Ivana Pultarová

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Applications of Mathematics, Vol. 50 (2005), No. 3, 323–329

Persistent URL: `http://dml.cz/dmlcz/134609`

THE STRENGTHENED C.B.S. INEQUALITY CONSTANT FOR SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL OPERATOR AND FOR HIERARCHICAL BILINEAR FINITE ELEMENT FUNCTIONS*

Ivana Pultarová, Praha

Abstract. We estimate the constant in the strengthened Cauchy-Bunyakowski-Schwarz inequality for hierarchical bilinear finite element spaces and elliptic partial differential equations with coefficients corresponding to anisotropy (orthotropy). It is shown that there is a nontrivial universal estimate, which does not depend on anisotropy. Moreover, this estimate is sharp and the same as for hierarchical linear finite element spaces.

Keywords: Cauchy-Bunyakowski-Schwarz inequality, multilevel preconditioning, elliptic partial differential equation

MSC 2000: 65N22, 65N12, 74S05

1. Introduction

We consider an elliptic partial differential equation

$$\int_\Omega \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial y^2} \,\mathrm{d}x\,\mathrm{d}y = f$$

on a domain Ω, and its discretized form. Let V be a finite element space and a, $a\colon V \times V \to \mathbb{R}$, the corresponding symmetric positive definite bilinear form. Then we consider the subspaces $U, W \subset V$, $V = U \oplus W$ and aim at finding the value γ, $0 \leqslant \gamma < 1$, which enters the strengthened Cauchy-Bunyakowski-Schwarz (C.B.S.) inequality

$$|a(u,w)| \leqslant \gamma \sqrt{a(u,u)}\sqrt{a(w,w)},$$

* This research was supported by the Grant Agency of Czech Republic under the contract No. 201/02/0595.

for all $u \in U$ and $w \in W$. When dealing with a hierarchy of finite element spaces, usually U is associated with a coarser grid, while W corresponds to a natural coarse grid complement.

As shown e.g. in [1], [2], [3], γ determines the convergence rate of two-level and multi-level algorithms for solving discretized elliptic problems. For two-level multiplicative methods, the condition number κ of the resulting preconditioned system can be estimated by

$$\kappa \approx \frac{1}{1-\gamma^2}.$$

The extensive survey of these methods and their convergence properties can be found in [1] and in the references therein.

The value γ depends of course on the bilinear form a and on the type of finite elements used. Usually, the constant γ increases in case of physical and numerical anisotropy. Thus there is a question, if γ can be bounded by a nontrivial (less than unity) universal estimate. The universal estimates are known e.g. for heat conduction and elasticity operators and linear elements, [1], [3]. Throughout this paper, a hierarchical decomposition of bilinear finite element spaces is considered.

2. The strengthened C.B.S. constant for hierarchical decomposition of bilinear finite element spaces

We are interested in the value of γ for an anisotropic orthotropic elliptic partial differential operator in 2D and its discretization based on the application of hierarchical bilinear finite element spaces. Let the corresponding symmetric positive definite bilinear form be

$$a(u,v) = \int_\Omega \alpha \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \beta \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,\mathrm{d}x\,\mathrm{d}y, \tag{1}$$

where α and β are positive piecewise constant functions.

For the discretization, we use the rectangular bilinear finite elements. We deal with two successive divisions of the domain: the finer set of nodal points differs from the coarser one by adding centres of edges and centres of elements. It defines the spaces U and V. The complement space W is generated by the basis functions corresponding to the added nodes. We assume that the problem coefficients α and β are constant on the elements of the coarses division (macroelements).

The upper bound for the constant γ equals to the maximum of γ_E, where γ_E are C.B.S. constants for macroelements E belonging to the coarse division of the domain Ω in which the equation is considered. This allows us to deal with only one reference rectangle divided into four subelements.

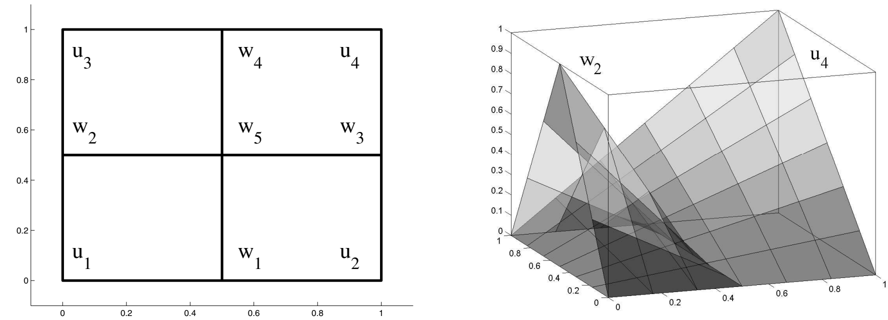

Figure 1. Numbering of nine finite element functions (left). Plot of $\mathbf{u}_4$ and $\mathbf{w}_2$ (right).

The macroelement is a part of the supports of four basis functions of U, which belong to the coarser grid. Let us denote them by $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$. Five basis functions from the set W, connected with the finer grid, which are nonzero on this element, are denoted by $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ and $\mathbf{w}_5$. For the notation see Fig. 1. Let all mentioned basis functions have the maximum nodal value equal to one.

Theorem 1. *The universal estimate for the strengthened C.B.S. constant for the operator* (1) *with piecewise constant coefficients and for the hierarchical decomposition of bilinear finite element spaces is equal to* $\sqrt{\frac{3}{4}}$. *Moreover, this estimate is sharp.*

Proof. Let us consider the division of the reference square $[0,h]\times[0,h]$, $h>0$, according to the previous notes. Let u, $u\in U$, be a linear combination of the basis functions belonging to the coarser grid, $u = u_1\mathbf{u}_1 + u_2\mathbf{u}_2 + u_3\mathbf{u}_3 + u_4\mathbf{u}_4$ and let w, $w\in W$, be a linear combination of the basis functions belonging to the finer grid, $w = w_1\mathbf{w}_1 + w_2\mathbf{w}_2 + w_3\mathbf{w}_3 + w_4\mathbf{w}_4 + w_5\mathbf{w}_5$, see Fig. 1. These nine coefficients u_i, $i = 1,2,3,4$, and w_i, $i = 1,2,3,4,5$, enter the expressions $a(u,u)$, $a(w,w)$ and $a(u,w)$:

$$\begin{aligned} a(u,u) &= \frac{\alpha}{3}(u_1^2+u_2^2+u_3^2+u_4^2-2u_1u_2-2u_3u_4+u_1u_3+u_2u_4-u_1u_4-u_2u_3)\\ &\quad+\frac{\beta}{3}(u_1^2+u_2^2+u_3^2+u_4^2-2u_1u_3-2u_2u_4+u_1u_2+u_3u_4-u_1u_4-u_2u_3),\end{aligned}$$

$$\begin{aligned} a(w,w) &= \frac{\alpha}{3}\big(2(w_1^2+w_2^2+w_3^2+w_4^2)+4w_5^2-w_1w_2-w_1w_3-w_2w_4-w_3w_4\\ &\qquad\qquad+2w_1w_5+2w_4w_5-4w_2w_5-4w_3w_5\big)\\ &\quad+\frac{\beta}{3}\big(2(w_1^2+w_2^2+w_3^2+w_4^2)+4w_5^2-w_1w_2-w_1w_3-w_2w_4-w_3w_4\\ &\qquad\qquad-4w_1w_5-4w_4w_5+2w_2w_5+2w_3w_5\big),\end{aligned}$$

$$a(u,w)=\frac{\alpha}{4}(u_1w_2-u_1w_3-u_2w_2+u_2w_3+u_3w_2-u_3w_3-u_4w_2+u_4w_3)$$
$$+\frac{\beta}{4}(u_1w_1-u_1w_4+u_2w_1-u_2w_4-u_3w_1+u_3w_4-u_4w_1+u_4w_4).$$

Let us note that the above equalities match without any dependence on h. After appropriate rearrangements, we have

$$\tag{2} a(u,u)=\frac{\alpha}{6}\big((u_1-u_2+u_3-u_4)^2+(u_1-u_2)^2+(u_3-u_4)^2\big)+\frac{\beta}{6}\big((u_1+u_2-u_3-u_4)^2+(u_1-u_3)^2+(u_2-u_4)^2\big),$$

$$a(w,w)=\frac{\alpha}{3}\Big(\frac{1}{2}(2w_5+w_1-w_2-w_3)^2+\frac{1}{2}(2w_5+w_4-w_2-w_3)^2+\frac{3}{2}w_1^2+\frac{3}{2}w_4^2+(w_2-w_3)^2\Big)+\frac{\beta}{3}\Big(\frac{1}{2}(2w_5+w_2-w_1-w_4)^2+\frac{1}{2}(2w_5+w_3-w_1-w_4)^2+\frac{3}{2}w_2^2+\frac{3}{2}w_3^2+(w_1-w_4)^2\Big),$$

$$a(u,w)=\frac{\alpha}{4}(u_1-u_2+u_3-u_4)(w_2-w_3)+\frac{\beta}{4}(u_1+u_2-u_3-u_4)(w_1-w_4).$$

Applying standard inequalities, the following estimates hold

$$\tag{3} a(u,u)\geqslant\frac{\alpha}{4}(u_1-u_2+u_3-u_4)^2+\frac{\beta}{4}(u_1+u_2-u_3-u_4)^2,$$

$$\tag{4} a(w,w)\geqslant\frac{\alpha}{3}\big((w_1-w_4)^2+(w_2-w_3)^2\big)+\frac{\beta}{3}\big((w_1-w_4)^2+(w_2-w_3)^2\big).$$

Using the Hölder inequality for (2), we have

$$\begin{aligned}|a(u,w)|^2&\leqslant\Big(\frac{\alpha}{4}(u_1-u_2+u_3-u_4)^2+\frac{\beta}{4}(u_1+u_2-u_3-u_4)^2\Big)\\&\quad\times\Big(\frac{\alpha}{4}(w_2-w_3)^2+\frac{\beta}{4}(w_1-w_4)^2\Big)\\&\leqslant\frac{3}{4}\Big(\frac{\alpha}{4}(u_1-u_2+u_3-u_4)^2+\frac{\beta}{4}(u_1+u_2-u_3-u_4)^2\Big)\\&\quad\times\frac{\alpha+\beta}{3}\big((w_2-w_3)^2+(w_1-w_4)^2\big)\\&\leqslant\frac{3}{4}a(u,u)a(w,w).\end{aligned}$$

To display the sharpness of the estimate, let us set for example $(u_1,u_2,u_3,u_4)=(1,1,0,0)$ and $(w_1,w_2,w_3,w_4,w_5)=(1,0,0,-1,0)$. Considering $\beta=1$ and $\alpha\to0^+$, the following limit reads

$$\frac{|a(u,w)|^2}{a(u,u)a(w,w)} \to \frac{3}{4}^- .$$

The proof is complete. $\square$

Theorem 2. *The universal estimate of the strengthened C.B.S. constant for the Laplace differential operator and for the hierarchical decomposition of bilinear finite element spaces is equal to* $\sqrt{\frac{3}{8}}$. *Again, this estimate is sharp.*

Proof. Considering $\alpha = \beta = 1$ and using the estimates (3) and (4) from the proof of Theorem 1 and the Hölder inequality for (2), $|a(u,w)|^2$ can be estimated by

$$\begin{aligned} |a(u,w)|^2 &\leqslant \frac{1}{16}\big((u_1 - u_2 + u_3 - u_4)^2 + (u_1 + u_2 - u_3 - u_4)\big)^2 \\ &\quad \times \big((w_2 - w_3)^2 + (w_1 - w_4)^2\big) \\ &\leqslant \frac{3}{8} a(u,u)a(w,w). \end{aligned}$$

Vectors $(u_1, u_2, u_3, u_4) = (1, 1, 0, 0)$ and $(w_1, w_2, w_3, w_4, w_5) = (1, 0, 0, -1, 0)$ yield the equality

$$|a(u,w)|^2 = \frac{3}{8} a(u,u)a(w,w)$$

which means that the estimate for γ cannot be improved. $\square$

We have also performed computer experiments for the symmetric positive definite elliptic equation with the mixed derivation term involved. As a result, the strengthened C.B.S. constant for this case seems to be bounded again by $\sqrt{\frac{3}{4}}$, i.e. it should not differ from the constant for the diagonal differential operator.

3. Comparison to linear elements

The universal estimates of the strengthened C.B.S. constants γ for linear finite elements are presented e.g. in [2], [5]. For two levels of hierarchy, the constant γ is $\sqrt{\frac{1}{2}}$ in the case of the isotropic Laplacian and rectangular finite elements, $\sqrt{\frac{3}{8}}$ for the isotropic Laplacian and equilateral triangles and $\sqrt{\frac{3}{4}}$ in the case of arbitrary anisotropy and shape of elements. Thus the estimates for γ for the hierarchical linear and bilinear finite element spaces are fully comparable.

4. Modifying the size of the subelements

In this part, we consider nonequilateral rectangular subelements, i.e. the finer grid nodes are placed on the edges of macroelements and inside them subject only to the condition that the subelement's edges are parallel to the macroelements edges. Let h_1, h_2, h_3 and h_4 be the sizes of the edges of four subelements, see Fig. 2, such that $h_1 + h_2 = 1$ and $h_3 + h_4 = 1$. According to our expectation, when the size of some of subelements degenerates, then the C.B.S. constant tends to 1 for appropriate values α and β. For example, when we choose $(u_1, u_2, u_3, u_4) = (1, 0, 0, 0)$ and $(w_1, w_2, w_3, w_4, w_5) = (1, 0, 0, -1, 0)$, then for $\beta = 1$, $h_3 = 0.5$, $h_1 \to 0^+$ and $\alpha = h_1^2 \to 0^+$,

$$\frac{|a(u,w)|^2}{a(u,u)a(w,w)} = \frac{(2-h_1)^2(1-h_1)}{(1+h_1)(4-3h_1)},$$

and thus

$$\frac{|a(u,w)|^2}{a(u,u)a(w,w)} \to 1^-.$$

More details on the nontrivial universal estimates of the strengthened C.B.S. constants for the isotropic and anisotropic forms of the elliptic partial differential operator (1) and the hierarchical bilinear finite elements with nonequilateral subelements will appear in a subsequent paper.

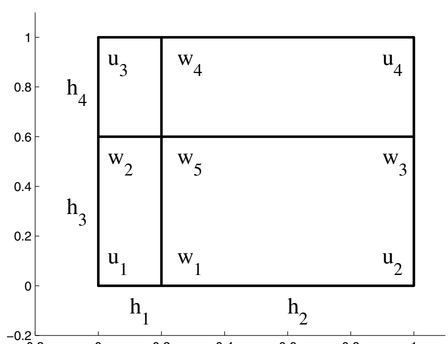

Figure 2. Modifying the size of the subelements.

5. Summary and discussion

We have investigated the value of the constant γ in the strengthened C.B.S. inequality, when considering an elliptic partial differential operator and two level bilinear finite element functions with rectangular supports. Bilinear finite elements are quite popular in the finite element method community but the hierarchical decomposition seems to be considered only in [4] for the bilinear-biquadratic case. We have shown that the universal estimate of the strengthened C.B.S. constant γ does not differ from the estimate of the strengthened C.B.S. constant in case of linear finite elements. This qualifies the problems discretized by bilinear finite elements to be preconditioned successfully with multilevel methods.

Still, some estimates for bilinear finite elements have not been found. The upper bound of γ in case of a differential operator with the mixed derivatives term has not been determined analytically yet, thought computer experiments show that the value should not differ from the value for the diagonal differential operator considered in this paper. Further, we consider only the most important case of the refinement of multplicity $m = 2$ (dividing macroelement into m^2 elements). But it could also be interesting to find the upper bound for γ for the decomposition with higher refinement multiplicity m and compare it with the estimate for the linear finite elements. In the latter case, the estimate $\gamma = \sqrt{\frac{m^2-1}{m^2}}$ can be found in [2].

Acknowledgement. The author thanks Professor R. Blaheta for helpful discussions.

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Author's address: *I. Pultarová*, Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, CZ-166 29 Praha 6, Czech Republic, e-mail: `ivana@mat.fsv.cvut.cz`.