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THE NEUMANN PROBLEM FOR SOME DEGENERATE ELLIPTIC EQUATIONS

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Abstract. In the paper we study the equation $Lu = f$, where L is a degenerate elliptic operator, with Neumann boundary condition in a bounded open set Ω. We prove existence and uniqueness of solutions in the space $H(\Omega)$ for the Neumann problem.

Keywords: Neumann problem, degenerate elliptic equations

MSC 2000: 35J70, 35J25

1. Introduction

In this paper we prove existence and uniqueness of solutions in the space $H(\Omega)$ (see Definition 2.2) for the Neumann problem

$$\text{(P)}\qquad \begin{cases} Lu(x) = f(x) & \text{on } \Omega, \\ \langle \mathcal{A}(x)\nabla u(x), \vec{\eta}(x)\rangle = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is a degenerate elliptic operator

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u(x)) + \sum_{i=1}^{n} b_i(x)D_i u(x) + g(x)u(x) + \theta u(x)v(x) \tag{1.1}$$

with $D_j = \partial/\partial x_j$ $(j = 1, \ldots, n)$, θ is a constant, the coefficients a_{ij}, b_i and g are measurable, real-valued functions, the coefficient matrix $\mathcal{A}(x) = (a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$|\xi|^2\omega(x) \leqslant \langle \mathcal{A}(x)\xi, \xi\rangle \leqslant |\xi|^2 v(x) \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$ where Ω is a bounded open set with piecewise smooth boundary (i.e., $\partial\Omega \in C^{0,1}$), ω and v are weight functions (that is, locally integrable and nonnegative functions on $\mathbb{R}^n$), $\vec{\eta}(x) = (\eta_1(x), \ldots, \eta_n(x))$ is the unit outward normal to $\partial\Omega$ at x, $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\mathbb{R}^n$ and the symbol ∇ indicates the gradient.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [2] and [3]).

A class of weights which is particularly well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [4]). These weights have found many useful applications in harmonic analysis (see [5]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [6] or [7]). There are, in fact, many interesting examples of weights (see [8] for p-admissible weights). In this paper we will consider only A_p-weights.

The following theorem will be proved in Section 3.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega \in C^{0,1}$. Suppose that*

(H1) $\omega \in A_2$, $v \in A_2$ *and* (1.2) *holds;*

(H2) $f/v \in L^2(\Omega, v)$;

(H3) $b_i/\omega \in L^\infty(\Omega)$ $(i = 1, \ldots, n)$ *and* $g/v \in L^\infty(\Omega)$.

Then there exists a constant $\mathbf{C} > 0$ such that for all $\theta \geqslant \mathbf{C}$ the Neumann problem (P) *has a unique solution $u \in H(\Omega)$. Moreover, we have*

$$\|u\|_{H(\Omega)} \leqslant 2\left\|\frac{f}{v}\right\|_{L^2(\Omega,v)}.$$

Example 1.2. Consider the domain $\Omega = \{(x,y) \in \mathbb{R}^2 \colon\ x^2 + y^2 < 1\}$, the weight functions

$$\omega(x,y) = (x^2+y^2)^{-1/3} \quad \text{and} \quad v(x,y) = (x^2+y^2)^{-1/2}$$

and the coefficient matrix

$$\mathcal{A}(x,y) = \begin{pmatrix} (x^2+y^2)^{-1/3} & 0 \\ 0 & (x^2+y^2)^{-1/2} \end{pmatrix}.$$

For all $\xi \in \mathbb{R}^2$ and almost every $(x,y) \in \Omega$ we have

$$\frac{1}{(x^2+y^2)^{1/3}}|\xi|^2 \leqslant \langle \mathcal{A}(x,y)\xi, \xi\rangle \leqslant \frac{1}{(x^2+y^2)^{1/2}}|\xi|^2.$$

If $(x,y)\in\partial\Omega=\{(x,y)\in\mathbb{R}^2\colon\ x^2+y^2=1\}$, then $\vec{\eta}(x,y)=(x,y)$ is the unit outward normal to $\partial\Omega$. By Theorem 1.1 the Neumann problem

$$\begin{cases} Lu(x,y)=(x^2+y^2)^{-3/8}\cos(xy) & \text{on } \Omega,\\ \langle\mathcal{A}(x,y)\cdot\nabla u,\vec{\eta}\rangle=0 & \text{on } \partial\Omega\end{cases}$$

where

$$\begin{aligned} Lu(x,y)= &-\Big[\frac{\partial}{\partial x}\Big(\frac{1}{(x^2+y^2)^{1/3}}\frac{\partial u}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(\frac{1}{(x^2+y^2)^{1/2}}\frac{\partial u}{\partial y}\Big)\Big]\\ &+\frac{\sin(xy)}{(x^2+y^2)^{1/3}}\frac{\partial u}{\partial x}+\frac{\cos(xy)}{(x^2+y^2)^{1/4}}\frac{\partial u}{\partial y}\\ &+\frac{u(x,y)\sin(xy)}{(x^2+y^2)^{1/3}}+\theta\frac{u(x,y)}{(x^2+y^2)^{1/2}}\end{aligned}$$

has a unique solution $u\in H(\Omega)$ (if $\theta\geqslant 2$).

2. Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0<\omega<\infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1<p<\infty$, or that ω is an A_p-weight, if there is a constant $C=C_{p,\omega}$ such that

$$\Big(\frac{1}{|B|}\int_B\omega(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B\omega^{1/(1-p)}(x)\,\mathrm{d}x\Big)^{p-1}\leqslant C$$

for all balls $B\subset\mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1<q\leqslant p$, then $A_q\subset A_p$ (see [5], [8] or [9] for more information about A_p-weights). The weight ω satisfies the doubling condition if $\omega(2B)\leqslant C\omega(B)$ for all balls $B\subset\mathbb{R}^n$, where $\omega(B)=\int_B\omega(x)\,\mathrm{d}x$ and $2B$ denotes the ball with the same center as B which is twice as large. If $\omega\in A_p$, then ω is doubling (see Corollary 15.7 in [8, p. 299]).

As an example of an A_p-weight, the function $\omega(x)=|x|^\alpha$, $x\in\mathbb{R}^n$, is in A_p if and only if $-n<\alpha<n(p-1)$ (see Corollary 4.4, Chapter IX in [10, p. 236]).

Given an open subset Ω of $\mathbb{R}^n$, we will denote by $L^p(\Omega,\omega)$ $(1\leqslant p<\infty)$ the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)}=\Big(\int_\Omega|f(x)|^p\omega(x)\,\mathrm{d}x\Big)^{1/p}<\infty.$$

If $\omega\in A_p$, $1<p<\infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega)\subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [10, p. 4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega, \omega)$ as the set of functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leqslant |\alpha| \leqslant k$. The norm of u in $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x) \, \mathrm{d}x + \sum_{1 \leqslant |\alpha| \leqslant k} \int_\Omega |D^\alpha u(x)|^p \omega(x) \, \mathrm{d}x \right)^{1/p}. \tag{2.1}$$

If $\omega \in A_p$, then $W^{k,p}(\Omega, \omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Proposition 3.5 in [11, p. 416] or Corollary 2.1.6 in [10, p. 18]). The space $W_0^{k,p}(\Omega, \omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega,\omega)} = \left(\sum_{1 \leqslant |\alpha| \leqslant k} \int_\Omega |D^\alpha u(x)|^p \omega(x) \, \mathrm{d}x \right)^{1/p}.$$

The spaces $W^{k,p}(\Omega, \omega)$ and $W_0^{k,p}(\Omega, \omega)$ are Banach spaces and for $k = 1$ and $p = 2$ the spaces $W^{1,2}(\Omega, \omega)$ and $W_0^{1,2}(\Omega, \omega)$ are Hilbert spaces.

It is evident that the weight functions ω which satisfy $0 < c_1 \leqslant \omega(x) \leqslant c_2$ for $x \in \Omega$ give nothing new (the space $\mathrm{W}^{\mathrm{k,p}}(\Omega, \omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we shall be interested above all in such weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both).

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set. We define the space $H(\Omega)$ as the closure of $C^\infty(\bar{\Omega})$ with respect to the norm

$$\|u\|_{H(\Omega)} = \left(\int_\Omega |u|^2 v \, \mathrm{d}x + \int_\Omega \langle \mathcal{A} \nabla u, \nabla u \rangle \, \mathrm{d}x \right)^{1/2},$$

where $\mathcal{A} = (a_{ij})$ is the coefficient matrix of the operator L defined in (1.1), $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$ and the symbol ∇ indicates the gradient.

The space $H(\Omega)$ is a Hilbert space with the inner product

$$a(u, \varphi) = \left(\int_\Omega u \varphi v \, \mathrm{d}x + \int_\Omega \langle \mathcal{A} \nabla u, \nabla \varphi \rangle \, \mathrm{d}x \right)^{1/2}.$$

Facts about $H(\Omega)$ are given in [1, p. 1115].

R e m a r k 2.3. By the degeneracy condition (1.2) we have

$$\int_\Omega |\nabla u|^2 \omega \, \mathrm{d}x \leqslant \int_\Omega \langle \mathcal{A} \nabla u, \nabla u \rangle \, \mathrm{d}x \leqslant \int_\Omega |\nabla u|^2 v \, \mathrm{d}x.$$

Therefore, $W^{1,2}(\Omega, v) \subset H(\Omega) \subset W^{1,2}(\Omega, \omega)$.

Note also that since $\mathcal{A}$ is symmetric, $|\langle \mathcal{A}x, y \rangle| \leqslant \langle \mathcal{A}x, x \rangle^{1/2} \langle \mathcal{A}y, y \rangle^{1/2}$.

Remark 2.4. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary $\partial\Omega \in C^{0,1}$. Using integration by parts with $u, \varphi \in H(\Omega)$, if u satisfies the boundary condition in problem (P), we have

$$\begin{aligned}\int_\Omega \varphi L u \,\mathrm{d}x = &\sum_{i,j=1}^n \int_\Omega a_{ij} D_i u D_j \varphi \,\mathrm{d}x + \sum_{i=1}^n \int_\Omega b_i \varphi D_i u \,\mathrm{d}x + \int_\Omega g u \varphi \,\mathrm{d}x \\ &+ \theta \int_\Omega u \varphi v \,\mathrm{d}x + \underbrace{\sum_{i,j=1}^n \int_{\partial\Omega} a_{ij} \frac{\partial u}{\partial x_j} \eta_i \varphi \,\mathrm{d}x}_{=0} \\ = {}& B(u, \varphi) + \theta \int_\Omega u \varphi v \,\mathrm{d}x,\end{aligned}$$

where

$$B(u, \varphi) = \sum_{i,j=1}^n \int_\Omega a_{ij} D_i u D_j \varphi \,\mathrm{d}x + \sum_{i=1}^n \int_\Omega b_i \varphi D_i u \,\mathrm{d}x + \int_\Omega g u \varphi \,\mathrm{d}x$$

is a bilinear form.

We introduce the following definition of solutions for the Neumann problem (P).

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\partial\Omega \in C^{0,1}$ and suppose that $f/v \in L^2(\Omega, v)$. A function $u \in H(\Omega)$ is a solution of the Neumann problem (P) if

$$\int_\Omega \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi \,\mathrm{d}x + \int_\Omega \left[\sum_{i=1}^n b_i D_i u + g u\right] \varphi \,\mathrm{d}x + \theta \int_\Omega u \varphi v \,\mathrm{d}x = \int_\Omega f \varphi \,\mathrm{d}x$$

for all $\varphi \in H(\Omega)$.

Lemma 2.6. *Suppose that $\omega \in A_2$, $v \in A_2$, $b_i/\omega \in L^\infty(\Omega)$ $(i = 1, \ldots, n)$ and $g/v \in L^\infty(\Omega)$. Then there exists a constant $\mathbf{C} > 0$ such that*

$$B(u, u) + \mathbf{C} \|u\|^2_{L^2(\Omega,\omega)} \geqslant \frac{1}{2} \|u\|^2_{H(\Omega)}$$

for all $u \in H(\Omega)$.

P r o o f. For all $u \in H(\Omega)$ we have

$$
\begin{aligned}
(2.2)\quad B(u,u) &= \sum_{i,j=1}^{n} \int_\Omega a_{ij} D_i u D_j u \,\mathrm{d}x + \sum_{i=1}^{n} \int_\Omega b_i u D_i u \,\mathrm{d}x + \int_\Omega g u^2 \,\mathrm{d}x \\
&= \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x + \sum_{i=1}^{n} \int_\Omega \frac{b_i}{\omega} \omega u D_i u \,\mathrm{d}x + \int_\Omega \frac{g}{v} u^2 v \,\mathrm{d}x \\
&\geqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - \Big(\max_{1\leqslant i\leqslant n} \Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)}\Big) \sum_{i=1}^{n} \int_\Omega |u|\,|D_i u| \omega \,\mathrm{d}x \\
&\quad - \Big\|\frac{g}{v}\Big\|_{L^\infty(\Omega)} \int_\Omega u^2 v \,\mathrm{d}x \\
&\geqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - C_1 \sum_{i=1}^{n} \Big(\int_\Omega u^2 \omega \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Omega |D_i u|^2 \omega \,\mathrm{d}x\Big)^{1/2} \\
&\quad - C_2 \int_\Omega u^2 v \,\mathrm{d}x \\
&\geqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - C_1 \Big(\int_\Omega u^2 v \,\mathrm{d}x\Big)^{1/2} \Big(\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x\Big)^{1/2} \\
&\quad - C_2 \|u\|^2_{L^2(\Omega,v)}
\end{aligned}
$$

where

$$
C_1 = \max_{1\leqslant i\leqslant n} \Big\|\frac{b_i}{\omega}\Big\|_{L^\infty(\Omega)} \quad \text{and} \quad C_2 = \Big\|\frac{g}{v}\Big\|_{L^\infty(\Omega)}.
$$

Using the elementary inequality

$$
ab \leqslant \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \text{for all } \varepsilon > 0,
$$

we obtain from (2.2)

$$
\begin{aligned}
(2.3)\quad B(u,u) &\geqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - C_1\Big(\varepsilon \|u\|^2_{L^2(\Omega,v)} + \frac{1}{4\varepsilon}\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x\Big) \\
&\quad - C_2 \|u\|_{L^2(\Omega,v)} \\
&= \Big(1 - \frac{C_1}{4\varepsilon}\Big) \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - (C_1\varepsilon + C_2)\|u\|^2_{L^2(\Omega,v)}.
\end{aligned}
$$

If $C_1 > 0$, we can choose $\varepsilon > 0$ such that

$$
1 - \frac{C_1}{4\varepsilon} = \frac{1}{2}, \quad \text{that is,} \quad \varepsilon = \frac{C_1}{2}.
$$

Thus, (2.3) transforms to

$$\begin{aligned}
B(u,u) &\geqslant \frac{1}{2}\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - \Big(\frac{C_1^2}{2} + C_2\Big)\|u\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\bigg(\int_\Omega u^2 v \,\mathrm{d}x + \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x\bigg) - \Big(\frac{C_1^2}{2} + C_2 + \frac{1}{2}\Big)\|u\|^2_{L^2(\Omega,v)} \\
&= \frac{1}{2}\|u\|^2_{H(\Omega)} - \mathbf{C}\|u\|^2_{L^2(\Omega,v)},
\end{aligned}$$

where $\mathbf{C} = \frac{1}{2}C_1^2 + C_2 + \frac{1}{2} > 0$. Therefore,

$$B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,v)} \geqslant \frac{1}{2}\|u\|^2_{H(\Omega)}.$$

If $C_1 = 0$ (that is, $b_i(x) \equiv 0$, $i = 1,\dots,n$) then (2.2) reduces to

$$\begin{aligned}
B(u,u) &\geqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x - C_2\|u\|^2_{L^2(\Omega,v)} \\
&\geqslant \frac{1}{2}\bigg(\int_\Omega |u|^2 v \,\mathrm{d}x + \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \,\mathrm{d}x\bigg) - \Big(C_2 + \frac{1}{2}\Big)\|u\|^2_{L^2(\Omega,v)} \\
&= \frac{1}{2}\|u\|^2_{H(\Omega)} - \mathbf{C}\|u\|^2_{L^2(\Omega,v)}.
\end{aligned}$$

Therefore, we also have

$$B(u,u) + \mathbf{C}\|u\|^2_{L^2(\Omega,v)} \geqslant \frac{1}{2}\|u\|^2_{H(\Omega)}$$

for all $u \in H(\Omega)$, where $\mathbf{C} = \frac{1}{2}C_1^2 + C_2 + \frac{1}{2}$. □

3. Proof of Theorem 1.1

We define a bilinear form

$$\tilde{B}\colon\ H(\Omega)\times H(\Omega) \longrightarrow \mathbb{R}, \quad \tilde{B}(u,\varphi) = B(u,\varphi) + \theta\int_\Omega u\varphi v \,\mathrm{d}x$$

and a linear mapping

$$T\colon\ H(\Omega) \longrightarrow \mathbb{R}, \quad T(\varphi) = \int_\Omega f\varphi \,\mathrm{d}x.$$

Then $u \in H(\Omega)$ is a solution of the Neumann problem (P) if

$$\tilde{B}(u,\varphi) = T(\varphi)$$

for all $\varphi \in H(\Omega)$.

Step 1. If $\theta \geqslant \mathbf{C}$ then $\tilde{B}$ is coercive, that is, there exists a constant $c > 0$ such that $\tilde{B}(u, u) \geqslant c\|u\|^2_{H(\Omega)}$ for all $u \in H(\Omega)$. In fact, by Lemma 2.6 there exists a constant $\mathbf{C} > 0$ such that

$$B(u, u) + \mathbf{C}\|u\|^2_{L^2(\Omega, v)} \geqslant \frac{1}{2}\|u\|_{H(\Omega)}.$$

Hence, if $\theta \geqslant \mathbf{C}$, we have

$$\begin{aligned}\tilde{B}(u, u) &= B(u, u) + \theta \int_\Omega u^2 v \, \mathrm{d}x = B(u, u) + \theta\|u\|^2_{L^2(\Omega, v)} \\ &\geqslant B(u, u) + \mathbf{C}\|u\|^2_{L^2(\Omega, v)} \geqslant \frac{1}{2}\|u\|^2_{H(\Omega)}.\end{aligned}$$

Therefore, for $\theta \geqslant \mathbf{C}$ we have that

$$\tilde{B}(u, u) \geqslant \frac{1}{2}\|u\|^2_{H(\Omega)} \tag{3.1}$$

for all $u \in H(\Omega)$.

Step 2. $\tilde{B}$ is bounded. In fact, using the fact that the coefficient matrix $\mathcal{A} = (a_{ij})$ is symmetric, (H2) and (H3), we obtain

$$\begin{aligned}
|\tilde{B}(u, \varphi)| &\leqslant |B(u, \varphi)| + \theta\left|\int_\Omega u\varphi v \, \mathrm{d}x\right| \\
&\leqslant \int_\Omega |\langle \mathcal{A}\nabla u, \nabla\varphi\rangle| \, \mathrm{d}x + \sum_{i=1}^n \int_\Omega |b_i||\varphi| \, |D_i u| \, \mathrm{d}x + \int_\Omega |g||\varphi| \, |u| \, \mathrm{d}x + \theta \int_\Omega |u||\varphi| v \, \mathrm{d}x \\
&\leqslant \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle^{1/2} \langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle^{1/2} \, \mathrm{d}x + \sum_{i=1}^n \int_\Omega \frac{|b_i|}{\omega}|\varphi||D_i u|\omega \, \mathrm{d}x \\
&\quad + \int_\Omega \frac{|g|}{v}|\varphi||u| v \, \mathrm{d}x + \theta \int_\Omega |u||\varphi| v \, \mathrm{d}x \\
&\leqslant \left(\int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega \langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle \, \mathrm{d}x\right)^{1/2} \\
&\quad + \left(\max_{1\leqslant i\leqslant n}\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)}\right) \sum_{i=1}^n \left(\int_\Omega |\varphi|^2 \omega \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega |D_i u|^2 \omega \, \mathrm{d}x\right)^{1/2} \\
&\quad + \left\|\frac{g}{v}\right\|_{L^\infty(\Omega)} \left(\int_\Omega |u|^2 v \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\varphi|^2 v \, \mathrm{d}x\right)^{1/2} \\
&\quad + \theta\left(\int_\Omega |u|^2 v \, \mathrm{d}x\right)^{1/2} \left(\int_\Omega |\varphi|^2 v \, \mathrm{d}x\right)^{1/2} \\
&\leqslant \left(1 + \max_{1\leqslant i\leqslant n}\left\|\frac{b_i}{\omega}\right\|_{L^\infty(\Omega)} + \left\|\frac{g}{v}\right\|_{L^\infty(\Omega)} + \theta\right)\|u\|_{H(\Omega)}\|\varphi\|_{H(\Omega)} \\
&= \tilde{C}\|u\|_{H(\Omega)}\|\varphi\|_{H(\Omega)}
\end{aligned}$$

for all $u, \varphi \in H(\Omega)$.

Step 3. The linear mapping T is bounded (that is, $T \in [H(\Omega)]^*$). In fact,

$$
\begin{aligned}
|T(\varphi)| &\leqslant \int_\Omega |f||\varphi| \,\mathrm{d}x = \int_\Omega \frac{|f|}{v}|\varphi| v \,\mathrm{d}x \\
&\leqslant \left[\int_\Omega \left(\frac{|f|}{v}\right)^2 v \,\mathrm{d}x\right]^{1/2} \left[\int_\Omega |\varphi|^2 v \,\mathrm{d}x\right]^{1/2} \\
&\leqslant \left\|\frac{f}{v}\right\|_{L^2(\Omega,v)} \|\varphi\|_{H(\Omega)}
\end{aligned}
$$

for all $\varphi \in H(\Omega)$.

Therefore the bilinear form $\tilde{B}$ and the linear functional T satisfy the hypotheses of the Lax-Milgram theorem. Thus, for every f with $f/v \in L^2(\Omega, v)$, there is a unique solution $u \in H(\Omega)$ such that

$$\tilde{B}(u, \varphi) = T(\varphi)$$

for all $\varphi \in H(\Omega)$, that is, u is a unique solution of the Neumann problem (P).

In particular, by setting $\varphi = u$, we have

$$\tilde{B}(u, u) = \int_\Omega f u \,\mathrm{d}x.$$

Using the definition of $\tilde{B}$, we obtain

$$
\begin{aligned}
\tilde{B}(u, u) = B(u, u) + \theta \int_\Omega u^2 v \,\mathrm{d}x &= \int_\Omega \frac{f}{v} u v \,\mathrm{d}x \\
&\leqslant \|u\|_{L^2(\Omega,v)} \left\|\frac{f}{v}\right\|_{L^2(\Omega,v)} \\
&\leqslant \|u\|_{H(\Omega)} \left\|\frac{f}{v}\right\|_{L^2(\Omega,v)}.
\end{aligned}
$$

Using (3.1), we obtain

$$\frac{1}{2}\|u\|^2_{H(\Omega)} \leqslant \tilde{B}(u, u) \leqslant \|u\|_{H(\Omega)} \left\|\frac{f}{v}\right\|_{L^2(\Omega,v)}.$$

Therefore,

$$\|u\|_{H(\Omega)} \leqslant 2\left\|\frac{f}{v}\right\|_{L^2(\Omega,v)}.$$

□

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