_0(x) Qv(\nabla u(x))\,\mathrm{d}x.\end{aligned}$$

Hence,

$$\int_\Omega \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) g_0(x)\sigma(\mathrm{d}x) = 0$$

and because $g_0 > 0$ in Ω and for σ-a.a. $x \in \Omega$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) \geqslant 0$$

by (1.14) we get that for σ-a.a. $x \in \Omega$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) = 0. \tag{2.4}$$

Moreover, as $g_0 > 0$ in Ω we get that for a.a. $x \in \Omega$

$$d_\sigma(x) \int_{\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p} \hat{\nu}_x(\mathrm{d}s) = Qv(\nabla u(x)). \tag{2.5}$$

The assertion (2.3) follows by (2.4), (2.5), and (1.11). $\square$

Theorem 2.3. *Let $v \in V_p$ and $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$. Suppose that there is $g_0 \in C(\bar\Omega)$, $g_0 > 0$ such that for $k \to \infty$*

$$\int_\Omega g_0(x)v(\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega g_0(x)Qv(\nabla u(x))\,\mathrm{d}x. \tag{2.6}$$

Suppose further that $\{\nabla u_k\}$ generates a DiPerna-Majda measure $(\sigma,\hat\nu) \in \mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$ such that for σ-a.a. $x \in \partial\Omega$

$$\int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{v(s)}{1+|s|^p}\hat\nu_x(\mathrm{d}s) \geqslant 0. \tag{2.7}$$

Then

$$v(\nabla u_k) \to Qv(\nabla u) \tag{2.8}$$

weakly in $\mathrm{rca}(\bar\Omega)$ if $k \to \infty$, i.e., (2.6) holds for all $g \in C(\bar\Omega)$ in place of g_0.*

P r o o f. The proof is almost the same as the proof of Theorem 2.2. Notice that due to (2.7) and Theorem 1.3 formula (1.4) holds for σ-a.a. $x \in \bar\Omega$. $\square$

In view of Remark 2.1 and Theorem 1.2 we have the following consequence of Theorem 2.3.

Corollary 2.4. *Let $v \in V_p$ and $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$, $\{u_k - u\}_{k\in\mathbb{N}} \subset W^{1,p}_0(\Omega;\mathbb{R}^m)$. Suppose that*

$$\int_\Omega v(\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega Qv(\nabla u(x))\,\mathrm{d}x. \tag{2.9}$$

Then there is a subsequence of $\{u_k\}$ (not relabeled) such that

$$v(\nabla u_k) \to Qv(\nabla u) \quad \text{weakly* in } \mathrm{rca}(\bar\Omega). \tag{2.10}$$

The convergence in $\mathrm{rca}(\bar\Omega)$ can be strengthened if $v(\nabla u_k) \geqslant 0$ for all $k \in \mathbb{N}$. The following theorem was proved for $v \geqslant 0$ quasiconvex and $g_0 = 1$ in [15].

Theorem 2.5. *Let $v \in V_p$, $u_k \to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$, and for all $k \in \mathbb{N}$ let $v(\nabla u_k) \geqslant 0$ almost everywhere in Ω. Suppose that there is $g_0 \in C(\bar{\Omega})$, $g_0 > 0$ such that for $k \to \infty$*

$$\int_\Omega g_0(x) v(\nabla u_k(x))\,\mathrm{d}x \to \int_\Omega g_0(x) Qv(\nabla u(x))\,\mathrm{d}x. \tag{2.11}$$

Then there is a subsequence of $\{\nabla u_k\}$ (not relabeled) such that

$$v(\nabla u_k) \to Qv(\nabla u) \tag{2.12}$$

weakly in $L^1(\Omega)$ if $k \to \infty$, i.e., (2.11) holds for all $g \in L^\infty(\Omega)$ in place of g_0.

We will need the following lemma.

Lemma 2.6. *Let $f \in V_p$, $f \geqslant 0$, and $f/(1+|\cdot|^p) \in \mathcal{R}$. Let further $\{y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^{m\times n})$ generate $(\sigma,\hat{\nu}) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Then $\{f(y_k)\}_{k\in\mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$ if and only if*

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{f(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) = 0. \tag{2.13}$$

Proof. This lemma was proved in [25, Lemma 3.2.14 (i)] for $f = 1+|\cdot|^p$. The proof for general f is analogous. $\square$

Proof of Theorem 2.5. As $v(\nabla u_k) \geqslant 0$ we can replace v by $|v|$ and take $\mathcal{R}$ such that $|v|/(1+|\cdot|^p) \in \mathcal{R}$. Moreover, $Q|v| \geqslant Qv$ because $|v| \geqslant v$.

We have for a subsequence of $\{\nabla u_k\}$ (not relabeled) generating $(\sigma,\hat{\nu})$:

$$\begin{aligned}
\int_\Omega g_0(x) Qv(\nabla u(x))\,\mathrm{d}x &= \lim_{k\to\infty} \int_\Omega g_0(x)|v|(\nabla u_k(x))\,\mathrm{d}x \\
&= \int_\Omega \int_{\mathbb{R}^{m\times n}} \frac{|v|(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) d_\sigma(x) g_0(x)\,\mathrm{d}x \\
&\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{|v|(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) g_0(x)\sigma(\mathrm{d}x) \\
&\geqslant \int_\Omega Qv(\nabla u(x)) g_0(x)\,\mathrm{d}x \\
&\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^{m\times n}\setminus\mathbb{R}^{m\times n}} \frac{|v|(s)}{1+|s|^p}\hat{\nu}_x(\mathrm{d}s) g_0(x)\sigma(\mathrm{d}x) \\
&\geqslant \int_\Omega g_0(x) Qv(\nabla u(x))\,\mathrm{d}x.
\end{aligned}$$

Taking into account that $g_0 > 0$ and Lemma 2.6 we get that $\{|v|(\nabla u_k)\} = \{v(\nabla u_k)\}$ is weakly relatively compact in $L^1(\Omega)$. Applying [24, Theorem 6.2] we conclude that a subsequence has the weak limit $Qv(\nabla u)$. $\square$

2.1. Applications to determinants

Suppose now that $p = n > 1$ and consider $v = \det$. Clearly, $v \in V_n$ and because the determinant is quasiaffine (i.e. det as well as $-\det$ are both quasiconvex) (1.13) as well as (1.14) hold as equalities.

Combining (1.14) with Lemma 2.6 and Theorem 2.5 we have the following proposition.

Proposition 2.7. *Let $u_k \to u$ weakly in $W^{1,n}(\Omega;\mathbb{R}^n)$, let $\{\nabla u_k\}_{k\in\mathbb{N}}$ generate $(\sigma,\hat{\nu}) \in \mathcal{GDM}^n(\Omega;\mathbb{R}^{n\times n})$, and let $\det/(1+|\cdot|^n) \in \mathcal{R}$. Let for all $k \in \mathbb{N}$ $\det\nabla u_k \geqslant 0$ almost everywhere in Ω. Then $\{\det\nabla u_k\}_{k\in\mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$ if and only if*

$$\int_{\partial\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}} \frac{\det s}{1+|s|^n}\hat{\nu}_x(\mathrm{d}s)\sigma(\mathrm{d}x) = 0. \tag{2.14}$$

If (2.14) holds then

$$\det\nabla u_k \to \det\nabla u \quad \textit{weakly in } L^1(\Omega).$$

Notice that (2.14) is satisfied if $\sigma(\partial\Omega) = 0$, if $u_k = u$ on $\partial\Omega$, or if $\{\nabla u_k\}$ has a p-equiintegrable extension; cf. Theorem 1.2. Hence, Proposition 2.7 generalizes [15, Theorem 4.1] and [22, Corollary 1.2].

Dropping the requirement $\det\nabla u_k \geqslant 0$ we have the following consequence of Theorem 2.3.

Proposition 2.8. *Let $u_k \to u$ weakly in $W^{1,n}(\Omega;\mathbb{R}^n)$, let $\{\nabla u_k\}_{k\in\mathbb{N}}$ generate $(\sigma,\hat{\nu}) \in \mathcal{GDM}^n(\Omega;\mathbb{R}^{n\times n})$, and let $\det/(1+|\cdot|^n) \in \mathcal{R}$. If (2.14) holds then*

$$\det\nabla u_k \to \det\nabla u \quad \textit{weakly}^* \textit{ in } \operatorname{rca}(\bar{\Omega}). \tag{2.15}$$

In particular, (2.15) is satisfied if $u_k = u$ on $\partial\Omega$, if $\sigma(\partial\Omega) = 0$, or if $\{\nabla u_k\}$ has a p-equiintegrable extension.

Finally, we may even give up (2.14) to hold and by Theorem 1.6 with $g_0 = \operatorname{dist}(\cdot,\partial\Omega)$ we obtain the following fact mentioned already in [3].

Proposition 2.9. *Let $u_k \to u$ weakly in $W^{1,n}(\Omega;\mathbb{R}^n)$. Then possibly for a subsequence it holds*

$$\det\nabla u_k \to \det\nabla u \quad \textit{weakly}^* \textit{ in } \operatorname{rca}(\Omega). \tag{2.16}$$

2.2. Biting lemma for quasiconvex integrands

It is well known that boundedness of a sequence in $L^1(\Omega;\mathbb{R}^m)$ is not sufficient for the existence of a weakly converging subsequence. Nevertheless, Brooks and Chacon showed that removing nested sets of vanishing Lebesgue measure we can recover weak L^1 convergence.

Lemma 2.10 ([5]). *Let $\{y_k\}\subset L^1(\Omega;\mathbb{R}^{m\times n})$ be bounded. Then there is $y\in L^1(\Omega;\mathbb{R}^{m\times n})$ and measurable sets $\{\Omega_j\}_{j\in\mathbb{N}}$, $\Omega_{j+1}\subset\Omega_j\subset\Omega$, $j\in\mathbb{N}$, $|\Omega_j|\to 0$ such that for $k\to\infty$*

$$y_k\to y \quad \text{weakly in } L^1(\Omega\setminus\Omega_j) \text{ for every fixed } j\in\mathbb{N}.$$

Ball and Zhang [4] showed that if $v\in C(\mathbb{R}^{m\times n})$, $|v|\leqslant C(1+|\cdot|^p)$ is quasiconvex and $u_k\to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ then there are measurable sets $\{\Omega_j\}_{j\in\mathbb{N}}$, $\Omega_{j+1}\subset\Omega_j\subset\Omega$, $j\in\mathbb{N}$, $|\Omega_j|\to 0$ such that for any $j\in\mathbb{N}$

$$\liminf_{k\to\infty}\int_{\Omega_j} v(\nabla u_k(x))\,\mathrm{d}x\geqslant\int_{\Omega_j} v(\nabla u(x))\,\mathrm{d}x.$$

The next theorem shows that Ω_j can be chosen as arbitrarily thin "boundary layers" of Ω.

Theorem 2.11. *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a Lipschitz boundary such that $0\in\Omega$. Let $u_k\to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ and let $v\in V_p$. Then for every $0<\varepsilon<1$ there is a subsequence of $\{u_k\}$ which we do not relabel and $\varepsilon<\delta<1$ such that*

$$\lim_{k\to\infty}\int_{\delta\Omega} v(\nabla u_k(x))\,\mathrm{d}x\geqslant\int_{\delta\Omega} Qv(\nabla u(x))\,\mathrm{d}x, \tag{2.17}$$

where $\delta\Omega=\{y\in\mathbb{R}^n;\ y/\delta\in\Omega\}$.

Proof. Take $\varepsilon\in\,]0,1[$. We will assume that $\{\nabla u_k\}$ generates $(\sigma,\hat\nu)\in\mathcal{GDM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^{m\times n})$. Notice that $\delta\Omega\subset\Omega$. Using [13, Lemma 3.6] we get that $\sigma(\partial\delta\Omega)>0$ only for at most countably many values of δ. Thus we take $\delta>\varepsilon$ such that $\sigma(\partial\delta\Omega)=0$. Then using [13, Lemma 3.2] we have that the restriction of $\{u_k\}$ on $\delta\Omega$ has the property that $\{\nabla u_k|_{\delta\Omega}\}$ generates $(\sigma,\hat\nu)|_{\delta\Omega}$. Then (2.17) follows by (1.11) in view of Theorem 1.2. $\square$

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