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ON DISCONTINUOUS GALERKIN METHOD AND SEMIREGULAR FAMILY OF TRIANGULATIONS*

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Abstract. Discretization of second order elliptic partial differential equations by discontinuous Galerkin method often results in numerical schemes with penalties. In this paper we analyze these penalized schemes in the context of quite general triangular meshes satisfying only a semiregularity assumption. A new (modified) penalty term is presented and theoretical properties are proven together with illustrative numerical results.

Keywords: discontinuous Galerkin method, elliptic equations, penalty method, semiregular family of triangulations

MSC 2000: 65N30, 65N12, 65N15

Introduction

Anisotropic (possibly flat) triangular elements are very popular for example in Computational Fluid Dynamics. They seem to have a great potential for the approximation of functions with anisotropic behaviour, as for example in boundary layers. Here we study penalized numerical schemes arising from the discretization of diffusion problem using the discontinuous Galerkin method on triangular meshes satisfying a *semiregularity* assumption, which is equivalent to the *maximum angle condition* in two-dimensional space.

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1. Model problem

We consider the following model problem:

$$(1)\qquad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded polygonal domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary $\partial\Omega$. Let the given function f be sufficiently regular.

Notation

Let $\mathcal{T}_h$ be a *conforming* triangulation of $\bar{\Omega}$. This means that the intersection of any two triangles is either an empty set or one vertex or the whole *edge*. We shall denote individual triangles of $\mathcal{T}_h$ by E and put $h_E = \operatorname{diam}(E)$.

Let $\mathcal{E}_h$ stand for the set of all *edges* of $\mathcal{T}_h$. Moreover, we distinguish the sets of *internal edges* ($\mathcal{E}_h^I$) and *boundary edges* ($\mathcal{E}_h^B$). The length of an edge $e \in \mathcal{E}_h$ will be denoted by h_e and the area of any triangle $E \in \mathcal{T}_h$ by $|E|$.

We assume that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is *semiregular*, which means that there exists a constant $M > 0$ (independent of h) such that

$$(2)\qquad Mh_E \geqslant R_E \quad \forall\, E \in \mathcal{T}_h,$$

where R_E is the radius of a circle *circumscribed* to E.

Note that in the finite element theory, usually the following stronger assumption is considered. The family of triangulations is called *regular* if the *aspect ratio condition* is satisfied with a constant $m > 0$ (independent of h) such that

$$(3)\qquad mh_E \leqslant r_E \quad \forall\, E \in \mathcal{T}_h,$$

where r_E denotes the radius of the *inscribed* circle in E. This condition is equivalent to the well-known *minimum angle condition.* Any *regular* family of triangulations is also *semiregular*, but the converse is not true, cf. [5].

For $k = 1, 2, \ldots$ let us define the so-called *broken Sobolev space* as

$$H^k(\mathcal{T}_h) = \{v \in L^2(\Omega)\colon\ v|_E \in H^k(E)\ \forall\, E \in \mathcal{T}_h\}$$

with an appropriate norm defined later. We shall also define its finite dimensional subspace

$$V_h = V_h^p(\mathcal{T}_h) = \{v \in L^2(\Omega)\colon\ v|_E \in \mathcal{P}^p(E)\ \forall\, E \in \mathcal{T}_h\},$$

where $\mathcal{P}^p(E)$ is the space of polynomials of degree at most $p \geqslant 1$.

For easier notation we introduce *average* and *jump* operators as follows. For $e \in \mathcal{E}_h^I$ we denote by E_1 and E_2 the two triangles sharing the edge e. Then for $u \in H^1(\mathcal{T}_h)$ the *average* operator $\{\cdot\}$ is defined by

$$\{u\} = \frac{1}{2}((u|_{E_1})|_e + (u|_{E_2})|_e) \quad \text{for } e \in \mathcal{E}_h^I, \ \{u\} = u|_e, \ e \in \mathcal{E}_h^B$$

and the *jump* operator $[\![\cdot]\!]$ by

$$[\![u]\!] = (u|_{E_1})|_e - (u|_{E_2})|_e \quad \text{for } e \in \mathcal{E}_h^I, \ [\![u]\!] = u|_e, \ e \in \mathcal{E}_h^B.$$

The unit normal vector $\boldsymbol{n}$ is then defined as outward to E_1, i.e., it points from E_1 to E_2.

Discontinuous Galerkin formulation

We introduce discontinuous Galerkin formulation of problem (1) with the aid of a bilinear form $a_h \colon V_h \times V_h \to \mathbb{R}$,

$$(4) \qquad a_h(u,v) = \sum_{E\in\mathcal{T}_h} \int_E \nabla u \cdot \nabla v \,\mathrm{d}x - \sum_{e\in\mathcal{E}_h} \int_e (\{\nabla u\} \cdot \boldsymbol{n}[\![v]\!] \pm \{\nabla v\} \cdot \boldsymbol{n}[\![u]\!]) \,\mathrm{d}S + \sum_{e\in\mathcal{E}_h} \beta_e \int_e [\![u]\!]\,[\![v]\!] \,\mathrm{d}S,$$

and the linear functional

$$F(v) = \int_\Omega f v \,\mathrm{d}x.$$

Here the penalty parameters β_e have the form $\beta_e = \sigma_e \gamma_e$, where σ_e is a user-defined positive constant specified later (which may differ from one edge to the other) and γ_e reflects the geometrical structure of the mesh. For reasons that will become clear later we set

$$(5) \qquad \gamma_e = h_e \sum_{E\colon e\subset\partial E} \frac{1}{|E|}, \ e \in \mathcal{E}_h^I, \quad \text{and} \quad \gamma_e = \frac{4h_e}{|E|}, \ e \in \mathcal{E}_h^B,$$

instead of the usual $\gamma_e = h_e^{-1}$.

The plus sign in the second term on the right-hand side of (4) implies a symmetric version of discretization similar to the one presented in [2], [8], [3], while the minus sign results in a nonsymmetric version, cf. [3], [6].

Our discrete formulation then becomes: Find $u_h \in V_h$ such that

$$(6) \qquad a_h(u_h, v) = F(v) \quad \forall\, v \in V_h.$$

It is possible to prove the consistency of such a formulation with problem (1). This means that any solution u of (1) satisfies formulation (6), and vice versa provided a certain regularity of the solution is assumed, see, e.g., [3], [6].

An important consequence of consistency is the relation

$$a_h(u - u_h, v) = 0 \quad \forall\, v \in V_h. \tag{7}$$

2. Existence and uniqueness of solution

We will use the following norm for the *broken Sobolev space* $H^1(\mathcal{T}_h)$:

$$|\!|\!| u |\!|\!| = \Big(\sum_{E \in \mathcal{T}_h} |u|^2_{1,E} + \sum_{e \in \mathcal{E}_h} \gamma_e \| [\![u]\!] \|^2_e \Big)^{1/2} \tag{8}$$

with γ_e defined by (5). Here we use the standard notation: $\|v\|_K$ is the L^2-norm of the function v over the set K, $|v|_{1,K} = \|\nabla v\|_K$ is a seminorm, etc.

First of all we offer an overview of useful inequalities and relations and prove several variants of scaling and trace inequalities with constants independent of the *aspect ratio condition*. We will use subscripts to distinguish constants and explicitly express possible dependencies. Otherwise C denotes a generic constant having different values at different places in general.

Lemma 2.1 (Scaling inequality). *For each function $w \in \mathcal{P}^p(E)$ and each triangle E with an edge e we have*

$$\|w\|_e \leqslant C_1(p) \sqrt{\frac{h_e}{|E|}}\, \|w\|_E \tag{9}$$

with C_1 depending only on the reference element and the polynomial degree.

P r o o f. We will use a "hat" sign to indicate quantities transformed by one-to-one linear affine mapping from an arbitrary element E to the unit triangle $\hat{E}$ with vertices $(0,0)$, $(1,0)$, $(1,1)$ and denote by $\hat{e}$ its side with end points $(0,0)$, $(1,0)$. Then

$$\|w\|_e^2 = \int_e w^2 \,\mathrm{d}S = h_e \int_{\hat{e}} \hat{w}^2 \,\mathrm{d}\hat{S} \leqslant C h_e \int_{\hat{E}} \hat{w}^2 \,\mathrm{d}\hat{x} = C \frac{h_e}{2|E|} \int_E w^2 \,\mathrm{d}x,$$

where the inequality can be found for example in [2], $C_1(p) = \sqrt{C(p)/2}$. □

Lemma 2.2 (Trace inequality). *For every $w \in H^1(E)$ and every edge $e \subset \partial E$ we have*

$$\|w\|_e^2 \leqslant C_2 \frac{h_e}{|E|}(\|w\|_E^2 + h_E^2 |w|_{1,E}^2),$$

where C_2 is independent of E.

Proof. We shall use transformations as in the previous lemma. The inequality

$$\|w\|_{\hat{e}}^2 \leqslant C(\|w\|_{\hat{E}}^2 + |w|_{1,\hat{E}}^2)$$

is a trace theorem on the reference element. The factor h_E arises when transforming the seminorm to an arbitrary element. □

Now we derive a useful estimate. For $u, v \in V_h^p(\mathcal{T}_h)$ we have

$$\begin{aligned}\int_e \{\nabla u\} \cdot \boldsymbol{n}[\![v]\!] \,\mathrm{d}S &= \frac{1}{2}\Big(\int_{e\in E_1} \nabla u \cdot \boldsymbol{n}[\![v]\!] \,\mathrm{d}S + \int_{e\in E_2} \nabla u \cdot \boldsymbol{n}[\![v]\!] \,\mathrm{d}S\Big) \\ &\leqslant \frac{1}{2}\|[\![v]\!]\|_e(\|\nabla u \cdot \boldsymbol{n}\|_{e\in E_1} + \|\nabla u \cdot \boldsymbol{n}\|_{e\in E_2}) \\ &\leqslant \frac{1}{2} C_1(p)\|[\![v]\!]\|_e\Big(\sqrt{\frac{h_e}{|E_1|}}\|\nabla u\|_{E_1} + \sqrt{\frac{h_e}{|E_2|}}\|\nabla u\|_{E_2}\Big),\end{aligned}$$

where the first inequality is a consequence of the Cauchy-Schwarz inequality, the other is a scaling inequality. Further, using Young's inequality for $\varepsilon > 0$, $2ab \leqslant \varepsilon a^2 + b^2/\varepsilon$, we have estimates of the type

$$\int_e \{\nabla u\} \cdot \boldsymbol{n}[\![v]\!] \,\mathrm{d}S \leqslant \frac{\varepsilon C_1^2}{4}\|[\![v]\!]\|_e^2 \sum_{E\colon e\subset\partial E} \frac{h_e}{|E|} + \frac{1}{4\varepsilon} \sum_{E\colon e\subset\partial E} |u|_{1,E}^2. \tag{10}$$

Similar bounds are valid also for boundary edges.

To prove existence and uniqueness of a solution of the discrete problem (6) we need to verify that the bilinear form $a_h(\cdot,\cdot)$ is coercive and continuous (i.e., bounded) on the space V_h, (cf. [4], the Lax-Milgram lemma).

For the nonsymmetric bilinear form the coercivity is obvious, because

$$a_h(v,v) \geqslant \min_{e\in\mathcal{E}_h}(1,\sigma_e)|\!|\!|v|\!|\!|^2$$

not only for all $v \in V_h$ but even for all $v \in H^1(\mathcal{T}_h)$.

Anyway, we have to be careful in obtaining the coercivity of the symmetric form. Using definition (4), inequalities (10) with $\varepsilon := 3\delta/2$ and γ_e as in (5), we find that

$$a_h(v,v) \geqslant \sum_{E\in\mathcal{T}_h}\Big(1 - \frac{1}{\delta}\Big)|v|_{1,E}^2 + \sum_{e\in\mathcal{E}_h} \gamma_e \|[\![v]\!]\|_e^2\Big(\sigma_e - \frac{3\delta C_1^2}{4}\Big). \tag{11}$$

To keep the terms in the round brackets positive, which is necessary for the coercivity of the symmetric bilinear form, we have to satisfy

$$\delta > 1 \quad \text{and} \quad \sigma_e > \frac{3\delta}{4} C_1^2 \qquad \text{for } e \in \mathcal{E}_h. \tag{12}$$

The continuity of both bilinear forms is a consequence of multiple use of the Cauchy-Schwarz inequality together with estimates of type (10). We conclude that

$$\begin{aligned} a_h(u,v) &\leqslant \Big[(1+C)\sum_{E\in\mathcal{T}_h}|u|_{1,E}^2 + \sum_{e\in\mathcal{E}_h}(\sigma_e+1)\gamma_e\|[\![u]\!]\|_e^2\Big]^{1/2} \\ &\quad\times\Big[(1+C)\sum_{E\in\mathcal{T}_h}|v|_{1,E}^2 + \sum_{e\in\mathcal{E}_h}(\sigma_e+1)\gamma_e\|[\![v]\!]\|_e^2\Big]^{1/2} \\ &\leqslant \max_{e\in\mathcal{E}_h}(1+C,1+\sigma_e)|\!|\!|u|\!|\!|\,|\!|\!|v|\!|\!|, \end{aligned}$$

where the constant $C = 3C_1^2/4$, in particular, is independent of the mesh and depends only on the polynomial degree. Thus, we can summarize:

Theorem 2.3. *The discrete problem* (6) *has for each right-hand side* $f \in L^2(\Omega)$ *exactly one solution provided the penalty parameters* σ_e *are positive for nonsymmetric formulation or satisfy inequalities* (12) *in the case of symmetric formulation.*

3. A priori error estimates

A priori error estimates can be obtained in a standard way by converting the problem of an error to the approximation properties of the finite element spaces. Here we use the following local error estimate:

Theorem 3.1. *Assume that the triangle* E *satisfies the maximum angle condition and let* $u \in H^{p+1}(E) \cap C(\overline{E})$. *Then the difference between the function* u *and its Lagrangian interpolation* u_I *can be estimated by*

$$|u-u_I|_{m,E} \leqslant Ch_E^{p+1-m}|u|_{p+1,E}, \quad m = 0,1,2. \tag{13}$$

Proof. For a more general version see [1, p. 47]. The case $p = 1$ and $m = 2$, which is not covered in the original proof, is simply the equality $|u-u_I|_{2,E} = |u|_{2,E}$. □

Theorem 3.2 (Energy error estimate). *Let $\{\mathcal{T}_h\}_{h>0}$ be a semiregular family of triangulations and let u_h be the unique solution of (6). If the weak solution u of (1) satisfies $u \in H^{p+1}(\Omega) \cap H_0^1(\Omega)$ then*

$$\|\!|u - u_h|\!\|^2 \leqslant \sum_{E\in\mathcal{T}_h} C h_E^{2p} |u|_{p+1,E}^2 \tag{14}$$

with the constant C specified in the proof.

Proof. For a nonsymmetric version we get

$$\begin{aligned}\sum_{E\in\mathcal{T}_h} |u-u_h|_{1,E}^2 + \sum_{e\in\mathcal{E}_h} \beta_e \|[\![u-u_h]\!]\|_e^2 &= a_h(u-u_h, u-u_h)\\ &= a_h(u-u_h, u-u_I),\end{aligned}$$

where we have used identity (7). Then we can estimate (using the fact that u and u_I are continuous) by the Cauchy-Schwarz and Young inequalities, denoting $\sigma_0 = \min\limits_{e\in\mathcal{E}_h}(\sigma_e, 1)$,

$$\begin{aligned}a_h(u-u_h, u-u_I) &= \sum_{E\in\mathcal{T}_h} \int_E \nabla(u-u_h)\cdot\nabla(u-u_I)\,\mathrm{d}x\\ &\quad+ \sum_{e\in\mathcal{E}_h} \int_e \{\nabla(u-u_I)\}\cdot \boldsymbol{n}[\![u-u_h]\!]\,\mathrm{d}S\\ &\leqslant \Big(1-\frac{\sigma_0}{2}\Big)\sum_{E\in\mathcal{T}_h}|u-u_h|_{1,E}^2 + \frac{1}{4(1-\frac{\sigma_0}{2})}\sum_{E\in\mathcal{T}_h}|u-u_I|_{1,E}^2\\ &\quad+ \sum_{e\in\mathcal{E}_h}\Big(\sigma_e - \frac{\sigma_0}{2}\Big)\gamma_e\|[\![u-u_h]\!]\|_e^2\\ &\quad+ \sum_{e\in\mathcal{E}_h}\frac{1}{16(\sigma_e-\frac{\sigma_0}{2})}\bigg(\sum_{E\colon e\subset\partial E}\frac{|E|}{h_e}\|\nabla(u-u_I)\cdot\boldsymbol{n}\|_{e\in E}^2\bigg).\end{aligned}$$

Putting terms with $u-u_h$ to the left-hand side and using the trace inequality

$$\begin{aligned}\|\!|u-u_h|\!\|^2 &\leqslant \frac{1}{2\sigma_0(1-\sigma_0/2)}\sum_{E\in\mathcal{T}_h}|u_I-u|_{1,E}^2\\ &\quad+\sum_{e\in\mathcal{E}_h}\frac{C_2}{8\sigma_0(\sigma_e-\sigma_0/2)}\bigg(\sum_{E\colon e\subset\partial E}|u_I-u|_{1,E}^2 + h_E^2|u_I-u|_{2,E}^2\bigg)\end{aligned}$$

and the local estimate (13) completes the proof.

A symmetric scheme is again a little bit more difficult. First we have to estimate the error $u_h - u_I$, because we have proved the coercivity estimate only for the discrete space. Then the desired result comes from the triangle inequality

$$|\!|\!| u - u_h |\!|\!| \leqslant |\!|\!| u - u_I |\!|\!| + |\!|\!| u_I - u_h |\!|\!|.$$

We start from the coercivity estimate (11) with $v := u_I - u_h$ and $\delta > 1$ such that the inequality $1 - 1/\delta < \sigma_e - 3\delta C_1^2/4$ holds for all σ_e. Such a choice is always possible. Then we proceed as follows:

$$\begin{aligned}
&a_h(u_I - u_h, u_I - u_h) = a_h(u_I - u, u_I - u_h) \\
&\leqslant \left(\frac{\delta-1}{2\delta}\right) \sum_{E\in\mathcal{T}_h} |u_I - u_h|_{1,E}^2 + \frac{1}{2(1-\frac{1}{\delta})} \sum_{E\in\mathcal{T}_h} |u - u_I|_{1,E}^2 \\
&\quad + \sum_{e\in\mathcal{E}_h} \left(\sigma_e - \frac{3\delta C_1^2}{4} - \frac{\delta-1}{2\delta}\right) \gamma_e \| [\![u - u_h]\!] \|_e^2 \\
&\quad + \frac{1}{16} \sum_{e\in\mathcal{E}_h} \left(\sigma_e - \frac{3\delta C_1^2}{4} - \frac{\delta-1}{2\delta}\right)^{-1} \left(\sum_{E:\, e\subset\partial E} \frac{|E|}{h_e} \|\nabla(u-u_I)\cdot \boldsymbol{n}\|_{e\in E}^2 \right),
\end{aligned}$$

where the equality is a consequence of the Galerkin orthogonality (7). This estimate and (11) give

$$\begin{aligned}
|\!|\!| u_I - u_h |\!|\!|^2 \leqslant & \left(\frac{\delta}{\delta-1}\right)^2 \sum_{E\in\mathcal{T}_h} |u_I - u|_{1,E}^2 \\
& + \frac{1}{8} \sum_{e\in\mathcal{E}_h} \frac{\delta}{\delta-1} \left(\sigma_e - \frac{3\delta C_1^2}{4} - \frac{\delta-1}{2\delta}\right)^{-1} \\
& \times \left(\sum_{E:\, e\subset\partial E} \frac{|E|}{h_e} \|\nabla(u-u_I)\cdot \boldsymbol{n}\|_{e\in E}^2 \right).
\end{aligned}$$

Then the final result is a consequence of Lemma 2.2, the local estimate (13) and the triangle inequality. □

One can see that the constant in the bound (14) depends on the stabilization parameters in the following way. It blows up at lower bounds ($\sigma_e = 0$ for the non-symmetric version, $\sigma_e = 3C_1^2/4$ for the symmetric discretization) and monotonically decreases to a positive constant as σ_e's tend to infinity.

The usual Aubin-Nitsche trick can be used to derive an error estimate in the L^2-norm. However, the optimal rate of convergence is available only for the symmetric scheme.

Theorem 3.3 (L^2 error estimate). *Let $\{\mathcal{T}_h\}_{h>0}$ be a semiregular family of triangulations of a convex domain Ω and let u_h be the unique solution of* (6) *with the symmetric bilinear form. If the weak solution of* (1) *satisfies $u \in H^{p+1}(\Omega) \cap H^1_0(\Omega)$, then*

$$\|u - u_h\|_\Omega \leqslant Ch^{p+1}|u|_{p+1,\Omega}, \tag{15}$$

where $h = \max_{E\in\mathcal{T}_h} h_E$.

Proof. We consider an auxiliary problem

$$-\Delta\psi_g = g \text{ in } \Omega, \quad \psi_g = 0 \text{ on } \partial\Omega,$$

for which the regularity estimate $|\psi_g|_{2,\Omega} \leqslant C\|g\|_\Omega$ holds provided $g \in L^2(\Omega)$ and Ω is convex. The symmetry of the bilinear form implies that

$$a_h(v, \psi_g) = (v, g) \quad \forall\, v \in H^2(\mathcal{T}_h).$$

Using the characterization of the norm and the Galerkin orthogonality (7) with ψ_I (linear Lagrangian interpolation of ψ_g), we get for $g \neq 0$

$$\begin{aligned}
\|u - u_h\|_\Omega &= \sup_{g\in L^2(\Omega)} \frac{(u - u_h, g)}{\|g\|} = \sup_{g\in L^2(\Omega)} \frac{a_h(u - u_h, \psi_g)}{\|g\|} \\
&= \sup_{g\in L^2(\Omega)} \frac{a_h(u - u_h, \psi_g - \psi_I)}{\|g\|} \\
&\leqslant \sup_{g\in L^2(\Omega)} \frac{1}{\|g\|}\Bigg(\Bigg[\sum_{E\in\mathcal{T}_h} |u - u_h|^2_{1,E}\Bigg]^{1/2} \Bigg[\sum_{E\in\mathcal{T}_h} |\psi_g - \psi_I|^2_{1,E}\Bigg]^{1/2} \\
&\quad + C\Bigg[\sum_{e\in\mathcal{E}_h} \gamma_e \|[\![u - u_h]\!]\|^2_e\Bigg]^{1/2} \Bigg[\sum_{E\in\mathcal{T}_h} |\psi_g - \psi_I|^2_{1,E} + h_E^2|\psi_g|^2_{2,E}\Bigg]^{1/2}\Bigg) \\
&\leqslant C|\!|\!|u - u_h|\!|\!| \sup_{g\in L^2(\Omega)} \frac{1}{\|g\|}\Bigg[\sum_{E\in\mathcal{T}_h} h_E^2|\psi_g|^2_{2,E}\Bigg]^{1/2} \\
&\leqslant C|\!|\!|u - u_h|\!|\!| \sup_{g\in L^2(\Omega)} \frac{1}{\|g\|} h|\psi_g|_{2,\Omega} \\
&\leqslant Ch|\!|\!|u - u_h|\!|\!| \leqslant C(C_2, \Omega)h^{p+1}|u|_{p+1}.
\end{aligned}$$

□

For the nonsymmetric version such a technique cannot be used. Additional terms arising from the nonsymmetry of the scheme appear. The suboptimal estimate

$$\|u - u_h\|_\Omega \leqslant Ch^p |u|_{p+1,\Omega}$$

holds as one can see from the inequality

$$\|w\|_\Omega \leqslant C(\Omega) |||w|||, \quad w \in H^1(\mathcal{T}_h),$$

which can be proved in a similar way as in [2] for our penalty term (5) and, hence, the energy norm (8).

4. Lower bound for stabilization

Let us look more carefully at the scaling inequality, Lemma 2.1, which is essential for relation (12). We sketch an algorithm for computation of the constant $C_1(p)$ and present its values for a few "low" p.

We need an estimate on the reference element $\hat{E} = ABC$ and $\hat{e} = AB$:

$$\|w\|_{\hat{e}}^2 \leqslant C_3^2 \|w\|_{\hat{E}}^2, \quad w \in \mathcal{P}^p(\hat{E}),$$

where $A = (0,0)$, $B = (1,0)$, $C = (1,1)$. This problem can be formulated in the following way: Find a function w for which $\|w\|_{\hat{E}}^2$ is minimal under the additional condition $\|w\|_{\hat{e}}^2 = 1$. The unit value causes no loss of generality.

An important feature of polynomials of the form

$$w(x,y) = \sum_{i+j=0}^{n} a_{ij} x^i y^j$$

is that $\|w\|_{\hat{e}}^2$ and $\|w\|_{\hat{E}}^2$ can be expressed by

$$G(\{a_{ij}\}) := \|w\|_{\hat{e}}^2 = \sum_{k,l=0}^{n} a_{k0} a_{l0} \frac{1}{k+l+1},$$

$$H(\{a_{ij}\}) := \|w\|_{\hat{E}}^2 = \sum_{k+l=0}^{n} \sum_{q+r=0}^{n} a_{kl} a_{qr} \frac{1}{(l+r+1)(k+l+q+r+2)}$$

as functions of unknown parameters a_{ij}, $i+j = 0,\ldots,n$. Clearly,

$$\|w\|_{\hat{E}}^2 = \int_0^1 \int_0^x \left[\sum_{k+l=0}^{n} a_{kl} x^k y^l \right]^2 \mathrm{d}y \, \mathrm{d}x$$

$$= \int_0^1 \int_0^x \left[\sum_{k+l=0}^{n} \sum_{q+r=0}^{n} a_{kl} a_{qr} x^{k+q} y^{l+r} \right] \mathrm{d}y \, \mathrm{d}x$$

and the resulting expression for the function H appears after integration. In the same way it is possible to obtain the above identity for the function G.

Then the minimum of $H(\{a_{ij}\})$ under the condition $G(\{a_{ij}\}) = 1$ is found with the aid of the constrained extremal theory, cf. [9]. Solution of such a problem is done in a fairly standard way, hence we skip details. Note that all the partial derivatives of G and H with respect to the variables a_{ij} are linear functions of a_{ij}'s. Thus, the solution of linear systems is the main ingredient.

Here we present Tab. 1 with the values of C_1^2 for $p = 0, \ldots, 8$ computed using MAPLE software. We recall that $C_1^2 = C_3^2/2$. Note that the value of $C_1^2(p)$ corresponds to the number of degrees of freedom associated with the space of polynomials of degree p on triangles.

p	0	1	2	3	4	5	6	7	8
$C_1^2(p)$	1	3	6	10	15	21	28	36	45

Table 1. Values of the constant $C_1^2(p)$ for different polynomial degrees.

It is clear that the values in Tab. 1 have the same asymptotic behaviour as follows from [7, Theorem 4.76]:

$$\|w\|_{\hat{e}}^2 \leqslant Cp^2 \|w\|_{\hat{E}}^2, \quad w \in \mathcal{P}^p(\hat{E}).$$

R e m a r k 1. These results can be used also for the original DG schemes on *regular* families of triangulations. Setting $\beta_e = \sigma_e/h_e$, we can easily see that it is essential to be able to estimate the ratio $h_e^2/|E|$, cf. Section 2. This is possible under the *aspect ratio condition.* Clearly $h_e^2/|E| \leqslant h_E/r_E \leqslant 1/m$, where m and r_E are defined by (3). Thus, the symmetric bilinear form with penalty $\beta_e = \sigma_e/h_e$ is coercive if

$$\sigma_e > \frac{3}{2m} C_1^2, \quad e \in \mathcal{E}_h^I, \tag{16}$$

and similarly, up to a constant, for boundary edges. For triangulations consisting of equilateral triangles this bound can be improved to $\sigma_e > 2\sqrt{3}\, C_1^2$.

Inequality (16) also indicates why the standard penalty term is not sufficient for our purposes. The use of penalty $\beta_e = \sigma_e/h_e$ would cause that all important constants (boundedness, lower bound for stabilization of the symmetric scheme, error estimates) would, in fact, depend on the *minimal angle* in an unpleasant way.

R e m a r k 2. Scaling inequality (9) is applied to the normal derivative of $u \in \mathcal{P}^p(E)$, which in this case belongs to the space $\mathcal{P}^{p-1}(E)$. Thus, for the scheme with piecewise polynomials of degree p, one should be in fact interested in the value $C_1(p-1)$.

5. Numerical results

In this section we present some numerical results concerning the numerical schemes presented above.

Both the symmetric and nonsymmetric schemes were implemented in *Fortran* programming language for piecewise polynomial approximation up to degree $p = 4$. The resulting linear system was solved with the restarted GMRES method with incomplete LU preconditioning. For known solutions the errors in L^2 and energy norms were estimated using suitable quadrature formulae.

Two types of computation meshes were used. The former was a sequence of *regular* unstructured triangular grids consisting of (almost) equilateral triangles. The latter sequence is similar to the one used in [5], see Fig. 1. It is clear that this sequence of meshes satisfies only the *semiregularity* condition. Then the experimental order of convergence (EOC) was computed using log-linear regression. See Tabs. 2–5.

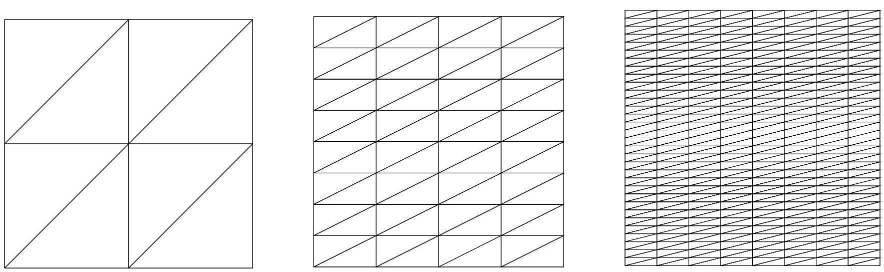

Figure 1. Semiregular family of triangulations.

Meshes	Regular		Semiregular	
Norm	L^2	Energy	L^2	Energy
Symmetric	$0.38h^{1.99}$	$1.30h^{1.01}$	$0.42h^{2.00}$	$1.48h^{0.96}$
Nonsymmetric	$0.15h^{2.00}$	$1.17h^{0.98}$	$0.44h^{2.02}$	$0.96h^{0.96}$

Table 2. Linear finite elements.

Meshes	Regular		Semiregular	
Norm	L^2	Energy	L^2	Energy
Symmetric	$0.10h^{3.00}$	$1.57h^{1.99}$	$0.10h^{3.10}$	$0.39h^{2.13}$
Nonsymmetric	$0.19h^{1.95}$	$0.96h^{1.99}$	$0.07h^{2.11}$	$0.25h^{1.84}$

Table 3. Piecewise quadratic polynomial approximation.

Meshes	Regular		Semiregular	
Norm	L^2	Energy	L^2	Energy
Symmetric	$0.01h^{3.96}$	$0.15h^{2.91}$	$0.02h^{3.99}$	$0.22h^{3.19}$
Nonsymmetric	$0.02h^{3.97}$	$0.18h^{2.98}$	$0.03h^{4.00}$	$0.28h^{3.24}$

Table 4. Piecewise cubic polynomials.

Meshes	Regular		Semiregular	
Norm	L^2	Energy	L^2	Energy
Symmetric	$0.001h^{5.09}$	$0.03h^{4.09}$	$0.002h^{5.10}$	$0.05h^{4.10}$
Nonsymmetric	$0.001h^{3.90}$	$0.04h^{3.99}$	$0.003h^{4.04}$	$0.08h^{4.19}$

Table 5. Piecewise polynomials of the fourth degree.

All features of the presented schemes, as estimated errors or the performance of the iterative solver of the linear system, depend on the choice of stabilization parameters σ_e.

Error in the energy norm behaves in all cases as predicted in Theorem 3.2 (decreasing with increasing penalty parameters).

In the case of the L^2-norm, the situation is different. For the symmetric version the computed error in the L^2-norm exhibits the minimal value for $\sigma_e \approx C_1^2(p)/2$. This minimum is the most significant for piecewise linear elements ($\sigma_e \approx 1.5$).

For the nonsymmetric scheme, however, the minimum of the L^2-error was observed only for $p = 1$ (again at $\sigma_e \approx 1.5$). In other cases the error is decreasing with increasing penalty. The well-known improvement of EOC in the L^2-norm is observed for $p = 1$ and $p = 3$.

Let us look at practical results concerning the presented results for a lower bound of stabilization of the symmetric scheme. For linear elements $p = 1$ we choose $\sigma_e > 3/4$ (according to Tab. 1 and Remark 2 from the previous section). For computations on *regular* families of triangulations, values $\sigma_e \geqslant 0.85$ gave good results. For higher $p \geqslant 2$ the values $\sigma_e = 2.25$ ($p = 2$), $\sigma_e = 4.5$ ($p = 3$) and $\sigma_e = 7.5$ ($p = 4$) were already sufficient.

6. Conclusions

In this paper the existing theory of discontinuous Galerkin method for elliptic problems was extended to the case of *semiregular* families of triangulations. The modified penalty term was presented together with related modifications of theoretical results.

A lower bound for stabilization of the symmetric scheme was found with the aid of the theory of constrained extremes. A lower bound for stabilization of the original symmetric scheme with penalty term $\beta_e = \sigma_e/h_e$ on *regular* families of triangulations was also presented together with explicit dependence on the *aspect ratio* constant.

Theoretical properties were illustrated by numerical results. The well-known improvement of the order of accuracy of the nonsymmetric discretization was observed for odd polynomial degrees $p = 1$ and $p = 3$.

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