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APPROXIMATION OF AN EIGENVALUE PROBLEM ASSOCIATED WITH THE STOKES PROBLEM BY THE STREAM FUNCTION-VORTICITY-PRESSURE METHOD*

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Abstract. By means of eigenvalue error expansion and integral expansion techniques, we propose and analyze the stream function-vorticity-pressure method for the eigenvalue problem associated with the Stokes equations on the unit square. We obtain an optimal order of convergence for eigenvalues and eigenfuctions. Furthermore, for the bilinear finite element space, we derive asymptotic expansions of the eigenvalue error, an efficient extrapolation and an a posteriori error estimate for the eigenvalue. Finally, numerical experiments are reported.

Keywords: eigenvalue problem, Stokes problem, stream function-vorticity-pressure method, asymptotic expansion, extrapolation, a posteriori error estimates

MSC 2000: 65N30, 65N25, 35Q30

1. Introduction

There are various approximation methods for solving the Stokes problem, see Bercovier and Pironneau [2], Brezzi et al. [6], Girault and Raviart [10], Glowinski and Pironneau [11], Křížek [15], Rannacher and Turek [23], Stenberg [25], Verfurth [26], Ye [30], Zhou et al. [31], Wang et al. [27], and references cited therein. In [20] and [22] the authors describe an eigenvalue problem associated with the Stokes problem as follows:

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Find λ, $\mathbf{u}$ and p satisfying

$$(1)\qquad \begin{cases} -\Delta\mathbf{u} + \mathbf{grad}\, p = \lambda\mathbf{u} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \Gamma = \partial\Omega, \end{cases}$$

where $\Omega = (0,1) \times (0,1)$.

The approximation of the eigenvalue involves problems that are of great interest and have been extensively investigated (see, e.g., [1], [3]–[5], [7], [12], [14], [18], [20]–[22], [24], [28], [29], and references cited therein).

In this paper we consider a stream function-vorticity-pressure method to solve eigenvalue problem (1).

If we introduce the stream function ψ for the velocity ($\mathbf{u} = \mathbf{curl}\,\psi = (\partial_2\psi, -\partial_1\psi)$), based on the identities

$$(2)\qquad \mathbf{curl}(\operatorname{curl}\mathbf{u}) = -\Delta\mathbf{u} + \mathbf{grad}(\operatorname{div}\mathbf{u}), \quad \operatorname{curl}(\mathbf{curl}\,\psi) = -\Delta\psi,$$

where $\operatorname{curl}\mathbf{u} = -\partial_2 u_1 + \partial_1 u_2$, problem (1) can be expressed as the following buckling plate problem:

Find λ, ψ satisfying

$$(3)\qquad \begin{cases} -\Delta^2\psi = \lambda\Delta\psi & \text{in } \Omega, \\ \psi = \dfrac{\partial\psi}{\partial\mathbf{n}} = 0 & \text{on } \Gamma, \end{cases}$$

where $\mathbf{n}$ is the outward unit normal.

From now on, we shall use the standard notation as those in Ciarlet [8], for example, the notation of the Sobolev spaces, product, norms, seminorms, discretized norms, etc.

We then consider the following mixed formulation for (3):

Find $\lambda, (\psi, \omega) \in H_0^1(\Omega) \times H^1(\Omega)$ such that $|\psi|_{1,\Omega} = 1$ and

$$(4)\qquad \begin{cases} a(\omega,\theta) + b(\theta,\psi) = 0 & \forall\,\theta \in H^1(\Omega), \\ b(\omega,\varphi) = -\lambda s(\psi,\varphi) & \forall\,\varphi \in H_0^1(\Omega), \end{cases}$$

and find $p \in H^1(\Omega)$ such that

$$(5)\qquad \begin{cases} (\mathbf{grad}\, p, \mathbf{grad}\, q) = \lambda(\mathbf{u} - \mathbf{curl}\,\omega, \mathbf{grad}\, q) & \forall\, q \in H^1(\Omega), \\ \displaystyle\int_\Omega p\,\mathrm{d}x = 0, \end{cases}$$

where $\omega = -\Delta\psi$ and

$$\begin{aligned}
&a(\omega,\theta) = (\omega,\theta) && \text{for } \omega,\theta \in H^1(\Omega),\\
&b(\omega,\varphi) = -(\mathbf{curl}\,\omega, \mathbf{curl}\,\varphi) && \text{for } \omega \in H^1(\Omega),\ \varphi \in H_0^1(\Omega),\\
&s(\psi,\varphi) = (\mathbf{curl}\,\psi, \mathbf{curl}\,\varphi) && \text{for } \psi,\varphi \in H_0^1(\Omega).
\end{aligned}$$

Problem (4) has an eigenvalue sequence $\{\lambda_j\}$ (see [1]):

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k \leqslant \lambda_{k+1} \leqslant \ldots, \qquad \lim_{k\to\infty} \lambda_k = \infty,$$

and the associated eigenfuctions

$$(\psi_1,\omega_1),\ (\psi_2,\omega_2),\ \ldots,\ (\psi_k,\omega_k),\ (\psi_{k+1},\omega_{k+1}),\ \ldots,$$

where $(\mathbf{curl}\,\psi_i, \mathbf{curl}\,\psi_j) = \delta_{ij}$, $\omega_k = -\Delta\psi_k$.

It is well known that if (λ,ψ) is an eigenpair of (3) and $\omega = -\Delta\psi$ then (λ,ψ,ω) is an eigenpair of (4), and if (λ,ψ,ω) is an eigenpair of (4) then (λ,ψ) is an eigenpair of (3) and $\omega = -\Delta\psi$.

Assume that $T_h = \{e\}$ is a regular family of rectangular meshes on Ω with mesh size h, where $e = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$. Introduce finite element spaces

$$\begin{aligned}
S_h^k &= \{v \in H^1(\Omega)\colon\ v|_e \in Q_k\ \forall\, e \in T_h\},\\
\overset{\circ}{S}{}_h^k &= H_0^1(\Omega) \cap S_h^k,
\end{aligned}$$

where $Q_k = \operatorname{span}\{x^i y^j,\ 0 \leqslant i,\ j \leqslant k\}$.

The approximation schemes for (4) are:

Find $\lambda^h, (\psi^h,\omega^h) \in \overset{\circ}{S}{}_h^k \times S_h^k$ such that $|\psi^h|_{1,\Omega} = 1$,

$$\text{(6)} \qquad \begin{cases} a(\omega^h,\theta_h) + b(\theta_h,\psi^h) = 0 & \forall\,\theta_h \in S_h^k,\\ b(\omega^h,\varphi_h) = -\lambda^h s(\psi^h,\varphi_h) & \forall\,\varphi_h \in \overset{\circ}{S}{}_h^k, \end{cases}$$

and find $p_h \in S_h^k$ such that

$$\text{(7)} \qquad \begin{cases} (\mathbf{grad}\,p_h, \mathbf{grad}\,q_h) = \lambda(\mathbf{u}_h - \mathbf{curl}\,\omega^h, \mathbf{grad}\,q_h) & \forall\,q_h \in S_h^k,\\ \displaystyle\int_\Omega p_h\,\mathrm{d}x = 0, \end{cases}$$

with $\mathbf{u} = \mathbf{curl}\,\psi^h$.

Problem (6) has eigenvalues

$$0 < \lambda_1^h \leqslant \ldots \leqslant \lambda_i^h \leqslant \ldots \leqslant \lambda_{N(h)}^h$$

and the corresponding eigenfunctions

$$(\psi_1^h, \omega_1^h), \ldots, (\psi_i^h, \omega_i^h), \ldots, (\psi_{N(h)}^h, \omega_{N(h)}^h),$$

where $(\psi_i^h, \psi_j^h) = \delta_{ij}$ and $\omega_k^h = -\Delta_h \psi_k^h$ (where Δ_h is a suitable discretization of Δ).

For the sake of using the approximation theory of spectrum to analyze the above approximation scheme, we define the associated source problem for (4) and the approximation source problem for (6), respectively: For $g \in H^1(\Omega)$, find $Ag \in H^1(\Omega)$, $Bg \in H_0^1(\Omega)$ such that

$$\begin{cases} a(Ag, \theta) + b(\theta, Bg) = 0 & \forall\, \theta \in H^1(\Omega), \\ b(Ag, \varphi) = -s(g, \varphi) & \forall\, \varphi \in H_0^1(\Omega), \end{cases} \tag{8}$$

and find $A_h g \in S_h^k$, $B_h g \in \overset{\circ}{S}{}_h^k$ such that

$$\begin{cases} a(A_h g, \theta_h) + b(\theta_h, B_h g) = 0 & \forall\, \theta_h \in S_h^k, \\ b(A_h g, \varphi_h) = -s(g, \varphi_h) & \forall\, \varphi_h \in \overset{\circ}{S}{}_h^k. \end{cases} \tag{9}$$

We can define adjoint operators A^*, B^* of A, B as those in [20]. Clearly $B^* = B$, $A^* = A$, and thus A and B are selfadjoint. Problems (8)–(9) are uniquely solvable and $B, B_h\colon\ H^1(\Omega) \to H^1(\Omega)$ are compact.

So the eigenvalues of (3) can be characterized in terms of the operator B. If λ, (ω, ψ) is an eigenpair of (3) then $\lambda B\psi = \psi$, $\psi \neq 0$, and if $\lambda B\psi = \psi$, $\psi \neq 0$, then there exists an $\omega \in H^1(\Omega)$ such that $\lambda, (\omega, \psi)$ is an eigenpair of (3). In a similar way we see that the eigenvalues of (6) are the reciprocals of the eigenvalues of B_h.

From now on, we denote by $(\lambda, \psi, \omega) = (\lambda_j, \psi_j, \omega_j)$ the jth eigensolution of (4), and by $(\lambda^h, \psi^h, \omega^h) = (\lambda_j^h, \psi_j^h, \omega_j^h)$ its corresponding discretized eigensolution of (6).

An outline of the paper goes as follows. In Section 2 we introduce the 'vertices-edges-element' interpolation and describe the integral expansions. In Section 3, we prove the error estimate for the source problem. Then, in Section 4, we first give an optimal order estimate for the eigenvalue and eigenfuntion, then construct the error expansion for the eigenvalue in Lemma 5. By using the technique of eigenvalue error expansion and integral expansions, we demonstrate the eigenvalue asymptotic expansion. Furthermore, in Section 5, we get an efficient extrapolation and an a posteriori error estimate for the eigenvalue on a square mesh.

Finally, in Section 6, we present a numerical experiment to show a good performance of the methods and illustrate the theoretical results.

Throughout this paper, the symbol $C(v_1, v_2, \ldots)$ denotes a generic positive constant which is dependent on $v_1, v_2, \ldots$, and independent of h. In addition, we shall assume that ψ is sufficiently smooth, and the smoothness requirements will be shown by the norms in our analysis.

2. Integral expansions

Starting from the next section we will use integral expansion techniques to analyze the eigenvalue problem associated with the Stokes problem and its source problem. In this section we will introduce some integral expansions.

Define 'vertices-edges-element' interpolation operator (see [19]) $I_h^k\colon C(\overline{\Omega}) \to S_h^k$ by the following conditions:

For $k = 1$

$$I_h^1 u(Z_i) = u(Z_i), \quad i = 1, 2, 3, 4.$$

For $k \geqslant 2$

$$\int_{l_i} (I_h^k u - u) v \, \mathrm{d}l = 0 \quad \forall\, v \in P_{k-2}(l_i), \; i = 1, 2, 3, 4,$$
$$\int_e (I_h^k u - u) v \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \forall\, v \in Q_{k-2}(e),$$

where Z_i, l_i ($i = 1, 2, 3, 4$) are the vertices and edges of e, and $P_{k-2}(l_i)$ is a polynomial space on l_i of degree no more than $k - 2$.

We have the following integral expansions (see [17]):

Lemma 1. *For $v \in S_h^1$ we have integral expansions*

$$\int_e (\omega - I_h^1 \omega) v \, \mathrm{d}x \, \mathrm{d}y = -\frac{1}{3} \int_e (h_e^2 \omega_{xx} + k_e^2 \omega_{yy}) v \, \mathrm{d}x \, \mathrm{d}y + O(h^3) |\omega|_{3,e} \|v\|_{0,e}, \tag{10}$$

$$\int_e (\omega - I_h^1 \omega)_x v_x \, \mathrm{d}x \, \mathrm{d}y = -\frac{1}{3} k_e^2 \int_e \omega_{xyy} v_x \, \mathrm{d}x \, \mathrm{d}y + \frac{4}{45} k_e^4 \int_e \omega_{xyyy} v_{xy} \, \mathrm{d}x \, \mathrm{d}y + O(h^4) |\omega|_{5,e} |v|_{1,e}. \tag{11}$$

Lemma 2. *For $v \in S_h^2$ we have*

$$(12)\qquad \int_e (\omega - I_h^2\omega) v \,\mathrm{d}x\,\mathrm{d}y = -\frac{h_e^4}{45}\int_e \omega_{xxx} v_x \,\mathrm{d}x\,\mathrm{d}y - \frac{k_e^4}{45}\int_e \omega_{yyy} v_y \,\mathrm{d}x\,\mathrm{d}y + O(h^4)|\omega|_{4,e}\|v\|_{0,e},$$

$$(13)\qquad \int_e (\omega - I_h^2\omega)_x v_x \,\mathrm{d}x\,\mathrm{d}y = -\frac{k_e^4}{45}\int_e \omega_{xyyy} v_{xy} \,\mathrm{d}x\,\mathrm{d}y + \frac{32k_e^6}{3^3 5^2 7}\int_e \omega_{xyyyy} v_{xyy} \,\mathrm{d}x\,\mathrm{d}y + O(h^6)|\omega|_{6,e}\|v\|_{0,e}.$$

Lemma 3. *For $v \in S_h^k$, $k \geqslant 3$, we have*

$$(14)\qquad \int_e (\omega - I_h^k\omega) v \,\mathrm{d}x\,\mathrm{d}y = -C_1\int_e (h_e^{2k}\partial_x^{k+1}\omega\cdot\partial_x^{k-1}v + k_e^{2k}\partial_y^{k+1}\omega\cdot\partial_y^{k-1}v) \,\mathrm{d}x\,\mathrm{d}y + O(h^{2k+2})|\omega|_{k+3,e}|v|_{k,e},$$

$$(15)\qquad \int_e (\omega - I_h^k\omega)_x v_x \,\mathrm{d}x\,\mathrm{d}y = -C_1 k_e^{2k}\int_e \partial_x\partial_y^{k+1}\omega\cdot\partial_x\partial_y^{k-1}v \,\mathrm{d}x\,\mathrm{d}y + O(h^{2k+2})|\omega|_{k+4,e}|v|_{k+1,e},$$

where

$$C_1 = \frac{1}{(2k-1)!!(2k+1)!!}.$$

From Lemmas 1–3, using Green's formulas and the inverse inequalities in the finite element space S_h^k, we can easily obtain

Lemma 4. *For $v \in S_h^1$,*

$$(16)\qquad (\mathbf{curl}(\omega - I_h^1\omega), \mathbf{curl}\, v) = \begin{cases} O(h^2)\|\omega\|_{3,\Omega}\|v\|_{1,\Omega}, \\ O(h^2)\|\omega\|_{4,\Omega}\|v\|_{0,\Omega}, \quad \text{for } \dfrac{\partial\omega}{\partial n} = 0 \text{ on } \Gamma. \end{cases}$$

For $v \in S_h^k$ $(k \geqslant 2)$,

$$(17)\qquad (\mathbf{curl}(\omega - I_h^k\omega), \mathbf{curl}\, v) = \begin{cases} O(h^{k+1})\|\omega\|_{k+2,\Omega}\|v\|_{1,\Omega}, \\ O(h^{k+2})\|\omega\|_{k+3,\Omega}\|v\|_{1,\Omega} \quad \text{for } v \in \mathring{S}_h^k, \\ O(h^{k+2})\|\omega\|_{k+3,\Omega}\|v\|_{1,\Omega} \quad \text{for } \dfrac{\partial\omega}{\partial n} = 0 \text{ on } \Gamma. \end{cases}$$

3. Theoretical analysis of the source problem

In this section we will demonstrate the error estimate for the source problem (8) by the approximation scheme (9).

Theorem 1. *For operators A, A_h, B, B_h defined by* (8) *and* (9)*, respectively, we have*

$$\|(A-A_h)g\|_{0,\Omega} \leqslant Ch^{k+1}, \tag{18}$$

$$\|(A-A_h)g\|_{1,\Omega} \leqslant Ch^{k}, \tag{19}$$

$$\|(B-B_h)g\|_{0,\Omega} \leqslant Ch^{k+1}, \tag{20}$$

$$\|(B-B_h)g\|_{1,\Omega} \leqslant Ch^{k+1}. \tag{21}$$

Proof. From (8) and (9) we get

$$\begin{aligned}\|(A-A_h)g\|_{0,\Omega}^2 = {} & (Ag - I_h^k(Ag), Ag - A_hg) \\ & + (\mathbf{curl}(Bg - I_h^k(Bg)), \mathbf{curl}(A_hg - I_h^k(Ag))) \\ & + (\mathbf{curl}(Ag - I_h^k(Ag)), \mathbf{curl}(I_h^k(Bg) - B_hg)).\end{aligned}$$

It follows from the above equation and Lemma 4, $\frac{\partial(Bg)}{\partial n}|_\Gamma = 0$ and the approximation properties of the finite element space S_h^k that

$$\begin{aligned}\|(A-A_h)g\|_{0,\Omega}^2 \leqslant {} & Ch^{k+1}\|Bg\|_{k+3,\Omega}\|Ag - A_hg\|_{0,\Omega} + Ch^{2k+2}\|Bg\|_{k+3,\Omega}^2 \\ & + Ch^{k+1}\|Bg\|_{k+4,\Omega}\|I_h^k(Bg) - B_hg\|_{0,\Omega}.\end{aligned} \tag{22}$$

Furthermore, we have by using the Poincaré inequality, (8), (9) and Lemma 4 that

$$\begin{aligned}\|I_h^k(Bg) - B_hg\|_{1,\Omega}^2 & \leqslant C(\mathbf{curl}(I_h^k(Bg) - B_hg), \mathbf{curl}(I_h^k(Bg) - B_hg)) \\ & = C[(\mathbf{curl}(I_h^k(Bg) - Bg), \mathbf{curl}(I_h^k(Bg) - B_hg)) \\ & \quad + (Ag - A_hg, I_h^k(Bg) - B_hg)] \\ & \leqslant Ch^{k+1}\|Bg\|_{k+3,\Omega}\|I_h^k(Bg) - B_hg\|_{1,\Omega} \\ & \quad + C\|Ag - A_hg\|_{0,\Omega}\|I_h^k(Bg) - B_hg\|_{0,\Omega},\end{aligned}$$

which implies

$$\|I_h^k(Bg) - B_hg\|_{1,\Omega}^2 \leqslant Ch^{2k+2}\|Bg\|_{k+3,\Omega}^2 + C\|Ag - A_hg\|_{0,\Omega}^2. \tag{23}$$

Using a similar argument, we have

$$\begin{aligned}\|Bg - B_hg\|_{1,\Omega}^2 \leqslant {} & Ch^{k+1}\|Bg\|_{k+1,\Omega}\|Bg - B_hg\|_{1,\Omega} \\ & + C\|Ag - A_hg\|_{0,\Omega}\|I_h^k(Bg) - B_hg\|_{0,\Omega}.\end{aligned}$$

Then

$$\|Bg - B_h g\|_{1,\Omega}^2 \leqslant Ch^{2k+2}\|Bg\|_{k+1,\Omega}^2 + C\|Ag - A_h g\|_{0,\Omega}^2. \tag{24}$$

From (22) and (23) we prove (18). Combing (22) with (24) implies (20) and (21). Finally, using the approximation properties of the interpolation operator I_h^k and (18), we obtain (19). □

4. Theoretical analysis for the eigenvalue problem

Assuming here that all eigenvalues have ascent and their geometric multiplicity is one, in this section we will give theoretical analysis of the eigenvalue problem.

From Theorem 6.1 and Theorem 6.2 in [20], and from the error estimate for the source problem in the preceding section, we have the following error estimate for the discretized eigenvalue problem described by (4) and (6).

Theorem 2. *Assume that* $(\lambda, \psi, \omega) \in \mathbb{R} \times H_0^1(\Omega) \times H^1(\Omega)$ *is the solution of* (4) *and* $(\lambda^h, \psi^h, \omega^h) \in \mathbb{R} \times \overset{\circ}{S}{}_h^k \times S_h^k$ *is the solution of* (6)*. Then we have*

$$|\lambda - \lambda^h| \leqslant Ch^{2k}, \tag{25}$$

$$\|\psi - \psi^h\|_{1,\Omega} \leqslant Ch^{k+1}. \tag{26}$$

From Theorem 2 we can easily prove the following result:

Theorem 3. *Using the notation given by* (4), (5), (6) *and* (7), *we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\omega - \omega^h\|_{0,\Omega} \leqslant Ch^{k+1}, \tag{27}$$

$$\|p - p_h\|_{1,\Omega} \leqslant Ch^k. \tag{28}$$

P r o o f. From (4) and (6) we see

$$\begin{aligned}
\|\omega - \omega^h\|_{0,\Omega}^2 = {} & (\omega - I_h^k\omega, \omega - \omega^h) + (\mathbf{curl}(I_h^k\omega - \omega^h), \mathbf{curl}(\psi - I_h^k\psi)) \\
& + (\mathbf{curl}(I_h^k\omega - \omega), \mathbf{curl}(I_h^k\psi - \psi^h)) \\
& + \lambda^h(\mathbf{curl}(\psi - \psi^h), \mathbf{curl}(I_h^k\psi - \psi^h)) \\
& + (\lambda - \lambda^h)(\mathbf{curl}\,\psi, \mathbf{curl}(I_h^k\psi - \psi^h)).
\end{aligned}$$

Notice that $\mathbf{u}|_{\partial\Omega} = 0$, $\frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0$. It follows by using the integral expansions in Section 2, the approximation property for the interpolation I_h^k, the inverse inequality for the finite element space, Theorem 2 and the above equation that

$$\|\omega - \omega^h\|_{0,\Omega} \leqslant Ch^{k+1}. \tag{29}$$

Combining (29) with (26) leads to (27).

Introduce the operator $R_h\colon\ H^1(\Omega) \to S_h^k$ such that

$$\begin{cases} (\mathbf{grad}(p - R_h p), \mathbf{grad}\, q_h) = 0 \quad \forall\, q_h \in S_h^k, \\ (p - R_h p, 1) = 0. \end{cases} \tag{30}$$

We know from (30) and the approximation property for the interpolation that

$$\begin{aligned} (\mathbf{grad}(p - R_h p), \mathbf{grad}(p - R_h p)) &= (\mathbf{grad}(p - R_h p), \mathbf{grad}(p - I_h^k p)) \\ &\leqslant Ch^k |p - R_h p|_{1,\Omega}, \end{aligned}$$

which implies

$$|p - R_h p|_{1,\Omega} \leqslant Ch^k. \tag{31}$$

From (30) we see that $p \in L_0^2(\Omega)$, which implies $R_h p \in L_0^2(\Omega)$. Notice that $\|v\|_{1,\Omega} \leqslant C\big(|v|_{1,\Omega} + |\int_\Omega v|\big)$ for all $v \in H^1(\Omega)$. Then from (31) we obtain

$$\|p - R_h p\|_{1,\Omega} \leqslant Ch^k. \tag{32}$$

From (5), (7) and (30) we see that for $q_h \in S_h^k$

$$\begin{aligned} (\mathbf{grad}\,(R_h p - p_h), \mathbf{grad}\, q_h) &= (\mathbf{grad}(p - p_h), \mathbf{grad}\, q_h) \\ &= (\lambda - \lambda^h)(\mathbf{u} - \mathbf{curl}\,\omega, \mathbf{grad}\, q_h) + \lambda^h(\mathbf{u} - \mathbf{u}_h, \mathbf{grad}\, q_h) \\ &\quad + \lambda^h(\mathbf{curl}\,\omega^h - \mathbf{curl}\,\omega, \mathbf{grad}\, q_h). \end{aligned} \tag{33}$$

From (29) we can get $|\omega - \omega^h|_{1,\Omega} \leqslant Ch^k$. Using (25) and (27) and taking $q_h = R_h p - p_h$ in (33), we have

$$|R_h p - p_h|_{1,\Omega} \leqslant Ch^k.$$

Notice that, since $R_h p - p_h \in L_0^2(\Omega) \cap H^1(\Omega)$, an argument similar to that used to get (32) yields

$$\|R_h p - p_h\|_{1,\Omega} \leqslant Ch^k,$$

from which and (32) we prove (28). $\square$

For $(\psi,\omega)\in H_0^1(\Omega)\times H^1(\Omega)$ we define the projection $(M_h\psi,N_h\omega)\in \overset{\circ}{S}{}_h^k\times S_h^k$ by

$$
(34)\qquad \begin{cases} a(N_h\omega,\theta_h)+b(\theta_h,M_h\psi)=a(\omega,\theta_h)+b(\theta_h,\psi) & \forall\,\theta_h\in S_h^k, \\ b(N_h\omega,\varphi_h)=b(\omega,\varphi_h) & \forall\,\varphi_h\in \overset{\circ}{S}{}_h^k\,. \end{cases}
$$

Using a similar argument to that in the proof of Theorem 1, we derive the following theorem.

Theorem 4. *For the mixed projection defined by* (34) *we have*

$$
(35)\qquad \|\omega-N_h\omega\|_{0,\Omega}\leqslant Ch^{k+1},
$$

$$
(36)\qquad \|\omega-N_h\omega\|_{1,\Omega}\leqslant Ch^{k},
$$

$$
(37)\qquad \|\psi-M_h\psi\|_{0,\Omega}\leqslant Ch^{k+1},
$$

$$
(38)\qquad \|\psi-M_h\psi\|_{1,\Omega}\leqslant Ch^{k+1}.
$$

Set $D((\varphi_1,v_1),(\varphi_2,v_2))=a(v_1,v_2)+b(v_1,\varphi_2)+b(v_2,\varphi_1)$. Note that $s(\psi,\psi)=1$ and $s(\psi^h,\psi^h)=1$. From (4) and (6) we have

$$
(39)\qquad D((\psi,\omega),(\psi,\omega))=\,-\,\lambda,
$$

$$
(40)\qquad D((\psi^h,\omega^h),(\psi^h,\omega^h))=\,-\,\lambda^h,
$$

$$
(41)\qquad D((\psi,\omega),(\psi^h,\omega^h))=\,-\,\lambda s(\psi,\psi^h),
$$

and

$$
(42)\qquad \lambda|\psi-\psi^h|_{1,\Omega}^2=2\lambda-2\lambda s(\psi,\psi^h).
$$

Using (39)–(42), we have

$$
(43)\qquad \begin{aligned} D((\psi-\psi^h,u-u^h),(\psi-\psi^h,u-u^h)) &=\,-\,\lambda-\lambda^h+2\lambda s(\psi,\psi^h)\\ &=\lambda-\lambda^h-\lambda|\psi-\psi^h|_1^2. \end{aligned}
$$

From (4), (6), and (34), we get

$$
(44)\qquad \begin{aligned} &D((\psi-\psi^h,\omega-\omega^h),(\psi-\psi^h,\omega-\omega^h))\\ &\quad=\,-\,\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}\,\psi)\\ &\qquad-D((M_h\psi,N_h\omega),(\psi-I_h^k\psi,\omega-I_h^k\omega))\\ &\qquad+D((M_h\psi-\psi^h,N_h\omega-\omega^h),(M_h\psi-\psi^h,N_h\omega-\omega^h))\\ &\quad=\,-\,\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}\,\psi)-(\omega-I_h^k\omega,N_h\omega)\\ &\qquad+(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}(N_h\omega))+(\mathbf{curl}(\omega-I_h^k\omega),\mathbf{curl}(M_h\psi))\\ &\qquad+D((M_h\psi-\psi^h,N_h\omega-\omega^h),(M_h\psi-\psi^h,N_h\omega-\omega^h)). \end{aligned}
$$

From (43) and (44), we obtain the expansion for the eigenvalue error:

Lemma 5. *Assume that* $(\lambda,\psi,\omega)\in\mathbb{R}\times H_0^1(\Omega)\times H^1(\Omega)$ *is a solution of* (4) *and* $(\lambda^h,\psi^h,\omega^h)\in\mathbb{R}\times \overset{\circ}{S}{}_h^k\times S_h^k$ *is a solution of* (6). *Then we have the following expansion for the eigenvalue error:*

$$\begin{aligned}(45)\quad \lambda-\lambda^h=&\ \lambda|\psi-\psi^h|_1^2-\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}\,\psi)-(\omega-I_h^k\omega,N_h\omega)\\&+(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}(N_h\omega))+(\mathbf{curl}(\omega-I_h^k\omega),\mathbf{curl}(M_h\psi))\\&+D((M_h\psi-\psi^h,N_h\omega-\omega^h),(M_h\psi-\psi^h,N_h\omega-\omega^h)).\end{aligned}$$

From now on, for $k=1$, i.e., when we use the bilinear finite element space, we will apply integral expansion techniques described in Section 2 and the error estimate for the mixed projection demonstrated in Theorem 4 to estimate every term on the right-hand side of the expansion (45). Finally, we can get the asymptotic expansion for the eigenvalue error.

Theorem 2 implies

$$(46)\qquad |\psi-\psi^h|_{1,\Omega}^2\leqslant Ch^4.$$

Using the error estimate for the eigenfunction (26), the integral expansion in Lemma 1, and the approximation properties for the interpolation, we estimate the second term of (45) as follows:

$$\begin{aligned}(47)\quad -\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}\,\psi)=&-\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}(\psi-\psi^h))\\&-\lambda(\mathbf{curl}(\psi-I_h^k\psi),\mathbf{curl}\,\psi^h)\\=&\ \frac{\lambda}{3}\sum_e\int_e(k_e^2\psi_{xyy}\psi_x+h_e^2\psi_{yxx}\psi_y)\,\mathrm{d}x\,\mathrm{d}y+O(h^3).\end{aligned}$$

By an argument similar to that used to get (47), we can expand the third to fifth terms of (45). Using the error estimate for the mixed projections described in Theorem 4 we get

$$(48)\qquad -(\omega-I_h^k\omega,N_hu)=\frac{1}{3}\sum_e\int_e(h_e^2\omega_{xx}+k_e^2\omega_{yy})\omega\,\mathrm{d}x\,\mathrm{d}y+O(h^3)|u|_{3,\Omega}.$$

For the fourth term, we have

$$\begin{aligned}(49)\qquad (\mathbf{curl}(\psi-I_h^1\psi),\mathbf{curl}\,N_h\omega)=&-\frac{1}{3}\sum_e k_e^2\int_e\psi_{xyy}\omega_x\,\mathrm{d}x\,\mathrm{d}y\\&-\frac{1}{3}\sum_e h_e^2\int_e\psi_{yxx}\omega_y\,\mathrm{d}x\,\mathrm{d}y+O(h^3),\end{aligned}$$

and we rewrite the fifth term as

$$(\mathbf{curl}(\omega - I_h^1\omega), \mathbf{curl}\, M_h\psi) = -\frac{1}{3}\sum_e k_e^2 \int_e \omega_{xyy}\psi_x \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{3}\sum_e h_e^2 \int_e \omega_{yxx}\psi_y \,\mathrm{d}x\,\mathrm{d}y + O(h^3). \tag{50}$$

From (4), (6), and (34) we have: $\forall\,(\varphi, v) \in \overset{\circ}{S}{}_h^k \times S_h^k$

$$\begin{aligned}|D((M_h\psi - \psi^h, N_h\omega - \omega^h), (\varphi, v))| &= |\lambda s(\psi, \varphi) - \lambda^h s(\psi^h, \varphi)| \\ &= |(\lambda - \lambda^h)s(\psi, \varphi) + \lambda^h s(\psi - \psi^h, \varphi)| \\ &= O(1)(|\lambda - \lambda^h| + |\psi - \psi^h|_{1,\Omega}|\varphi|_{1,\Omega},\end{aligned}$$

from which and Theorems 2 and 4 we obtain the error estimate for the sixth term as follows:

$$\begin{aligned}&|D((M_h\psi - \psi^h, N_h\omega - \omega^h), (M_h\psi - \psi^h, N_h\omega - \omega^h))| \\ &\quad = O(1)(|\lambda - \lambda^h| + |\psi - \psi^h|_{1,\Omega})(|M_h\psi - \psi|_{1,\Omega} + |\psi - \psi^h|_{1,\Omega}) \\ &\quad = O(h^4).\end{aligned} \tag{51}$$

Combining the error estimates (46)–(51) with the expansion (45), we derive the following asymptotic expansion for the eigenvalue error:

Theorem 5. *When we use the bilinear finite element space, we have*

$$\begin{aligned}\lambda - \lambda^h = &\frac{\lambda}{3}\sum_e \int_e (k_e^2\psi_{xyy}\psi_x + h_e^2\psi_{yxx}\psi_y)\,\mathrm{d}x\,\mathrm{d}y \\ &+ \frac{1}{3}\sum_e \int_e (h_e^2\omega_{xx} + k_e^2\omega_{yy})\omega\,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{1}{3}\sum_e \int_e (k_e^2\psi_{xyy}\omega_x + h_e^2\psi_{yxx}\omega_y)\,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{1}{3}\sum_e \int_e (k_e^2\omega_{xyy}\psi_x + h_e^2\omega_{yxx}\psi_y)\,\mathrm{d}x\,\mathrm{d}y + O(h^3).\end{aligned} \tag{52}$$

For the uniform mesh ($h_e \equiv h_1,\ k_e \equiv h_2$), noting that $\omega = -\Delta\psi$, $\psi|_{\partial\Omega} = \frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0$ and using Green's formula, we can prove

Theorem 6. *For the uniform rectangular mesh, when we use the bilinear finite element space, the asymptotic expansion for the eigenvalue error has the form*

$$(53)\quad \begin{aligned}\lambda-\lambda^h = & -\frac{\lambda}{3}(h_1^2+h_2^2)\|\psi_{xy}\|_{0,\Omega}^2+\frac{1}{3}\left(h_1^2\int_\Omega \omega_{xx}\omega\,\mathrm{d}x\,\mathrm{d}y+h_2^2\int_\Omega \omega_{yy}\omega\,\mathrm{d}x\,\mathrm{d}y\right)\\ &+\frac{2}{3}(h_1^2+h_2^2)(\|\psi_{xxy}\|_{0,\Omega}^2+\|\psi_{xyy}\|_{0,\Omega}^2)+O(h^3).\end{aligned}$$

For the square mesh ($h_e\equiv k_e\equiv h$), noting that $\omega=-\Delta\psi$, $\Delta\omega=\lambda\Delta\psi$, $\psi|_{\partial\Omega}=\frac{\partial\psi}{\partial n}|_{\partial\Omega}=0$, and using Green's formula, from Theorem 5 we can prove

Theorem 7. *If T_h is a square mesh and if we use the bilinear finite element space, then*

$$(54)\quad \begin{aligned}\lambda-\lambda^h = & -\frac{\lambda h^2}{3}\int_\Omega(\psi_{xx}^2+\psi_{yy}^2+4\psi_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\\ &+\frac{4h^2}{3}\int_\Omega(\psi_{xxy}^2+\psi_{xyy}^2)\,\mathrm{d}x\,\mathrm{d}y+O(h^3)\|\psi\|_{6,\Omega}.\end{aligned}$$

5. Extrapolation and an a posteriori error estimate for eigenvalues

In this section, let T_h be a square partition on Ω with mesh size h. We assume that $T_{h/2}$ has been obtained from T_h by dividing each element into four squares. Let $(\lambda^{h/2},\psi^{h/2},\omega^{h/2})\in\mathbb{R}\times V_0^h\times V^h$ be an eigensolution approximation on the mesh $T_{h/2}$ by (6).

Denote by

$$(55)\quad \tilde\lambda^h=\frac{4\lambda^{h/2}-\lambda^h}{3}$$

the extrapolation of λ. By Theorem 7, we can get the following error estimate for the extrapolation $\tilde\lambda^h$ and an a posteriori error estimate for the eigenvalue:

Theorem 8. *We have*

$$(56)\quad \lambda-\tilde\lambda^h=O(h^3)\|\psi\|_{6,\Omega}$$

and thus,

$$(57)\quad \lambda-\lambda^{h/2}=\frac{\lambda^h-\lambda^{h/2}}{3}+O(h^3)\|\psi\|_{6,\Omega}$$

provides an a posteriori error estimate $\frac{1}{3}(\lambda^h - \lambda^{h/2})$ *for* $\lambda - \lambda^{h/2}$.

6. Numerical results

In this section we compute the first eigenvalue for the algebraic eigenvalue problem

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} X = \lambda \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} X, \tag{58}$$

which arises from the discrete scheme (6). Using the inverse power method and a nonlinear inexact Uzawa algorithm (see [13]), we compute the first eigenvalue as follows:

mesh	λ^h	$\lvert\lambda^h - \lambda\rvert/\lambda$	$\tilde{\lambda}^h$	$\lvert\tilde{\lambda}^h - \lambda\rvert/\lambda$
4×4	59.138354834623172	0.1298	52.485719795258298	$0.269e-02$
8×8	54.148878555099522	$0.345e-01$	52.336338480432033	$0.1596e-03$
16×16	52.789473499098904	$0.8497e-02$	52.343869992942331	$0.1569e-04$
32×32	52.455270869481474	$0.2113e-02$	52.344691794291471	$0.1196e-07$
64×64	52.372336563088972	$0.5281e-03$		

Table 1. Computation of the first eigenvalue of the Stokess problem.

Tab. 1 shows the numerical results obtained by using the stream function-vorticity-pressure method to approximate the eigenvalue problem associated with the Stokes problem discussed in Section 1 by bilinear elements on square meshes.

According to [3], [7] and [28], the most accurate approximation for the first eigenvalue given by Wieners [28] is 52.3446911. In Tab. 1 we take $\lambda = 52.3446911$.

Consequently, the theoretical results obtained in Sections 4 and 5 are well realized in practice. The extrapolation of the eigenvalue gives a more efficient approximation.

7. Conclusions

We have derived an optimal error estimate for the Stream Function-Vorticity-Pressure approximation of the eigenvalue problem associated with the Stokes problem. Further we have obtained an asymptotic expansion, an efficient extrapolation and an a posteriori error estimate for the eigenvalue. The main tools we have used here are the technique of eigenvalue error expansion (see [18]) and the technique of integral expansion which is a useful tool used to investigate superconvergence phenomena (see [16], [17] and references cited therein). Finally, the efficiency of the results has been illustrated by numerical experiments.

There are some possible future studies about the eigenvalue problem associated with the Stokes problem:

1. In this paper, we have only considered a unit square region and assumed the eigenfunction is smooth enough. The analysis on a region with smooth boundary will be the subject of a forthcoming paper.

2. Here, we solve the eigenvalue problem associated with the Stokes equation by the stream function and the vorticity. In the future, we will examine a finite element approximation for the primitive variables provided the so-called BB compatibility condition is true.

3. We have assumed here that all eigenvalues have ascent and their geometric multiplicity is one. The theoretical analysis for the ascent larger than one will be our future work.

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