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FINITE ELEMENT ANALYSIS OF FREE MATERIAL OPTIMIZATION PROBLEM

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Abstract. Free material optimization solves an important problem of structural engineering, i.e. to find the stiffest structure for given loads and boundary conditions. Its mathematical formulation leads to a saddle-point problem. It can be solved numerically by the finite element method. The convergence of the finite element method can be proved if the spaces involved satisfy suitable approximation assumptions. An example of a finite-element discretization is included.

Keywords: structural optimization, material optimization, topology optimization, finite elements

MSC 2000: 65N30, 74G15, 74P05

0. INTRODUCTION

The free material optimization solves one of the basic problems of structural engineering, viz. to find the stiffest structure for given set of loads and boundary conditions. Traditional methods of solution of this problem include variations of size and shape variables (cf. [14]). With the invention of composites and other advanced man-made materials it was natural to extend the variation to material properties. The basic problem setting was originated by the works of Bensøe et al. [6] and Ringertz [19], where it was suggested to represent material properties as elements of the unrestricted set of positive semi-definite constitutive tensors. The problem was also studied in [1], [4], [5], [7]. More details on engineering background can be found in [3], [6].

For simplicity of explanation the investigated structures are considered two-dimensional. Three dimensional structures could be approached in a similar way. The material properties of the structure are represented by a positive semi-definite constitutive tensor function. It means that the material is supposed to be nonhomogeneous and anisotropic. For example, composite materials can have these properties. The deformation of the body is described by the small strain tensor. Free material optimization means to optimize the constitutive tensor so that the optimal structure stands the static force load in the "easiest" way. This leads to a saddle-point problem. In two dimensions the constitutive tensor has six independent components. Yet the problem can be reformulated so that only the trace of the constitutive tensor remains an independent variable. Then the mathematical formulation of the optimization problem becomes similar to optimization of a variable thickness of a plate in two dimensions. An interesting point is that the optimal constitutive tensor can be reconstructed from its trace and from the deformation of the studied structure. Numerical examples, which can be found e.g. in [20], show that the norm of the constitutive tensor can be zero in some regions of the studied domain. This situation is interpreted as void material.

Numerical solution of the saddle-point problem can be obtained by the finite element method (cf. [10]). The implementation of the finite element discretization for the free material optimization problem can be found e.g. in [20]. Similar situation comes out of the optimization of a variable thickness of a two-dimensional plate (cf. [18]), where the convergence of the finite element approximation of the variable thickness optimization problem can be proved if suitable approximation properties of the spaces involved are assumed.

This article contributes to the finite element analysis of the free material optimization problem. This analysis is based on suitable approximation properties of the spaces involved, too.

1. Mathematical formulation

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain (an elastic body) with a Lipschitz boundary Γ, which is divided into disjoint parts Γ_0, Γ_f, Γ_c and Γ_R such that $\Gamma = \Gamma_0 \cup \Gamma_f \cup \Gamma_c \cup \Gamma_R$, the Hausdorff measure $\mathcal{H}_1(\Gamma_R) = 0$, Γ_0 is nonempty, and Γ_0, Γ_f, Γ_c are open in Γ. Γ_0 represents the fixed boundary, Γ_f is freely deformable, Γ_c denotes a region with a possible contact with an obstacle (cf. Fig. 1).

Deformation of the structure is described by a displacement vector $u \in V$, where

$$V := \{v \in [H^1(\Omega)]^2 \colon \, v|_\Gamma = 0 \text{ on } \Gamma_0\},$$

where $H^1(\Omega)$ is the Sobolev space.

Let $R \subset \mathbb{R}^2$ be a rigid foundation, which unilaterally supports the structure Ω. Frictionless contact between Ω and R can occur along Γ_c. The contact is handled in a

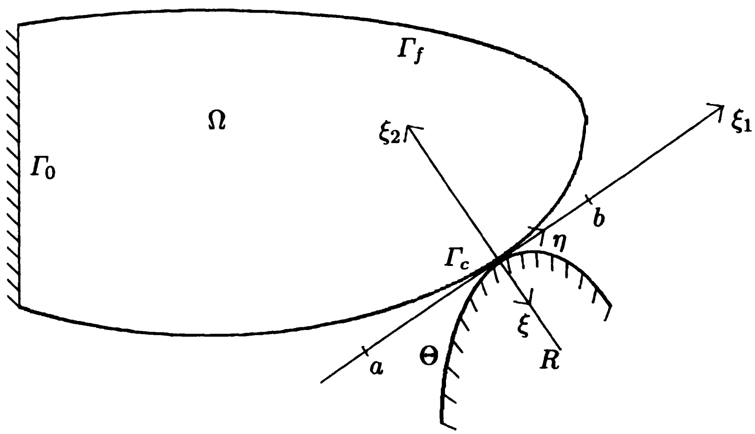

Figure 1. The loaded structure with unilateral contact.

local orthogonal coordinate system (ξ_1, ξ_2) with the origin at a fixed point of contact such that the axis ξ_1 is tangent both to the domain Ω and to the rigid foundation R. The contact boundaries are represented by continuous mappings $\psi, \varphi \in C([a,b])$ such that

$$\Gamma_c = \{(\xi_1, \xi_2) \colon \ \xi_2 = \psi(\xi_1), \ \ \xi_1 \in (a,b)\},$$

and the boundary Θ of the obstacle R is defined (cf. Fig. 1) as

$$\Theta = \{(\xi_1, \xi_2) \colon \ \xi_2 = \varphi(\xi_1), \ \ \xi_1 \in (a,b)\}.$$

The body Ω does not penetrate the foundation R. Let η, ξ be fixed unit vectors such that their coordinates in the local coordinate system (ξ_1, ξ_2) are $\eta = (1,0)$, $\xi = (0,-1)$. "Not penetrating the foundation R by the structure Ω" means that the displacement u will satisfy the inequality

$$\langle u([t, \psi(t)]), \xi \rangle \leqslant \psi(t) - \varphi(t) \quad \text{a.e. in } (a,b),$$

where $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^2$. Admissible displacement vectors are elements of the set

$$K := \{u \in V \colon \ \langle u([t, \psi(t)]), \xi \rangle \leqslant \psi(t) - \varphi(t) \quad \text{a.e. in } (a,b)\}.$$

K is closed and convex.

Assumption. K is not empty.

The small strain tensor is defined as

$$\hat{e}^u_{ij}(x) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x)\right), \quad i,j = 1,2, \;\; x \in \Omega, \;\; u \in K.$$

Let $\hat{\sigma}$ be the stress tensor with components $\hat{\sigma}_{ij} \in L^2(\Omega)$, $i,j = 1,2$.

Assumption. The system satisfies the linear Hooke's law with the elasticity 4-tensor (a tensor of the fourth order) $\hat{E}$, whose components $\hat{E}_{ijkl} \in L^\infty(\Omega)$, $i,j,k,l = 1,2$.

Symmetry of the stress tensor implies $\hat{E}_{ijkl} = \hat{E}_{jikl}$, $i,j,k,l = 1,2$. Without loss of generality it can be assumed that $\hat{E}_{ijkl} = \hat{E}_{klij}$, $i,j,k,l = 1,2$ (cf. [8] for rigorous physical arguments). The elasticity tensor $\hat{E}$ is assumed to be positive semi-definite. Thanks to the symmetry it is possible to rewrite Hooke's law representing the small strain tensor and the stress tensor by vectors and the elasticity tensor by a tensor of the second order, i.e.

$$\begin{aligned} e_u &:= \left(\hat{e}^u_{11}, \hat{e}^u_{22}, \sqrt{2}\hat{e}^u_{12}\right)^T, \\ \sigma &:= \left(\hat{\sigma}_{11}, \hat{\sigma}_{22}, \sqrt{2}\hat{\sigma}_{12}\right)^T, \\ E &:= \begin{pmatrix} \hat{E}_{1111} & \hat{E}_{1122} & \sqrt{2}\hat{E}_{1112} \\ \hat{E}_{2211} & \hat{E}_{2222} & \sqrt{2}\hat{E}_{2212} \\ \sqrt{2}\hat{E}_{1211} & \sqrt{2}\hat{E}_{1222} & 2\hat{E}_{1212} \end{pmatrix}. \end{aligned}$$

Then Hooke's law is equivalent to the equation

$$\sigma(x) = E(x)e_u(x) \quad \text{a.e. in } \Omega.$$

Similar simplification can be found in [20].

Assumption. Gravity has little effect on deformation of the structure, and it can be neglected. No other volume forces are considered.

The outer load will be described by $f \in [L^2(\Gamma_f)]^2$.

Classical formulation of the contact problem for elastic bodies. Find $\tilde{u} \in K$ such that

$$\begin{aligned} \operatorname{div}\hat{\sigma}(x) &= 0 && \text{a.e. in } \Omega, \\ \hat{\sigma}(x)n(x) &= f(x) && \text{a.e. on } \Gamma_f, \\ \langle \hat{\sigma}(x)n(x), \eta\rangle &= 0 && \text{a.e. on } \Gamma_c, \\ \langle \hat{\sigma}(x)n(x), \xi\rangle &\geqslant 0 && \text{a.e. on } \Gamma_c, \\ \langle \hat{\sigma}(x)n(x), \xi\rangle(\langle u(x), \xi\rangle - \psi(t) + \varphi(t)) &= 0 && \text{a.e. on } \Gamma_c, \end{aligned}$$

where n is the outer normal field of Γ, η and ξ are defined above, and in the last equation we have $x = [t, \psi(t)]$.

If we take into account the above assumptions about the elasticity tensor and the choice of K, the standard theory of elliptic partial differential equations (cf. [16], Sec. 3.2) confirms for coercive elasticity tensors existence of a displacement vector field $\tilde{u} \in K$ which solves the force balance equations in a weak sense. Let $A \cdot B$ denote the scalar product of matrices A and B. The elasticity tensor $\hat{E}$ is coercive if

$$\int_\Omega \hat{e}^u(x) \cdot \hat{E}(x)\hat{e}^u(x)\,\mathrm{d}x - \int_{\Gamma_f} \langle f(x), u(x)\rangle\,\mathrm{d}\Sigma \to +\infty \quad \text{for } \|u\|_V \to +\infty, \quad u \in K.$$

Weak formulation of the contact problem for elastic bodies. Find $\tilde{u} \in K$ such that the following inequality holds for each $v \in K$:

$$\int_\Omega \hat{e}^{v-\tilde{u}}(x) \cdot \hat{E}(x)\hat{e}^{\tilde{u}}(x)\,\mathrm{d}x \geqslant \int_{\Gamma_f} \langle f(x), v(x) - \tilde{u}(x)\rangle\,\mathrm{d}\Sigma.$$

If the elasticity tensor $\hat{E}$ is coercive, the above weak formulation is equivalent to the minimization of the potential energy:

$$\hat{\Pi}(E,u) := \frac{1}{2}\int_\Omega \langle E(x)e_u(x), e_u(x)\rangle\,\mathrm{d}x - \int_{\Gamma_f} \langle f(x), u(x)\rangle\,\mathrm{d}\Sigma$$

on K.

1.1. Optimal design of material

In the linear case, when there is no unilateral contact considered, the weak solution has the potential energy

$$\hat{\Pi}(E, \tilde{u}(E)) = -\frac{1}{2}\int_{\Gamma_f} \langle f(x), \tilde{u}(x)\rangle\,\mathrm{d}\Sigma =: -W(E).$$

The function $W(E)$ represents the work of external loads done to deform the structure. The work $W(E)$ can be understood as a measure of deformation of the structure. The aim of the material optimization is to find the stiffest structure possible. To this end, $W(E)$ is minimized for the given load f over the particular choice of material, i.e. the choice of $E_{ij} \in L^\infty(\Omega)$, $i, j = 1, 2, 3$, where E_{ij} are components of the tensor E defined above. The set of admissible materials is given by physical and engineering constraints. Symmetry and positive semi-definiteness of $E(x)$ was discussed above. To express the stiffness of the structure, the trace of the matrix $E(x)$ is taken into account. Let $\bar{t} > 0$ be a real number. Let the stiffness be bounded in this way:

$$0 \leqslant \mathrm{tr}(E(x)) \leqslant \bar{t} \quad \text{a.e. in } \Omega.$$

The engineering constraint (cost constraint), or figuratively speaking the limited amount ($\overline{V} > 0$) of the material used, is stated as follows:

$$\int_\Omega \operatorname{tr}(E(x))\,\mathrm{d}x \leqslant \overline{V}.$$

Without this limitation the optimal structure would be attained by a material such that $\operatorname{tr}(E(x)) = \bar{t}$ in Ω. This case is not interesting. Let $\hat{U}_{\text{ad}}$ be the set of admissible materials:

$$\hat{U}_{\text{ad}} := \Big\{ E \in [L^\infty(\Omega)]^{3\times 3} \colon\ E(x) \text{ is symmetric and positive semi-definite,}$$
$$0 \leqslant \operatorname{tr}(E(x)) \leqslant \bar{t} \quad \text{a.e. in } \Omega,\ \int_\Omega \operatorname{tr}(E(x))\,\mathrm{d}x \leqslant \overline{V} \Big\}.$$

The case $\operatorname{tr}(E(x)) = 0$ a.e. in A, where $A \subset \Omega$, $\operatorname{meas}(A) > 0$, can be interpreted as void material, because it implies $E_{ij}(x) = 0$, $i,j = 1,2,3$, a.e. in A. Indeed, as $E(x)$ is positive semi-definite, $E_{ii}(x) \geqslant 0$, $i = 1,2,3$. Thus we have $\operatorname{tr}(E(x)) = 0$ implies $E_{ii}(x) = 0$, $i = 1,2,3$. Let $\{a_1, a_2, a_3\}$ be the Euclidean base of $\mathbb{R}^3$. Then the inequality $\langle a_i \pm a_j, E(x)(a_i \pm a_j)\rangle \geqslant 0$, $i,j = 1,2,3$, implies

$$2|E_{ij}(x)| \leqslant E_{ii}(x) + E_{jj}(x), \quad i,j = 1,2,3. \tag{1.1}$$

Thus finally $E_{ij}(x) = 0$, $i,j = 1,2,3$.

To obtain the minimum of $W(E)$ means to reach the maximum of $\hat{\Pi}(E, \tilde{u}(E))$. When the unilateral contact is taken into account, the potential energy $\hat{\Pi}(E, \tilde{u}(E))$ is not equal to the work of the outer forces. Still, the potential energy (sometimes also called compliance) of the deformed structure can be taken as a measure of response of the structure to the outer forces. The task of the material optimization is to find $\tilde{E} \in \hat{U}_{\text{ad}}$ such that

$$\min_{u \in K} \hat{\Pi}(\tilde{E}, u) = \max_{E \in \hat{U}_{\text{ad}}} \inf_{u \in K} \hat{\Pi}(E, u). \tag{1.2}$$

Eventually, the optimization has got the form of a max-inf problem for $\hat{\Pi}(E,u)$ in $\hat{U}_{\text{ad}} \times K$.

2. Existence of a solution

The existence of a solution of the problem (1.2) is proved by means of a theorem on the existence of a saddle-point for a functional on a product of a space which is a topological dual of a Banach space, and a reflexive space (cf. [9]), and a lemma on the correspondence between the saddle-point problem and the max-inf problem (cf. [11]).

Let A, B be arbitrary sets. A pair $(\tilde{\varrho}, \tilde{u}) \in A \times B$ is a saddle-point of a function

$$\mathcal{L}\colon A \times B \to \mathbb{R} \overset{\text{def}}{\Longleftrightarrow} \mathcal{L}(\varrho, \tilde{u}) \leqslant \mathcal{L}(\tilde{\varrho}, \tilde{u}) \leqslant \mathcal{L}(\tilde{\varrho}, u) \quad \text{for all } (\varrho, u) \in A \times B.$$

Korn's inequality is necessary for the existence of a solution of problem (1.2).

Lemma 2.1. *There exists $C_K > 0$ such that*

$$\|e_u\|^2_{[L^2(\Omega)]^3} \geqslant C_K \|u\|^2_V \quad \textit{for all } u \in V. \tag{2.1}$$

The proof may be found e.g. in [17]. The space V (cf. Sec. 1) is chosen so that this inequality holds.

Let Z be an arbitrary Banach space and Z^* its topological dual space. Let X be a reflexive Banach space. A general theorem of existence of a saddle-point can be stated as follows.

Theorem 2.1. *Let $A \subset Z^*$ be convex, bounded, weakly* sequentially compact and non-empty, let $B \subset X$ be convex, closed and non-empty. Let a function $\mathcal{L}$: $A \times B \to \mathbb{R}$ satisfy*

(2.2) *for all $\varrho \in A$, $u \mapsto \mathcal{L}(\varrho, u)$ is convex and continuous,*

(2.3) *for all $u \in B$, $\varrho \mapsto \mathcal{L}(\varrho, u)$ is concave and weakly* upper semi-continuous.*

Let there exist $\varrho_0 \in A$ such that

$$\mathcal{L}(\varrho_0, u) \to +\infty \quad \textit{for } \|u\|_X \to +\infty, \ u \in B. \tag{2.4}$$

Then the function $\mathcal{L}$ possesses at least one saddle-point $(\tilde{\varrho}, \tilde{u}) \in A \times B$.

The proof can be found in [9]. The correspondence between max-inf and the saddle-point is the subject of the following lemma.

Lemma 2.2. *Let A, B and $\mathcal{L}$ be the same as in Theorem* 2.1. *Then*

$$
\begin{aligned}
&(\tilde{\varrho}, \tilde{u}) \in A \times B \ \text{ is a saddle-point of } \mathcal{L} \text{ in } A \times B\\
&\qquad \Longleftrightarrow \mathcal{L}(\tilde{\varrho}, \tilde{u}) = \max_{\varrho \in A} \inf_{u \in B} \mathcal{L}(\varrho, u) = \min_{u \in B} \max_{\varrho \in A} \mathcal{L}(\varrho, u),
\end{aligned}
$$

and $\tilde{\varrho}$ is the point where the maximum in the max-inf part is achieved, and $\tilde{u}$ is the point where the minimum in the min-max part is achieved.

The proof can be found in [11]. The theorem on the existence of a solution of the problem (1.2) follows.

Theorem 2.2. *There exists a solution of the problem* (1.2). *Moreover,*

$$\max_{E \in \hat{U}_{\mathrm{ad}}} \inf_{u \in K} \hat{\Pi}(E, u) = \min_{u \in K} \max_{E \in \hat{U}_{\mathrm{ad}}} \hat{\Pi}(E, u).$$

P r o o f. For $Z = [L^1(\Omega)]^{3 \times 3}$, the set $\hat{U}_{\mathrm{ad}} \subset Z^*$ is convex and non-empty. The norm in $[L^\infty(\Omega)]^{3\times 3}$ let be defined by

$$\|E\|_{[L^\infty(\Omega)]^{3\times 3}} := \operatorname*{ess\,sup}_{x \in \Omega} \max_{i,j=1,2,3} |E_{ij}(x)|.$$

Positive semi-definiteness of $E(x)$ implies $E_{ii}(x) \geqslant 0$, $i = 1, 2, 3$. Then Equation (1.1) implies that

$$\bar{t} \geqslant \operatorname*{ess\,sup}_{x \in \Omega} \operatorname{tr}(E(x)) \geqslant 2\|E\|_{[L^\infty(\Omega)]^{3\times 3}}.$$

Thus $\hat{U}_{\mathrm{ad}}$ is bounded. Let $E \in [L^\infty(\Omega)]^{3\times 3}$, and let $\{E^n\}_{n=0}^\infty \subset \hat{U}_{\mathrm{ad}}$ be a sequence such that $\lim\limits_{n\to\infty} \|E^n - E\|_{[L^\infty(\Omega)]^{3\times 3}} = 0$. As

$$\|E^n - E\|_{[L^\infty(\Omega)]^{3\times 3}} \geqslant \max_{i,j=1,2,3} |E^n_{ij}(x) - E_{ij}(x)| \geqslant \frac{1}{3} |\operatorname{tr}(E^n(x)) - \operatorname{tr}(E(x))|$$

holds for a.a. $x \in \Omega$, it follows that $E \in \hat{U}_{\mathrm{ad}}$. Thus the set $\hat{U}_{\mathrm{ad}}$ is closed, and the Banach theorem implies that it is weakly* sequentially compact, and thus it satisfies all the conditions for the set A of Theorem 2.1. For $X = V$, K satisfies those for B. From Schwartz's inequality and continuity of the trace operator in V we obtain

$$|\hat{\Pi}(E, u)| \leqslant \frac{1}{2} \|E\|_{[L^\infty(\Omega)]^{3\times 3}} \|u\|_V^2 + \|f\|_{[L^2(\Gamma_f)]^2} C_T \|u\|_V,$$

where C_T is the norm of the trace operator

$$T\colon V \to [L^2(\Gamma_f)]^2, \quad T(u) := u|_{\Gamma_f}.$$

Hence the mapping $u \mapsto \hat{\Pi}(E,u)$ is continuous. It is also convex, thus the assumption (2.2) is satisfied for $\hat{\Pi}$. The linear mapping $E \mapsto \hat{\Pi}(E,u)$ is continuous, too. Thus the set $\{(E,y) \in \hat{U}_{\mathrm{ad}} \times \mathbb{R};\ y \leqslant \hat{\Pi}(E,u)\}$ is closed and convex for all $u \in V$. It is a consequence of the Hahn-Banach theorem that this set is also weakly* closed, thus the mapping $E \mapsto \hat{\Pi}(E,u)$ is weakly* upper semi-continuous (cf. [11]), and the assumption (2.3) is satisfied for $\hat{\Pi}$. Let

$$E_0 := \begin{pmatrix} \tilde{t} & 0 & 0 \\ 0 & \tilde{t} & 0 \\ 0 & 0 & \tilde{t} \end{pmatrix},$$

where

$$\tilde{t} := \frac{1}{3}\min\left\{\bar{t}, \frac{\overline{V}}{\operatorname{meas}(\Omega)}\right\}.$$

Then $E_0 \in \hat{U}_{\mathrm{ad}}$, and

$$\hat{\Pi}(E_0,u) \geqslant \frac{1}{2}\tilde{t}\|e_u\|^2_{[L^2(\Omega)]^3} - \|f\|_{[L^2(\Gamma_f)]^2} C_T \|u\|_V.$$

Korn's inequality (2.1) gives

$$\hat{\Pi}(E_0,u) \geqslant \frac{1}{2}C_K\tilde{t}\|u\|^2_V - \|f\|_{[L^2(\Gamma_f)]^2} C_T \|u\|_V,$$

and thus the condition (2.4) is valid, too. Theorem 2.1 establishes the existence of a saddle-point, which implies the existence of a solution of the problem (1.2) by Lemma 2.2. Lemma 2.2 gives also the remaining statement of this theorem. □

3. Simplification of the problem

To compute the matrix $\tilde{E}$ directly would mean to work with its six unknown components. It turns out that this complication is avoidable, and the problem can be solved by finding one unknown function which can give all six unknown material variables. Theorem 2.2 will be a crucial result for the following sections. Some ideas of this section can be found in [20].

Let

$$(3.1) \qquad U_{\mathrm{ad}} := \left\{\varrho \in L^\infty(\Omega)\colon \int_\Omega \varrho(x)\,\mathrm{d}x \leqslant \overline{V},\ 0 \leqslant \varrho(x) \leqslant \bar{t} \quad \text{a.e. in } \Omega\right\},$$

where $\overline{V}$ and $\bar{t}$ are the same as in Sec. 1.1 above. And let for fixed $\varrho \in U_{\mathrm{ad}}$

$$U^{\varrho}_{\mathrm{ad}} := \{E \in \hat{U}_{\mathrm{ad}}\colon\ \operatorname{tr}(E(x)) = \varrho(x) \ \text{ a.e. in } \Omega\}.$$

The first step is the change of max and inf and the split of max.

Proposition 3.1.

$$\max_{E\in\hat U_{\mathrm{ad}}}\inf_{u\in K}\hat\Pi(E,u)$$
$$=\min_{u\in K}\max_{\varrho\in U_{\mathrm{ad}}}\left\{\int_\Omega\max_{E\in U^\varrho_{\mathrm{ad}}}\frac{1}{2}\langle E(x)e_u(x),e_u(x)\rangle\,\mathrm{d}x-\int_{\Gamma_f}\langle f(x),u(x)\rangle\,\mathrm{d}\Sigma\right\}.$$

Proof. From Theorem 2.2 it follows that max and inf can be interchanged and min can be written instead of inf; moreover, the set equality $\hat U_{\mathrm{ad}}=\bigcup\limits_{\varrho\in U_{\mathrm{ad}}}U^\varrho_{\mathrm{ad}}$ implies that the max can be split so that

$$\max_{E\in\hat U_{\mathrm{ad}}}\inf_{u\in K}\hat\Pi(E,u)=\min_{u\in K}\max_{\varrho\in U_{\mathrm{ad}}}\max_{E\in U^\varrho_{\mathrm{ad}}}\hat\Pi(E,u).$$

The last modification of putting $\max\limits_{E\in U^\varrho_{\mathrm{ad}}}$ behind $\int_\Omega$ is possible, because the definition of U^ϱ_{ad} involves only local properties of E. □

The following general properties of the trace of a matrix will be used in the simplification calculations.

Lemma 3.1. *For $A,B\in\mathbb{R}^{N\times N}$ and $a\in\mathbb{R}^N$ we have*
(i) $\mathrm{tr}(A+B)=\mathrm{tr}(A)+\mathrm{tr}(B)$;
(ii) $\mathrm{tr}(AB^T)=\sum\limits_{i,j=1}^{N}A_{ij}B_{ij}$;
(iii) $\mathrm{tr}(Aaa^T)=\langle Aa,a\rangle$;
iv) $\mathrm{tr}(aa^T)=\langle a,a\rangle$;
(v) $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

All equalities are consequences of straightforward calculations.

The following calculations lead to direct evaluation of

$$\max_{E\in U^\varrho_{\mathrm{ad}}}\langle E(x)e_u(x),e_u(x)\rangle\quad\text{a.e. in }\Omega.$$

Let $G(x):=e_u(x)e_u^T(x)$, let $\mu_i(x)$, $i=1,2,3$, be eigenvalues of $G(x)$, and let $s_i(x)$, $i=1,2,3$, be orthonormal eigenvectors of $G(x)$ such that

$$G(x)s_i(x)=\mu_i(x)s_i(x),\quad i=1,2,3.$$

$G(x)$ is a rank-1 matrix, and it follows immediately from its definition that $\mu_1(x)=|e_u(x)|^2$, $s_1(x)=e_u(x)/|e_u(x)|$, $\mu_2(x)=0$, $\mu_3(x)=0$ a.e. in Ω. Existence of orthonormal eigenvectors $s_i(x)$, $i=1,2,3$, follows from the symmetry of the real matrix $G(x)$. Equality (iii) of Lemma 3.1 implies that

$$\langle E(x)e_u(x),e_u(x)\rangle=\mathrm{tr}(E(x)G(x)).$$

The simplification is based on the following estimate.

Lemma 3.2. *The following inequality holds:*

$$\operatorname{tr}(E(x)G(x)) \leqslant \operatorname{tr}(E(x))|e_u(x)|^2.$$

Proof. Let $S(x) := (s_1(x), s_2(x), s_3(x))$ (the columns of $S(x)$ are the eigenvectors of $G(x)$). The eigenvectors $s_i(x)$, $i = 1, 2, 3$, of $G(x)$ are orthonormal; therefore

$$G(x) = S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^{-1}(x).$$

Then by substitution

$$\operatorname{tr}(E(x)G(x)) = \operatorname{tr}\left(E(x)S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^{-1}(x) \right).$$

The equality (v) of Lemma 3.1 gives

$$\operatorname{tr}(E(x)G(x)) = \operatorname{tr}\left(S^{-1}(x)E(x)S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Let $E^S(x) := S^{-1}(x)E(x)S(x)$, then by virtue of the equality (ii) of Lemma 3.1

$$\operatorname{tr}(E(x)G(x)) = E^S_{11}(x)|e_u(x)|^2.$$

As $S^{-1}(x) = S^T(x)$ and $E(x)$ is positive semi-definite, we have $E^S_{ii}(x) \geqslant 0$, $i = 1, 2, 3$, and

$$\operatorname{tr}(E(x)G(x)) \leqslant \sum_{i=1}^{N} E^S_{ii}(x)|e_u(x)|^2.$$

Moreover, the equality (v) of Lemma 3.1 implies

$$\operatorname{tr}(E(x)) = \operatorname{tr}(E^S(x)),$$

which is the last step to complete the proof. □

It can be concluded from the properties of $E^S(x)$, as they were mentioned in the course of the proof of Lemma 3.2, that if $|e_u(x)| \neq 0$, then $\operatorname{tr}(E(x)G(x)) = \operatorname{tr}(E(x))|e_u(x)|^2$ if and only if $E^S_{ii}(x) = 0$, $i = 2,3$. Then everything is ready for claiming that

$$\max_{E \in U^{\varrho}_{\mathrm{ad}}} \frac{1}{2}\langle E(x)e_u(x), e_u(x)\rangle = \frac{1}{2}\varrho(x)|e_u(x)|^2.$$

Maximum is achieved, for example, when

$$E^S(x) = \begin{pmatrix} \varrho(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$E(x) = \varrho(x)s_1(x)s_1^T(x) = \frac{\varrho(x)}{|e_u(x)|^2}e_u(x)e_u^T(x). \tag{3.2}$$

Symmetry and positive semi-definiteness of $E(x)$ ensures also that this maximizer is unique. Indeed, the most general form of $E^S(x)$ could be

$$E^S(x) = \begin{pmatrix} \varrho(x) & d(x) & b(x) \\ d(x) & 0 & c(x) \\ b(x) & c(x) & 0 \end{pmatrix},$$

where $d, b, c \in L^\infty(\Omega)$. Let $\{a_1, a_2, a_3\}$ be the Euclidean base of $\mathbb{R}^3$. It follows that

$$\langle a_2 \pm a_3, E^S(x)(a_2 \pm a_3)\rangle \geqslant 0 \Longrightarrow c(x) = 0 \quad \text{a.e. in } \Omega.$$

Then the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{\varrho(x) \pm \sqrt{\varrho(x)^2 + 4(d(x)^2 + b(x)^2)}}{2}.$$

Positive semi-definiteness of $E^S(x)$ now confirms that

$$d(x) = b(x) = 0 \quad \text{a.e. in } \Omega.$$

The simplification is eventually expressed by the equation

$$\max_{E \in \hat{U}_{\mathrm{ad}}} \inf_{u \in K} \hat{\Pi}(E, u) = \min_{u \in K} \max_{\varrho \in U_{\mathrm{ad}}} \Pi(\varrho, u),$$

where
$$\Pi(\varrho, u) := \int_\Omega \frac{1}{2}\varrho(x)|e_u(x)|^2 \, dx - \int_{\Gamma_f} \langle f(x), u(x)\rangle \, d\Sigma$$

(cf. Sec. 1.1 and Equation (3.1) for definitions of $\hat{U}_{\text{ad}}$ and U_{ad}).

The free material optimization problem (1.2) is reduced to the problem

$$\begin{aligned}
&\text{Find } \tilde{\varrho} \in U_{\text{ad}} \text{ and } \tilde{u} \in K \text{ such that} \\
&\Pi(\tilde{\varrho}, \tilde{u}) = \min_{u \in K} \Pi(\tilde{\varrho}, u) = \max_{\varrho \in U_{\text{ad}}} \inf_{u \in K} \Pi(\varrho, u) \\
&\qquad = \max_{\varrho \in U_{\text{ad}}} \Pi(\varrho, \tilde{u}) = \min_{u \in K} \max_{\varrho \in U_{\text{ad}}} \Pi(\varrho, u).
\end{aligned} \tag{3.3}$$

Also Π, U_{ad}, and K satisfy the assumptions of Theorem 2.1, hence according to Lemma 2.2 the solution $(\tilde{\varrho}, \tilde{u})$ of the problem (3.3) can be found as a saddle-point of Π. The constitutive tensor $\tilde{E}$ which solves the problem (1.2) can be reconstructed by means of Equation (3.2).

4. Galerkin approximation and its convergence

The discretization of the problem (3.3) is a standard procedure. It will be solved as a saddle-point problem. This treatment reminds considerably the saddle-point problem which arises from topology optimization done by variable thickness of a plate. The discretization is done similarly as in [18].

Let the involved sets be approximated as follows. Let $h > 0$ be a mesh parameter. For each h let U_h, V_h be finite-dimensional spaces such that $V_h \subset V$ (cf. Sec. 1), and $U_h \subset L^\infty(\Omega)$. Further let K_h, $U_{\text{ad},h}$ be closed, convex, and non-empty sets satisfying $K_h \subset V_h$ and $U_{\text{ad},h} \subset U_h$.

4.1. Abstract assumptions

(i) for each positive sufficiently small h let $U_{\text{ad},h} \subset U_{\text{ad}}$;

(ii) for each $\varrho \in U_{\text{ad}}$ there exists a sequence $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\text{ad},h}$ such that $\varrho_h \to \varrho$ for $h \to 0+$ a.e. in Ω ;

(iii) for each sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, weakly convergent in V to $v \in V$, let $v \in K$;

(iv) for each $v \in K$ there exists a sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, which converges to v in V;

(v) there are constants $M > 0$, $N > 0$ and $\alpha > 0$ and a set of coercive mappings $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\text{ad},h}$ such that for each $h > 0$ and $u \in V$

$$\Pi(\varrho_h, u) \geqslant M\|u\|_V^\alpha - N.$$

By (i) it is assumed that $U_{\mathrm{ad},h}$ is the inner approximation of U_{ad}. On the other hand, usually only an external approximation K_h of K is available. Therefore the assumptions (iii) and (iv) are necessary, and they will not be easy to acquire. Uniform coercivity, as stated in (v), will be needed for the existence of discrete solutions and later for the proof of existence of a convergent subsequence of discrete solutions.

4.2. Continuity of potential energy

Lemma 4.1. *Let assumptions* 4.1 *be satisfied. Let sequences* $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$ *and* $\{v_h\}_{h>0}$, $v_h \in V_h$ *converge so that*

$$\varrho_h \rightharpoonup^* \varrho \quad \text{for } h \to 0+ \text{ in } L^\infty(\Omega) \quad \text{and} \quad v_h \to v \quad \text{for } h \to 0+ \text{ in } V,$$

where $\varrho \in U_{\mathrm{ad}}$ *and* $v \in V$. *Then*

$$\Pi(\varrho_h, v_h) \to \Pi(\varrho, v) \text{ for } h \to 0+.$$

The proof can be found in [18], where a similar lemma is used for the analysis of variable thickness plate optimization problem. Let

$$A_\varrho(v,w) := \int_\Omega \varrho(x)\langle e_v(x), e_w(x)\rangle \,\mathrm{d}x.$$

Lemma 4.2. *Let assumptions* 4.1 *be satisfied. Let sequences* $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$, $\{v_h\}_{h>0}$, $v_h \in V_h$, *converge so that*

$$\varrho_h \to \varrho \quad \text{for } h \to 0+ \text{ a.e. in } \Omega \quad \text{and} \quad v_h \rightharpoonup v \quad \text{for } h \to 0+ \text{ in } V,$$

where $\varrho \in U_{\mathrm{ad}}$, *and* $v \in V$. *Then for each* $w \in V$

$$\lim_{h\to 0+} A_{\varrho_h}(v_h, w) = A_\varrho(v,w),$$

and

$$\liminf_{h\to 0+} \Pi(\varrho_h, v_h) \geqslant \Pi(\varrho, v).$$

Proof. See [18] for the proof.

4.3. Convergence of the solutions of the discretized problem

Fix $h > 0$ sufficiently small. To solve the discretization of the problem (3.3) is to

$$\text{find } \tilde\varrho_h \in U_{\mathrm{ad},h} \text{ and } \tilde u_h \in K_h, \text{ such that}$$
$$\Pi(\varrho_h, \tilde u_h) \leqslant \Pi(\tilde\varrho_h, \tilde u_h) \leqslant \Pi(\tilde\varrho_h, u_h) \quad \text{for each } \varrho_h \in U_{\mathrm{ad},h} \text{ and } u_h \in K_h. \tag{4.1}$$

Existence of a solution of the problem (4.1) is formulated in the following theorem.

Theorem 4.1. *Let assumption* 4.1 (v) *hold. Then the discretized problem* (4.1) *has a solution for each sufficiently small* $h > 0$.

Proof. The set $U_{ad,h}$ satisfies conditions assumed about A in Theorem 2.1, V_h satisfies those for B. The mapping Π satisfies conditions (2.2) and (2.3) concerning the functional $\mathcal{L}$ (cf. the proof of Theorem 2.2). As 4.1 (v) is assumed, the condition (2.4) is valid, too. Hence the proof is completed by Theorem 2.1. □

The convergence of the discrete solutions is established by the following theorem.

Theorem 4.2. *Let assumptions* 4.1 *be satisfied and let* $\{(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$ *be a sequence of solutions of the discretized problem* (4.1). *Then there is a subsequence* $\{(\tilde{\varrho}_{h'}, \tilde{u}_{h'})\}_{h'>0} \subset \{(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$ *and elements* $\tilde{\varrho} \in U_{\rm ad}$ *and* $\tilde{u} \in K$ *such that*

$$\tilde{\varrho}_{h'} \rightharpoonup^* \tilde{\varrho} \quad \text{for } h' \to 0+ \text{ in } L^\infty(\Omega) \quad \text{and} \quad \tilde{u}_{h'} \rightharpoonup \tilde{u} \quad \text{for } h' \to 0+ \text{ in } V.$$

Moreover, the couple $(\tilde{\varrho}, \tilde{u})$ *solves the problem* (3.9), *and*

$$\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \to \Pi(\tilde{\varrho}, \tilde{u}) \quad \text{for } h' \to 0+.$$

Proof. *Step 1.* The sequence $\{\tilde{\varrho}\}_{h>0}$ is bounded by the definition of $U_{\rm ad}$ and assumption 4.1 (i); therefore as $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$, Alaoglu Theorem confirms the existence of a subsequence $\{\tilde{\varrho}_{h'''}\}_{h'''>0} \subset \{\tilde{\varrho}_h\}_{h>0}$ and a mapping $\tilde{\varrho} \in L^\infty(\Omega)$ such that

$$\tilde{\varrho}_{h'''} \rightharpoonup^* \tilde{\varrho} \quad \text{for } h''' \to 0+ \text{ in } L^\infty(\Omega).$$

$U_{\rm ad}$ is weakly* closed, thus $\tilde{\varrho} \in U_{\rm ad}$.

Step 2. As K is considered non-empty (cf. Sec. 1), assumption 4.1 (iv) guarantees the existence of a bounded sequence $\{\tilde{v}_h\}_{h>0}$, $\tilde{v}_h \in K_h$, $\|\tilde{v}_h\|_V \leqslant C$. Elements of $U_{{\rm ad},h}$ are bounded by the definition of $U_{\rm ad}$ (cf. 4.1 (i)), hence there exists a constant C_1 such that for all sufficiently small $h > 0$

$$\Pi(\tilde{\varrho}_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, \tilde{v}_h) < C_1.$$

Step 3. Let the set $\{\tilde{\varrho}_h\}_{h>0}$ satisfy assumption 4.1 (v). Inequality

$$M\|\tilde{u}_h\|_V^\alpha - N \leqslant \Pi(\bar{\varrho}_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, \tilde{u}_h) < C_1,$$

following from the fact that $(\tilde{\varrho}_h, \tilde{u}_h)$ solves the problem (4.1), the assumption 4.1 (v), and step 2 give immediately the boundedness of $\{\tilde{u}_h\}_{h>0}$ in V. The space V is reflexive, hence the sequence $\{\tilde{u}_h\}_{h>0}$ as well as its subsequence $\{\tilde{u}_{h'''}\}_{h'''>0}$, where

h''' means the same choice of indexes as in step 1, is weakly sequentially compact, in the sense that there exists a subsequence $\{\tilde{u}_{h''}\}_{h''>0} \subset \{\tilde{u}_{h'''}\}_{h'''>0}$ and a mapping $\tilde{u} \in V$ such that

$$\tilde{u}_{h''} \rightharpoonup \tilde{u} \quad \text{for } h'' \to 0+ \text{ in } V.$$

Moreover, the assumption 4.1 (iii) ensures that $\tilde{u} \in K$.

Step 4. The inequality presented in step 3 guarantees also boundedness of the potential energy

$$-N \leqslant \Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''}) < C_1,$$

where $\{\Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''})\}_{h''>0}$ is a subsequence of $\{\Pi(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$, where the choice of indexes h'' is the same as in step 3. Then the Bolzano-Weierstrass theorem yields the existence of a real number β and a convergent subsequence $\{\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'})\}_{h'>0} \subset \{\Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''})\}_{h''>0}$ such that

$$\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \to \beta \quad \text{for } h' \to 0+.$$

Step 5. Let $(\varrho, u) \in U_{\mathrm{ad}} \times K$ be arbitrary. Then 4.1 (ii) implies the existence of a sequence $\{\varrho_{h'}\}_{h'>0}$, $\varrho_{h'} \in U_{\mathrm{ad},h'}$ such that

$$\varrho_{h'} \to \varrho \quad \text{for } h' \to 0+ \text{ a.e. in } \Omega,$$

and 4.1 (iv) leads to the existence of a sequence $\{u_{h'}\}_{h'>0}$, $u_{h'} \in K_{h'}$ such that

$$u_{h'} \to u \quad \text{for } h' \to 0+ \text{ in } V$$

(indexes h' were chosen coherently with step 4). The fact that $(\tilde{\varrho}_{h'}, \tilde{u}_{h'})$ solves the problem (4.1) gives

$$\Pi(\varrho_{h'}, \tilde{u}_{h'}) \leqslant \Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \leqslant \Pi(\tilde{\varrho}_{h'}, u_{h'}).$$

Lemma 4.1 and step 1 imply

$$\Pi(\tilde{\varrho}_{h'}, u_{h'}) \to \Pi(\tilde{\varrho}, u) \quad \text{for } h' \to 0+.$$

Lemma 4.2 and step 3 give

$$\Pi(\varrho, \tilde{u}) \leqslant \liminf_{h' \to 0+} \Pi(\varrho_{h'}, \tilde{u}_{h'}).$$

The two above facts together with step 4 imply

$$\Pi(\varrho, \tilde{u}) \leqslant \beta \leqslant \Pi(\tilde{\varrho}, u) \quad \text{for each } (\varrho, u) \in U_{\mathrm{ad}} \times K.$$

The above relation must hold for $u = \tilde{u}$ and also for $\varrho = \tilde{\varrho}$. These two choices yield

$$\Pi(\varrho, \tilde{u}) \leqslant \Pi(\tilde{\varrho}, \tilde{u}) \leqslant \Pi(\tilde{\varrho}, u).$$

This must be also valid when $u = \tilde{u}$ and simultaneously $\varrho = \tilde{\varrho}$. This leads to

$$\Pi(\tilde{\varrho}, \tilde{u}) = \beta.$$

□

5. Example of discretization

The aim of this section is to describe one of the possible discretizations which would satisfy assumptions 4.1. It is constructed for implementing on rectangular domains. The Sobolev space $H^1(\Omega)$ is approximated by Q_1 elements. The discretization is similar to that which is presented in [18]. In that article the use of the discretization is demonstrated on triangles.

5.1. Discretization

Assumption. The domain Ω can be divided into m squares Ω_i which are all of the same size and the sides of which are parallel to the chosen coordinate system.

(The standard iso-parametric concept can be used otherwise; cf. [10].) Let n be the number of distinct vertices (nodes of rectangulation) of Ω_i, $i = 1, \dots, m$. Let A_i, $i = 1, \dots, n_{cn}$ be the contact nodes, i.e. distinct vertices of Ω_i, $i = 1, \dots, m$, lying on $\overline{\Gamma}_c$ (cf. Sec. 1).

Assumption. There exists a partition of $[a, b]$, namely $a = t_1 < t_2 < \dots < t_{n_{cn}} = b$ such that $A_i = (t_i, \psi(t_i))$, $i = 1, \dots, n_{cn}$, in the local coordinate system (ξ_1, ξ_2).

Let $\{\mathcal{R}_h\}_{h>0}$ be a regular family of rectangulations (cf. [10]) of $\overline{\Omega}$ (each of them formed by squares Ω_i). For each $\mathcal{R}_h$ let

$$\begin{aligned} V_h &:= \{v_h \in [C(\overline{\Omega})]^2 \colon v_h|_{\Omega_i} \in [Q_1(\Omega_i)]^2 \quad \text{for each } \Omega_i \in \mathcal{R}_h, \ v_h \in V\}, \\ U_h &:= \{\varrho_h \in L^\infty(\Omega) \colon \varrho_h|_{\Omega_i} \in Q_0(\Omega_i) \quad \text{for each } \Omega_i \in \mathcal{R}_h\} \end{aligned}$$

($Q_i(\Omega_i)$ is the space of bilinear polynomials in Ω_i, $Q_0(\Omega_i)$ denotes constant functions in Ω_i, cf. [10]),

$$K_h := \{v_h \in V_h \colon \langle v_h(A_i), \xi(A_i)\rangle \leqslant \psi(t_i) - \varphi(t_i), \ \text{ for each } i = 1, \dots, n_{cn}\}$$

(t_i is a suitable ξ_1-coordinate in the local (ξ_1, ξ_2) coordinate system such that $(t_i, \psi(t_i)) = A_i$, $i = 1, \ldots, n_{cn}$, cf. Sec. 1 and Sec. 5.1). It is readily seen that in general K_h is an external approximation of K. Further let

$$U_{\mathrm{ad},h} := \left\{ \varrho_h \in U_h \colon \ 0 \leqslant \varrho_h \leqslant \bar{t} \quad \text{a.e. in } \Omega, \ \int_\Omega \varrho_h(x)\,\mathrm{d}x \leqslant \overline{V} \right\} = U_{\mathrm{ad}} \cap U_h$$

(cf. Sec. 3). The discretization of the problem (3.3) is:

$$\text{(5.1)} \qquad \begin{aligned} &\text{Find } \tilde{\varrho}_h \in U_{\mathrm{ad},h} \text{ and } \tilde{u}_h \in K_h \text{ such that} \\ \Pi(\tilde{\varrho}_h, \tilde{u}_h) &= \min_{u_h \in K_h} \Pi(\tilde{\varrho}_h, u_h) = \max_{\varrho_h \in U_{\mathrm{ad},h}} \inf_{u_h \in K_h} \Pi(\varrho_h, u_h) \\ &= \max_{\varrho_h \in U_{\mathrm{ad},h}} \Pi(\varrho_h, \tilde{u}_h) = \min_{u_h \in K_h} \max_{\varrho_h \in U_{\mathrm{ad},h}} \Pi(\varrho_h, u_h). \end{aligned}$$

According to Lemma 2.2 the solution of this discrete problem solves also the discrete problem (4.1).

5.2. Verification of assumptions 4.1

The following lemma is important for the validity of assumption 4.1 (iii).

Lemma 5.1. *Let f be a continuous function defined in $[a, b]$, $a, b \in \mathbb{R}$, $a < b$. Let $D_n\colon a \equiv t_0^n < t_1^n < \ldots < t_n^n \equiv b$ be a partition of $[a, b]$ whose norm tends to zero as $n \to +\infty$. Let $\{\tau_n\}_{n=1}^\infty$ be a sequence of piecewise linear continuous functions such that*

$$\tau_n(t_i^n) \geqslant f(t_i^n) \quad \text{for } i = 0, \ldots, n, \quad \text{and for all } n.$$

Let

$$\tau_n \to \tau \quad \text{for } n \to \infty \text{ a.e. in } [a, b].$$

Then

$$\tau \geqslant f \quad \text{a.e. in } [a, b].$$

The proof can be found in [13]. For the sake of simplicity, let the coordinate system (ξ_1, ξ_2) coincide with the Cartesian system (x_1, x_2). Then $\xi = (0, -1)$, and the non-penetrating condition is simplified to the inequality

$$u_2(x_1, \psi(x_1)) \geqslant \varphi(x_1) - \psi(x_1) \quad \text{(cf. Sec. 1)},$$

and thus

$$\begin{aligned} K &= \{v \in V \colon \ v_2(x_1, \psi(x_1)) \geqslant \varphi(x_1) - \psi(x_1), \ \text{for all } x_1 \in [a, b]\}, \\ K_h &= \{v_h \in V_h \colon \ v_{2,h}(t_i, \psi(t_i)) \geqslant \varphi(t_i) - \psi(t_i), \ \text{for each } i = 1, \ldots, n_{cn}\} \end{aligned}$$

(cf. Sec. 1).

The verification of the assumption 4.1 (iv) is based on the next lemma on smooth approximations in K.

Lemma 5.2. *Let the coordinate system (ξ_1, ξ_2) coincide with the Cartesian system (x_1, x_2), as described above. Let a continuous function $\varphi\colon [a,b] \to \mathbb{R}$ have an extension $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ such that*

(i) $\tilde{\varphi}|_{[a,b]} = \varphi$;

(ii) *$\tilde{\varphi}$ is sufficiently smooth;*

(iii) *there exists a neighborhood $\mathcal{U}_{\Gamma_0} \supset \Gamma_0$ such that $\mathcal{U}_{\Gamma_0} \subset R^C$, where*

$$R^C := \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_2 \geqslant \tilde{\varphi}(x_1),\ \ x_1 \in \mathbb{R}\}.$$

Then for any $v \in K$ there exists a sequence $\{\tilde{v}_\delta\}_{\delta \to 0+}$, $\tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2$ such that

$$\tilde{v}_\delta \to v \quad \text{for } \delta \to 0+ \text{ in } V.$$

P r o o f. The proof can be found in [18].

Proposition 5.1. *Let V_h, U_h, K_h and $U_{\mathrm{ad},h}$ be defined as described in Sec.* 5.1. *Let the assumptions of Lemma* 5.2 *be satisfied. Then the assumptions* 4.1 *are satisfied.*

P r o o f. *Step 1.* The assumption 4.1 (i) is satisfied by the choice of $U_{\mathrm{ad},h}$.

Step 2. For a given $\varrho \in U_{\mathrm{ad}}$ let

$$\varrho_i := \frac{1}{\operatorname{meas}(\Omega_i)} \int_{\Omega_i} \varrho(x)\,\mathrm{d}x \quad i = 1, \ldots, m$$

and

$$\varrho_h := \sum_{i=1}^{m} \varrho_i \chi_{\Omega_i},$$

where χ_{Ω_i} is the characteristic function of a square $\Omega_i \in \mathcal{R}_h$ (cf. Sec. 5.1). It is straightforward from the definition that $\varrho_h \in U_{\mathrm{ad},h}$. The validity of assumption 4.1 (ii) then follows from Lebesgue's point theorem (cf. [12]).

Step 3. Let the sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, converge weakly in V. The trace operator $T\colon\ V \to [L^2(\Gamma_c)]^2$ is compact; therefore there exists a subsequence $\{v_{h'}\}_{h'>0} \subset \{v_h\}_{h>0}$ such that

$$v_{h'} \to v \quad \text{for } h' \to 0+ \text{ in } [L^2(\Gamma_c)]^2.$$

For every sequence converging in $[L^2(\Gamma_c)]^2$ there exists a subsequence which converges a.e. in Γ_c. Thus there is a subsequence $\{v_{h''}\}_{h''>0} \subset \{v_{h'}\}_{h'>0}$ such that in the local coordinate system (ξ_1, ξ_2)

$$v_{h''}([\xi_1, \psi(\xi_1)]) \to v([\xi_1, \psi(\xi_1)]) \quad \text{for } h'' \to 0+ \text{ a.e. in } [a, b].$$

Further, continuity of the scalar product gives that

$$\langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi \rangle \to \langle v([\xi_1, \psi(\xi_1)]), \xi \rangle \quad \text{for } h'' \to 0+ \text{ a.e. in } [a, b].$$

Because $v_{h''} \in K_{h''}$, the mapping $\xi_1 \mapsto \langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi \rangle$ is piecewise linear in $[a, b]$. Lemma 5.1 used for $\{-\langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi \rangle\}_{h''>0}$ as the sequence $\{\tau_n\}_{n=1}^{\infty}$ and for $\varphi - \psi$ as the continuous function f (cf. Sec. 1) completes successfully the verification of assumption 4.1 (iii).

Step 4. Let v be an arbitrary element of K. According to Lemma 5.2 there exists a sequence $\{\tilde{v}_\delta\}_{\delta \to 0+}$, $\tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2$ such that

$$\tilde{v}_\delta \to v \quad \text{for } \delta \to 0+ \text{ in } V.$$

Classical approximation properties of V_h supply a piecewise bilinear interpolant $\pi_h \tilde{v}_\delta \in K_h$ such that

$$\pi_h \tilde{v}_\delta \to \tilde{v}_\delta \quad \text{in } V \text{ for } h \to 0+ \text{ for all } \tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2.$$

Moreover, thanks to the boundedness of the sequence $\{\tilde{v}_\delta\}_{\delta \to 0+}$, the convergence of $\{\{\pi_h \tilde{v}_\delta\}_{h>0}\}_{\delta \to 0+}$ is uniform in h. Then the sequence $\{v_h\}_{h>0}$, $v_h := \pi_h \tilde{v}_h$ satisfies assumption 4.1 (iv).

Step 5. Let

$$\varrho_h := \min\left\{ \frac{\overline{V}}{\operatorname{meas}(\Omega)}, \hat{t} \right\}, \quad \text{in } \Omega \text{ for each } h > 0.$$

The set $\{\varrho_h\}_{h>0}$ together with constants

$$\alpha = 2, \quad N = \frac{1}{2\varepsilon} \|f\|^2_{[L^2(\Gamma_f)]^2} \quad \text{and } M = \frac{1}{2}\left(C_K \min\left\{ \frac{\overline{V}}{\operatorname{meas}(\Omega)}, \bar{t} \right\} - \varepsilon C_T \right)$$

verifies assumption 4.1 (v) (cf. Sec. 1.1 for definitions of $\overline{V}$ and $\bar{t}$). C_K comes from Korn's inequality (2.1). C_T is the norm of the trace operator

$$T\colon V \to [L^2(\Gamma_f)]^2, \quad Tu = u|_{\Gamma_f}.$$

And ε is an arbitrary real number such that

$$0 < \varepsilon < \frac{C_{\mathcal{K}}}{C_T} \min\left\{ \frac{\overline{V}}{\operatorname{meas}(\Omega)}, \bar{t} \right\}.$$

Correctness of the estimate in assumption 4.1 (v) is based on Korn's inequality and Young's inequality. Derivation of the estimate involves similar ideas to those used in the proof of Theorem 2.2. □

6. Numerical solution

Numerical solution of the problem (5.1) leads to a so-called semi-definite program (cf. [15]). It can be solved by modern interior point polynomial time methods (cf. [15]), as well as by penalty/barrier multipliers method (cf. [2]).

An example computed by penalty/barrier multipliers method is presented in Figs. 2–4. A similar example can be found in [14, p. 325], where variable sheet thickness approach is used, and also in [2], where free material optimization approach is used.

The domain Ω is a square fixed at the top. It is pulled at the bottom by a constant downward lineload traction. It is supported by a rigid foundation consisting of two parts located symmetrically around the vertical axis of symmetry. The function that determines the boundary of the left part of the obstacle is defined as $\varphi_1(x_1) = -6.4\, x_1^4$ in the Cartesian system with the origin at the lower left corner of the domain Ω. The function that determines the boundary of the right part of the obstacle is defined symmetrically (cf. Fig. 2).

The calculated values of the function ϱ_h (cf. Sec. 5.1) for the mesh 30×30 are shown in Fig. 3. The values of ϱ_h are depicted by gradations of grey, i.e. full black corresponds to high values of ϱ_h, white corresponds to ϱ_h equal zero, which can be interpreted as void material (cf. Sec. 1.1). Fig. 4 presents directions and magnitudes of principal stresses in the finite elements.

The interested reader can find more on numerical realization of the free material optimization in [2], [15], [20].

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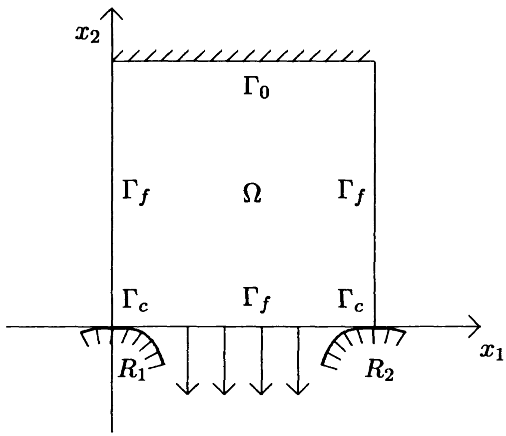

Figure 2. Boundary conditions.

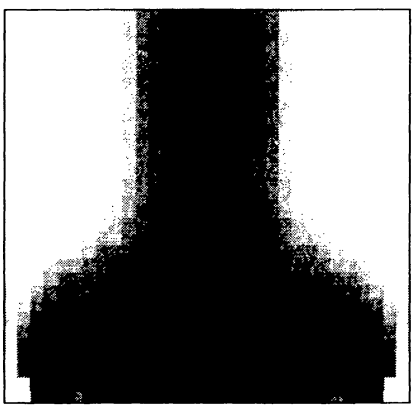

Figure 3. Values of the function ϱ_h.

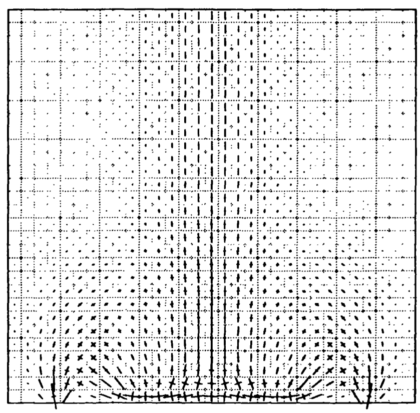

Figure 4. Principal stress directions and magnitudes.

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FINITE ELEMENT ANALYSIS OF FREE MATERIAL OPTIMIZATION PROBLEM

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Abstract. Free material optimization solves an important problem of structural engineering, i.e. to find the stiffest structure for given loads and boundary conditions. Its mathematical formulation leads to a saddle-point problem. It can be solved numerically by the finite element method. The convergence of the finite element method can be proved if the spaces involved satisfy suitable approximation assumptions. An example of a finite-element discretization is included.

Keywords: structural optimization, material optimization, topology optimization, finite elements

MSC 2000: 65N30, 74G15, 74P05

0. Introduction

The free material optimization solves one of the basic problems of structural engineering, viz. to find the stiffest structure for given set of loads and boundary conditions. Traditional methods of solution of this problem include variations of size and shape variables (cf. [14]). With the invention of composites and other advanced man-made materials it was natural to extend the variation to material properties. The basic problem setting was originated by the works of Bensøe et al. [6] and Ringertz [19], where it was suggested to represent material properties as elements of the unrestricted set of positive semi-definite constitutive tensors. The problem was also studied in [1], [4], [5], [7]. More details on engineering background can be found in [3], [6].

For simplicity of explanation the investigated structures are considered two-dimensional. Three dimensional structures could be approached in a similar way. The material properties of the structure are represented by a positive semi-definite constitutive tensor function. It means that the material is supposed to be nonhomogeneous and anisotropic. For example, composite materials can have these properties. The deformation of the body is described by the small strain tensor. Free material optimization means to optimize the constitutive tensor so that the optimal structure stands the static force load in the "easiest" way. This leads to a saddle-point problem. In two dimensions the constitutive tensor has six independent components. Yet the problem can be reformulated so that only the trace of the constitutive tensor remains an independent variable. Then the mathematical formulation of the optimization problem becomes similar to optimization of a variable thickness of a plate in two dimensions. An interesting point is that the optimal constitutive tensor can be reconstructed from its trace and from the deformation of the studied structure. Numerical examples, which can be found e.g. in [20], show that the norm of the constitutive tensor can be zero in some regions of the studied domain. This situation is interpreted as void material.

Numerical solution of the saddle-point problem can be obtained by the finite element method (cf. [10]). The implementation of the finite element discretization for the free material optimization problem can be found e.g. in [20]. Similar situation comes out of the optimization of a variable thickness of a two-dimensional plate (cf. [18]), where the convergence of the finite element approximation of the variable thickness optimization problem can be proved if suitable approximation properties of the spaces involved are assumed.

This article contributes to the finite element analysis of the free material optimization problem. This analysis is based on suitable approximation properties of the spaces involved, too.

1. Mathematical formulation

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain (an elastic body) with a Lipschitz boundary Γ, which is divided into disjoint parts Γ_0, Γ_f, Γ_c and Γ_R such that $\Gamma = \Gamma_0 \cup \Gamma_f \cup \Gamma_c \cup \Gamma_R$, the Hausdorff measure $\mathcal{H}_1(\Gamma_R) = 0$, Γ_0 is nonempty, and Γ_0, Γ_f, Γ_c are open in Γ. Γ_0 represents the fixed boundary, Γ_f is freely deformable, Γ_c denotes a region with a possible contact with an obstacle (cf. Fig. 1).

Deformation of the structure is described by a displacement vector $u \in V$, where

$$V := \{v \in [H^1(\Omega)]^2 \colon\ v|_\Gamma = 0 \ \text{ on } \ \Gamma_0\},$$

where $H^1(\Omega)$ is the Sobolev space.

Let $R \subset \mathbb{R}^2$ be a rigid foundation, which unilaterally supports the structure Ω. Frictionless contact between Ω and R can occur along Γ_c. The contact is handled in a

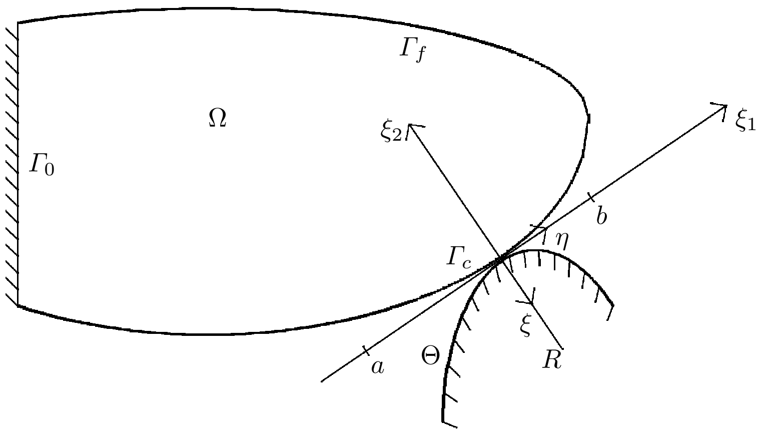

Figure 1. The loaded structure with unilateral contact.

local orthogonal coordinate system (ξ_1, ξ_2) with the origin at a fixed point of contact such that the axis ξ_1 is tangent both to the domain Ω and to the rigid foundation R. The contact boundaries are represented by continuous mappings $\psi, \varphi \in C([a, b])$ such that

$$\Gamma_c = \{(\xi_1, \xi_2)\colon\ \xi_2 = \psi(\xi_1),\ \ \xi_1 \in (a, b)\},$$

and the boundary Θ of the obstacle R is defined (cf. Fig. 1) as

$$\Theta = \{(\xi_1, \xi_2)\colon\ \xi_2 = \varphi(\xi_1),\ \ \xi_1 \in (a, b)\}.$$

The body Ω does not penetrate the foundation R. Let η, ξ be fixed unit vectors such that their coordinates in the local coordinate system (ξ_1, ξ_2) are $\eta = (1, 0)$, $\xi = (0, -1)$. "Not penetrating the foundation R by the structure Ω" means that the displacement u will satisfy the inequality

$$\langle u([t, \psi(t)]), \xi\rangle \leqslant \psi(t) - \varphi(t) \quad \text{a.e. in } (a, b),$$

where $\langle\cdot,\cdot\rangle$ stands for the scalar product in $\mathbb{R}^2$. Admissible displacement vectors are elements of the set

$$K := \{u \in V\colon\ \langle u([t, \psi(t)]), \xi\rangle \leqslant \psi(t) - \varphi(t) \quad \text{a.e. in } (a, b)\}.$$

K is closed and convex.

Assumption. K is not empty.

The small strain tensor is defined as

$$\hat{e}^u_{ij}(x) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x)\right), \quad i,j = 1,2, \;\; x \in \Omega, \;\; u \in K.$$

Let $\hat{\sigma}$ be the stress tensor with components $\hat{\sigma}_{ij} \in L^2(\Omega)$, $i,j = 1,2$.

Assumption. The system satisfies the linear Hooke's law with the elasticity 4-tensor (a tensor of the fourth order) $\hat{E}$, whose components $\hat{E}_{ijkl} \in L^\infty(\Omega)$, $i,j,k,l = 1,2$.

Symmetry of the stress tensor implies $\hat{E}_{ijkl} = \hat{E}_{jikl}$, $i,j,k,l = 1,2$. Without loss of generality it can be assumed that $\hat{E}_{ijkl} = \hat{E}_{klij}$, $i,j,k,l = 1,2$ (cf. [8] for rigorous physical arguments). The elasticity tensor $\hat{E}$ is assumed to be positive semi-definite. Thanks to the symmetry it is possible to rewrite Hooke's law representing the small strain tensor and the stress tensor by vectors and the elasticity tensor by a tensor of the second order, i.e.

$$\begin{aligned}
e_u &:= \left(\hat{e}^u_{11}, \hat{e}^u_{22}, \sqrt{2}\hat{e}^u_{12}\right)^T, \\
\sigma &:= \left(\hat{\sigma}_{11}, \hat{\sigma}_{22}, \sqrt{2}\hat{\sigma}_{12}\right)^T, \\
E &:= \begin{pmatrix} \hat{E}_{1111} & \hat{E}_{1122} & \sqrt{2}\hat{E}_{1112} \\ \hat{E}_{2211} & \hat{E}_{2222} & \sqrt{2}\hat{E}_{2212} \\ \sqrt{2}\hat{E}_{1211} & \sqrt{2}\hat{E}_{1222} & 2\hat{E}_{1212} \end{pmatrix}.
\end{aligned}$$

Then Hooke's law is equivalent to the equation

$$\sigma(x) = E(x)e_u(x) \quad \text{a.e. in } \Omega.$$

Similar simplification can be found in [20].

Assumption. Gravity has little effect on deformation of the structure, and it can be neglected. No other volume forces are considered.

The outer load will be described by $f \in [L^2(\Gamma_f)]^2$.

Classical formulation of the contact problem for elastic bodies. Find $\tilde{u} \in K$ such that

$$\begin{aligned}
\operatorname{div} \hat{\sigma}(x) &= 0 && \text{a.e. in } \Omega, \\
\hat{\sigma}(x)n(x) &= f(x) && \text{a.e. on } \Gamma_f, \\
\langle \hat{\sigma}(x)n(x), \eta\rangle &= 0 && \text{a.e. on } \Gamma_c, \\
\langle \hat{\sigma}(x)n(x), \xi\rangle &\geqslant 0 && \text{a.e. on } \Gamma_c, \\
\langle \hat{\sigma}(x)n(x), \xi\rangle(\langle u(x), \xi\rangle - \psi(t) + \varphi(t)) &= 0 && \text{a.e. on } \Gamma_c,
\end{aligned}$$

where n is the outer normal field of Γ, η and ξ are defined above, and in the last equation we have $x = [t, \psi(t)]$.

If we take into account the above assumptions about the elasticity tensor and the choice of K, the standard theory of elliptic partial differential equations (cf. [16], Sec. 3.2) confirms for coercive elasticity tensors existence of a displacement vector field $\tilde{u} \in K$ which solves the force balance equations in a weak sense. Let $A \cdot B$ denote the scalar product of matrices A and B. The elasticity tensor $\hat{E}$ is coercive if

$$\int_\Omega \hat{e}^u(x) \cdot \hat{E}(x)\hat{e}^u(x)\,\mathrm{d}x - \int_{\Gamma_f} \langle f(x), u(x)\rangle\,\mathrm{d}\Sigma \to +\infty \quad \text{for } \|u\|_V \to +\infty,\ u \in K.$$

Weak formulation of the contact problem for elastic bodies. Find $\tilde{u} \in K$ such that the following inequality holds for each $v \in K$:

$$\int_\Omega \hat{e}^{v-\tilde{u}}(x) \cdot \hat{E}(x)\hat{e}^{\tilde{u}}(x)\,\mathrm{d}x \geqslant \int_{\Gamma_f} \langle f(x), v(x) - \tilde{u}(x)\rangle\,\mathrm{d}\Sigma.$$

If the elasticity tensor $\hat{E}$ is coercive, the above weak formulation is equivalent to the minimization of the potential energy:

$$\hat{\Pi}(E, u) := \frac{1}{2}\int_\Omega \langle E(x)e_u(x), e_u(x)\rangle\,\mathrm{d}x - \int_{\Gamma_f} \langle f(x), u(x)\rangle\,\mathrm{d}\Sigma$$

on K.

1.1. Optimal design of material

In the linear case, when there is no unilateral contact considered, the weak solution has the potential energy

$$\hat{\Pi}(E, \tilde{u}(E)) = -\frac{1}{2}\int_{\Gamma_f} \langle f(x), \tilde{u}(x)\rangle\,\mathrm{d}\Sigma =: -W(E).$$

The function $W(E)$ represents the work of external loads done to deform the structure. The work $W(E)$ can be understood as a measure of deformation of the structure. The aim of the material optimization is to find the stiffest structure possible. To this end, $W(E)$ is minimized for the given load f over the particular choice of material, i.e. the choice of $E_{ij} \in L^\infty(\Omega)$, $i, j = 1, 2, 3$, where E_{ij} are components of the tensor E defined above. The set of admissible materials is given by physical and engineering constraints. Symmetry and positive semi-definiteness of $E(x)$ was discussed above. To express the stiffness of the structure, the trace of the matrix $E(x)$ is taken into account. Let $\bar{t} > 0$ be a real number. Let the stiffness be bounded in this way:

$$0 \leqslant \operatorname{tr}(E(x)) \leqslant \bar{t} \quad \text{a.e. in } \Omega.$$

The engineering constraint (cost constraint), or figuratively speaking the limited amount ($\overline{V} > 0$) of the material used, is stated as follows:

$$\int_\Omega \operatorname{tr}(E(x))\,\mathrm{d}x \leqslant \overline{V}.$$

Without this limitation the optimal structure would be attained by a material such that $\operatorname{tr}(E(x)) = \bar{t}$ in Ω. This case is not interesting. Let $\hat{U}_{\mathrm{ad}}$ be the set of admissible materials:

$$\hat{U}_{\mathrm{ad}} := \Big\{ E \in [L^\infty(\Omega)]^{3\times 3} \colon\ E(x) \ \text{is symmetric and positive semi-definite,} \\ 0 \leqslant \operatorname{tr}(E(x)) \leqslant \bar{t} \quad \text{a.e. in } \Omega,\ \int_\Omega \operatorname{tr}(E(x))\,\mathrm{d}x \leqslant \overline{V} \Big\}.$$

The case $\operatorname{tr}(E(x)) = 0$ a.e. in A, where $A \subset \Omega$, $\operatorname{meas}(A) > 0$, can be interpreted as void material, because it implies $E_{ij}(x) = 0$, $i,j = 1,2,3$, a.e. in A. Indeed, as $E(x)$ is positive semi-definite, $E_{ii}(x) \geqslant 0$, $i = 1,2,3$. Thus we have $\operatorname{tr}(E(x)) = 0$ implies $E_{ii}(x) = 0$, $i = 1,2,3$. Let $\{a_1, a_2, a_3\}$ be the Euclidean base of $\mathbb{R}^3$. Then the inequality $\langle a_i \pm a_j, E(x)(a_i \pm a_j)\rangle \geqslant 0$, $i,j = 1,2,3$, implies

$$2|E_{ij}(x)| \leqslant E_{ii}(x) + E_{jj}(x), \quad i,j = 1,2,3. \tag{1.1}$$

Thus finally $E_{ij}(x) = 0$, $i,j = 1,2,3$.

To obtain the minimum of $W(E)$ means to reach the maximum of $\hat{\Pi}(E, \tilde{u}(E))$. When the unilateral contact is taken into account, the potential energy $\hat{\Pi}(E, \tilde{u}(E))$ is not equal to the work of the outer forces. Still, the potential energy (sometimes also called compliance) of the deformed structure can be taken as a measure of response of the structure to the outer forces. The task of the material optimization is to find $\tilde{E} \in \hat{U}_{\mathrm{ad}}$ such that

$$\min_{u\in K} \hat{\Pi}(\tilde{E}, u) = \max_{E\in \hat{U}_{\mathrm{ad}}} \inf_{u\in K} \hat{\Pi}(E, u). \tag{1.2}$$

Eventually, the optimization has got the form of a max-inf problem for $\hat{\Pi}(E, u)$ in $\hat{U}_{\mathrm{ad}} \times K$.

2. Existence of a solution

The existence of a solution of the problem (1.2) is proved by means of a theorem on the existence of a saddle-point for a functional on a product of a space which is a topological dual of a Banach space, and a reflexive space (cf. [9]), and a lemma on the correspondence between the saddle-point problem and the max-inf problem (cf. [11]).

Let A, B be arbitrary sets. A pair $(\tilde{\varrho}, \tilde{u}) \in A \times B$ is a saddle-point of a function

$$\mathcal{L}\colon\ A \times B \to \mathbb{R} \overset{\text{def}}{\iff} \mathcal{L}(\varrho, \tilde{u}) \leqslant \mathcal{L}(\tilde{\varrho}, \tilde{u}) \leqslant \mathcal{L}(\tilde{\varrho}, u) \quad \text{for all } (\varrho, u) \in A \times B.$$

Korn's inequality is necessary for the existence of a solution of problem (1.2).

Lemma 2.1. *There exists $C_{\mathcal{K}} > 0$ such that*

$$\|e_u\|^2_{[L^2(\Omega)]^3} \geqslant C_{\mathcal{K}} \|u\|^2_V \quad \text{for all } u \in V. \tag{2.1}$$

The proof may be found e.g. in [17]. The space V (cf. Sec. 1) is chosen so that this inequality holds.

Let Z be an arbitrary Banach space and Z^* its topological dual space. Let X be a reflexive Banach space. A general theorem of existence of a saddle-point can be stated as follows.

Theorem 2.1. *Let $A \subset Z^*$ be convex, bounded, weakly* sequentially compact and non-empty, let $B \subset X$ be convex, closed and non-empty. Let a function $\mathcal{L}$: $A \times B \to \mathbb{R}$ satisfy*

(2.2) *for all* $\varrho \in A$, $\quad u \mapsto \mathcal{L}(\varrho, u)$ *is convex and continuous,*

(2.3) *for all* $u \in B$, $\varrho \mapsto \mathcal{L}(\varrho, u)$ *is concave and weakly* upper semi-continuous.*

Let there exist $\varrho_0 \in A$ such that

$$\mathcal{L}(\varrho_0, u) \to +\infty \quad \text{for } \|u\|_X \to +\infty,\ u \in B. \tag{2.4}$$

Then the function $\mathcal{L}$ possesses at least one saddle-point $(\tilde{\varrho}, \tilde{u}) \in A \times B$.

The proof can be found in [9]. The correspondence between max-inf and the saddle-point is the subject of the following lemma.

Lemma 2.2. *Let A, B and $\mathcal{L}$ be the same as in Theorem* 2.1. *Then*

$$\begin{aligned}&(\tilde{\varrho},\tilde{u})\in A\times B \text{ is a saddle-point of } \mathcal{L} \text{ in } A\times B\\ &\quad\Longleftrightarrow \mathcal{L}(\tilde{\varrho},\tilde{u})=\max_{\varrho\in A}\inf_{u\in B}\mathcal{L}(\varrho,u)=\min_{u\in B}\max_{\varrho\in A}\mathcal{L}(\varrho,u),\end{aligned}$$

and $\tilde{\varrho}$ is the point where the maximum in the max-inf part is achieved, and $\tilde{u}$ is the point where the minimum in the min-max part is achieved.

The proof can be found in [11]. The theorem on the existence of a solution of the problem (1.2) follows.

Theorem 2.2. *There exists a solution of the problem* (1.2). *Moreover,*

$$\max_{E\in\hat{U}_{\mathrm{ad}}}\inf_{u\in K}\hat{\Pi}(E,u)=\min_{u\in K}\max_{E\in\hat{U}_{\mathrm{ad}}}\hat{\Pi}(E,u).$$

P r o o f. For $Z=[L^1(\Omega)]^{3\times3}$, the set $\hat{U}_{\mathrm{ad}}\subset Z^*$ is convex and non-empty. The norm in $[L^\infty(\Omega)]^{3\times3}$ let be defined by

$$\|E\|_{[L^\infty(\Omega)]^{3\times3}}:=\operatorname*{ess\,sup}_{x\in\Omega}\max_{i,j=1,2,3}|E_{ij}(x)|.$$

Positive semi-definiteness of $E(x)$ implies $E_{ii}(x)\geqslant 0$, $i=1,2,3$. Then Equation (1.1) implies that

$$\bar{t}\geqslant\operatorname*{ess\,sup}_{x\in\Omega}\operatorname{tr}(E(x))\geqslant 2\|E\|_{[L^\infty(\Omega)]^{3\times3}}.$$

Thus $\hat{U}_{\mathrm{ad}}$ is bounded. Let $E\in[L^\infty(\Omega)]^{3\times3}$, and let $\{E^n\}_{n=0}^\infty\subset\hat{U}_{\mathrm{ad}}$ be a sequence such that $\lim\limits_{n\to\infty}\|E^n-E\|_{[L^\infty(\Omega)]^{3\times3}}=0$. As

$$\|E^n-E\|_{[L^\infty(\Omega)]^{3\times3}}\geqslant\max_{i,j=1,2,3}|E^n_{ij}(x)-E_{ij}(x)|\geqslant\frac{1}{3}|\operatorname{tr}(E^n(x))-\operatorname{tr}(E(x))|$$

holds for a.a. $x\in\Omega$, it follows that $E\in\hat{U}_{\mathrm{ad}}$. Thus the set $\hat{U}_{\mathrm{ad}}$ is closed, and the Banach theorem implies that it is weakly* sequentially compact, and thus it satisfies all the conditions for the set A of Theorem 2.1. For $X=V$, K satisfies those for B. From Schwartz's inequality and continuity of the trace operator in V we obtain

$$|\hat{\Pi}(E,u)|\leqslant\frac{1}{2}\|E\|_{[L^\infty(\Omega)]^{3\times3}}\|u\|_V^2+\|f\|_{[L^2(\Gamma_f)]^2}C_T\|u\|_V,$$

where C_T is the norm of the trace operator

$$T\colon\ V\to[L^2(\Gamma_f)]^2,\quad T(u):=u|_{\Gamma_f}.$$

Hence the mapping $u \mapsto \hat{\Pi}(E,u)$ is continuous. It is also convex, thus the assumption (2.2) is satisfied for $\hat{\Pi}$. The linear mapping $E \mapsto \hat{\Pi}(E,u)$ is continuous, too. Thus the set $\{(E,y) \in \hat{U}_{\rm ad} \times \mathbb{R};\ y \leqslant \hat{\Pi}(E,u)\}$ is closed and convex for all $u \in V$. It is a consequence of the Hahn-Banach theorem that this set is also weakly* closed, thus the mapping $E \mapsto \hat{\Pi}(E,u)$ is weakly* upper semi-continuous (cf. [11]), and the assumption (2.3) is satisfied for $\hat{\Pi}$. Let

$$E_0 := \begin{pmatrix} \tilde{t} & 0 & 0 \\ 0 & \tilde{t} & 0 \\ 0 & 0 & \tilde{t} \end{pmatrix},$$

where

$$\tilde{t} := \frac{1}{3} \min\Big\{ \bar{t}, \frac{\overline{V}}{\operatorname{meas}(\Omega)} \Big\}.$$

Then $E_0 \in \hat{U}_{\rm ad}$, and

$$\hat{\Pi}(E_0,u) \geqslant \frac{1}{2}\tilde{t}\|e_u\|^2_{[L^2(\Omega)]^3} - \|f\|_{[L^2(\Gamma_f)]^2} C_T \|u\|_V.$$

Korn's inequality (2.1) gives

$$\hat{\Pi}(E_0,u) \geqslant \frac{1}{2}C_{\mathcal{K}}\tilde{t}\|u\|^2_V - \|f\|_{[L^2(\Gamma_f)]^2} C_T \|u\|_V,$$

and thus the condition (2.4) is valid, too. Theorem 2.1 establishes the existence of a saddle-point, which implies the existence of a solution of the problem (1.2) by Lemma 2.2. Lemma 2.2 gives also the remaining statement of this theorem. $\square$

3. Simplification of the problem

To compute the matrix $\tilde{E}$ directly would mean to work with its six unknown components. It turns out that this complication is avoidable, and the problem can be solved by finding one unknown function which can give all six unknown material variables. Theorem 2.2 will be a crucial result for the following sections. Some ideas of this section can be found in [20].

Let

$$U_{\rm ad} := \Big\{ \varrho \in L^\infty(\Omega) \colon \int_\Omega \varrho(x)\,{\rm d}x \leqslant \overline{V},\ 0 \leqslant \varrho(x) \leqslant \bar{t} \quad \text{a.e. in } \Omega \Big\}, \tag{3.1}$$

where $\overline{V}$ and $\bar{t}$ are the same as in Sec. 1.1 above. And let for fixed $\varrho \in U_{\rm ad}$

$$U^\varrho_{\rm ad} := \{E \in \hat{U}_{\rm ad} \colon \operatorname{tr}(E(x)) = \varrho(x) \ \text{ a.e. in } \Omega\}.$$

The first step is the change of max and inf and the split of max.

Proposition 3.1.

$$\max_{E\in\hat U_{\mathrm{ad}}}\ \inf_{u\in K}\hat\Pi(E,u)$$
$$=\min_{u\in K}\max_{\varrho\in U_{\mathrm{ad}}}\left\{\int_\Omega \max_{E\in U^{\varrho}_{\mathrm{ad}}}\frac12\langle E(x)e_u(x),e_u(x)\rangle\,\mathrm{d}x-\int_{\Gamma_f}\langle f(x),u(x)\rangle\,\mathrm{d}\Sigma\right\}.$$

P r o o f. From Theorem 2.2 it follows that max and inf can be interchanged and min can be written instead of inf; moreover, the set equality $\hat U_{\mathrm{ad}}=\bigcup\limits_{\varrho\in U_{\mathrm{ad}}}U^{\varrho}_{\mathrm{ad}}$ implies that the max can be split so that

$$\max_{E\in\hat U_{\mathrm{ad}}}\ \inf_{u\in K}\hat\Pi(E,u)=\min_{u\in K}\max_{\varrho\in U_{\mathrm{ad}}}\max_{E\in U^{\varrho}_{\mathrm{ad}}}\hat\Pi(E,u).$$

The last modification of putting $\max\limits_{E\in U^{\varrho}_{\mathrm{ad}}}$ behind $\int_\Omega$ is possible, because the definition of $U^{\varrho}_{\mathrm{ad}}$ involves only local properties of E. □

The following general properties of the trace of a matrix will be used in the simplification calculations.

Lemma 3.1. *For $A,B\in\mathbb{R}^{N\times N}$ and $a\in\mathbb{R}^N$ we have*

(i) $\mathrm{tr}(A+B)=\mathrm{tr}(A)+\mathrm{tr}(B)$;

(ii) $\mathrm{tr}(AB^T)=\sum\limits_{i,j=1}^{N}A_{ij}B_{ij}$;

(iii) $\mathrm{tr}(Aaa^T)=\langle Aa,a\rangle$;

iv) $\mathrm{tr}(aa^T)=\langle a,a\rangle$;

(v) $\mathrm{tr}(AB)=\mathrm{tr}(BA)$.

All equalities are consequences of straightforward calculations.

The following calculations lead to direct evaluation of

$$\max_{E\in U^{\varrho}_{\mathrm{ad}}}\langle E(x)e_u(x),e_u(x)\rangle\quad\text{a.e. in }\Omega.$$

Let $G(x):=e_u(x)e_u^T(x)$, let $\mu_i(x)$, $i=1,2,3$, be eigenvalues of $G(x)$, and let $s_i(x)$, $i=1,2,3$, be orthonormal eigenvectors of $G(x)$ such that

$$G(x)s_i(x)=\mu_i(x)s_i(x),\quad i=1,2,3.$$

$G(x)$ is a rank-1 matrix, and it follows immediately from its definition that $\mu_1(x)=|e_u(x)|^2$, $s_1(x)=e_u(x)/|e_u(x)|$, $\mu_2(x)=0$, $\mu_3(x)=0$ a.e. in Ω. Existence of orthonormal eigenvectors $s_i(x)$, $i=1,2,3$, follows from the symmetry of the real matrix $G(x)$. Equality (iii) of Lemma 3.1 implies that

$$\langle E(x)e_u(x),e_u(x)\rangle=\mathrm{tr}(E(x)G(x)).$$

The simplification is based on the following estimate.

Lemma 3.2. *The following inequality holds:*

$$\operatorname{tr}(E(x)G(x)) \leqslant \operatorname{tr}(E(x))|e_u(x)|^2.$$

P r o o f. Let $S(x) := (s_1(x), s_2(x), s_3(x))$ (the columns of $S(x)$ are the eigenvectors of $G(x)$). The eigenvectors $s_i(x)$, $i = 1, 2, 3$, of $G(x)$ are orthonormal; therefore

$$G(x) = S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^{-1}(x).$$

Then by substitution

$$\operatorname{tr}(E(x)G(x)) = \operatorname{tr} \left(E(x)S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} S^{-1}(x) \right).$$

The equality (v) of Lemma 3.1 gives

$$\operatorname{tr}(E(x)G(x)) = \operatorname{tr} \left(S^{-1}(x)E(x)S(x) \begin{pmatrix} |e_u(x)|^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right).$$

Let $E^S(x) := S^{-1}(x)E(x)S(x)$, then by virtue of the equality (ii) of Lemma 3.1

$$\operatorname{tr}(E(x)G(x)) = E^S_{11}(x)|e_u(x)|^2.$$

As $S^{-1}(x) = S^T(x)$ and $E(x)$ is positive semi-definite, we have $E^S_{ii}(x) \geqslant 0$, $i = 1, 2, 3$, and

$$\operatorname{tr}(E(x)G(x)) \leqslant \sum_{i=1}^{N} E^S_{ii}(x)|e_u(x)|^2.$$

Moreover, the equality (v) of Lemma 3.1 implies

$$\operatorname{tr}(E(x)) = \operatorname{tr}(E^S(x)),$$

which is the last step to complete the proof. □

It can be concluded from the properties of $E^S(x)$, as they were mentioned in the course of the proof of Lemma 3.2, that if $|e_u(x)| \neq 0$, then $\operatorname{tr}(E(x)G(x)) = \operatorname{tr}(E(x))|e_u(x)|^2$ if and only if $E^S_{ii}(x) = 0$, $i = 2, 3$. Then everything is ready for claiming that

$$\max_{E \in U^{\varrho}_{\mathrm{ad}}} \frac{1}{2}\langle E(x)e_u(x), e_u(x)\rangle = \frac{1}{2}\varrho(x)|e_u(x)|^2.$$

Maximum is achieved, for example, when

$$E^S(x) = \begin{pmatrix} \varrho(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$E(x) = \varrho(x)s_1(x)s_1^T(x) = \frac{\varrho(x)}{|e_u(x)|^2}e_u(x)e_u^T(x). \tag{3.2}$$

Symmetry and positive semi-definiteness of $E(x)$ ensures also that this maximizer is unique. Indeed, the most general form of $E^S(x)$ could be

$$E^S(x) = \begin{pmatrix} \varrho(x) & d(x) & b(x) \\ d(x) & 0 & c(x) \\ b(x) & c(x) & 0 \end{pmatrix},$$

where $d, b, c \in L^\infty(\Omega)$. Let $\{a_1, a_2, a_3\}$ be the Euclidean base of $\mathbb{R}^3$. It follows that

$$\langle a_2 \pm a_3, E^S(x)(a_2 \pm a_3)\rangle \geqslant 0 \Longrightarrow c(x) = 0 \quad \text{a.e. in } \Omega.$$

Then the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \frac{\varrho(x) \pm \sqrt{\varrho(x)^2 + 4(d(x)^2 + b(x)^2)}}{2}.$$

Positive semi-definiteness of $E^S(x)$ now confirms that

$$d(x) = b(x) = 0 \quad \text{a.e. in } \Omega.$$

The simplification is eventually expressed by the equation

$$\max_{E \in \hat{U}_{\mathrm{ad}}} \inf_{u \in K} \hat{\Pi}(E, u) = \min_{u \in K} \max_{\varrho \in U_{\mathrm{ad}}} \Pi(\varrho, u),$$

where

$$\Pi(\varrho, u) := \int_\Omega \frac{1}{2}\varrho(x)|e_u(x)|^2 \,\mathrm{d}x - \int_{\Gamma_f} \langle f(x), u(x)\rangle \,\mathrm{d}\Sigma$$

(cf. Sec. 1.1 and Equation (3.1) for definitions of $\hat{U}_{\mathrm{ad}}$ and U_{ad}).

The free material optimization problem (1.2) is reduced to the problem

$$\begin{aligned} &\text{Find } \tilde{\varrho} \in U_{\mathrm{ad}} \text{ and } \tilde{u} \in K \text{ such that} \\ \Pi(\tilde{\varrho}, \tilde{u}) &= \min_{u\in K} \Pi(\tilde{\varrho}, u) = \max_{\varrho\in U_{\mathrm{ad}}} \inf_{u\in K} \Pi(\varrho, u) \\ &= \max_{\varrho\in U_{\mathrm{ad}}} \Pi(\varrho, \tilde{u}) = \min_{u\in K} \max_{\varrho\in U_{\mathrm{ad}}} \Pi(\varrho, u). \end{aligned} \tag{3.3}$$

Also Π, U_{ad}, and K satisfy the assumptions of Theorem 2.1, hence according to Lemma 2.2 the solution $(\tilde{\varrho}, \tilde{u})$ of the problem (3.3) can be found as a saddle-point of Π. The constitutive tensor $\tilde{E}$ which solves the problem (1.2) can be reconstructed by means of Equation (3.2).

4. Galerkin approximation and its convergence

The discretization of the problem (3.3) is a standard procedure. It will be solved as a saddle-point problem. This treatment reminds considerably the saddle-point problem which arises from topology optimization done by variable thickness of a plate. The discretization is done similarly as in [18].

Let the involved sets be approximated as follows. Let $h > 0$ be a mesh parameter. For each h let U_h, V_h be finite-dimensional spaces such that $V_h \subset V$ (cf. Sec. 1), and $U_h \subset L^\infty(\Omega)$. Further let K_h, $U_{\mathrm{ad},h}$ be closed, convex, and non-empty sets satisfying $K_h \subset V_h$ and $U_{\mathrm{ad},h} \subset U_h$.

4.1. Abstract assumptions

(i) for each positive sufficiently small h let $U_{\mathrm{ad},h} \subset U_{\mathrm{ad}}$;

(ii) for each $\varrho \in U_{\mathrm{ad}}$ there exists a sequence $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$ such that $\varrho_h \to \varrho$ for $h \to 0+$ a.e. in Ω ;

(iii) for each sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, weakly convergent in V to $v \in V$, let $v \in K$;

(iv) for each $v \in K$ there exists a sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, which converges to v in V;

(v) there are constants $M > 0$, $N > 0$ and $\alpha > 0$ and a set of coercive mappings $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$ such that for each $h > 0$ and $u \in V$

$$\Pi(\varrho_h, u) \geqslant M\|u\|_V^\alpha - N.$$

By (i) it is assumed that $U_{\mathrm{ad},h}$ is the inner approximation of U_{ad}. On the other hand, usually only an external approximation K_h of K is available. Therefore the assumptions (iii) and (iv) are necessary, and they will not be easy to acquire. Uniform coercivity, as stated in (v), will be needed for the existence of discrete solutions and later for the proof of existence of a convergent subsequence of discrete solutions.

4.2. Continuity of potential energy

Lemma 4.1. *Let assumptions* 4.1 *be satisfied. Let sequences* $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$ *and* $\{v_h\}_{h>0}$, $v_h \in V_h$ *converge so that*

$$\varrho_h \rightharpoonup^* \varrho \quad \text{for } h \to 0+ \text{ in } L^\infty(\Omega) \quad \text{and} \quad v_h \to v \quad \text{for } h \to 0+ \text{ in } V,$$

where $\varrho \in U_{\mathrm{ad}}$ *and* $v \in V$. *Then*

$$\Pi(\varrho_h, v_h) \to \Pi(\varrho, v) \text{ for } h \to 0+.$$

The proof can be found in [18], where a similar lemma is used for the analysis of variable thickness plate optimization problem. Let

$$A_\varrho(v, w) := \int_\Omega \varrho(x)\langle e_v(x), e_w(x)\rangle \,\mathrm{d}x.$$

Lemma 4.2. *Let assumptions* 4.1 *be satisfied. Let sequences* $\{\varrho_h\}_{h>0}$, $\varrho_h \in U_{\mathrm{ad},h}$, $\{v_h\}_{h>0}$, $v_h \in V_h$, *converge so that*

$$\varrho_h \to \varrho \quad \text{for } h \to 0+ \text{ a.e. in } \Omega \quad \text{and} \quad v_h \rightharpoonup v \quad \text{for } h \to 0+ \text{ in } V,$$

where $\varrho \in U_{\mathrm{ad}}$, *and* $v \in V$. *Then for each* $w \in V$

$$\lim_{h \to 0+} A_{\varrho_h}(v_h, w) = A_\varrho(v, w),$$

and

$$\liminf_{h \to 0+} \Pi(\varrho_h, v_h) \geqslant \Pi(\varrho, v).$$

P r o o f. See [18] for the proof.

4.3. Convergence of the solutions of the discretized problem

Fix $h > 0$ sufficiently small. To solve the discretization of the problem (3.3) is to

$$\text{find } \tilde{\varrho}_h \in U_{\mathrm{ad},h} \text{ and } \tilde{u}_h \in K_h, \text{ such that}$$

$$\Pi(\varrho_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, u_h) \quad \text{for each } \varrho_h \in U_{\mathrm{ad},h} \text{ and } u_h \in K_h. \tag{4.1}$$

Existence of a solution of the problem (4.1) is formulated in the following theorem.

Theorem 4.1. *Let assumption* 4.1 (v) *hold. Then the discretized problem* (4.1) *has a solution for each sufficiently small* $h > 0$.

P r o o f. The set $U_{ad,h}$ satisfies conditions assumed about A in Theorem 2.1, V_h satisfies those for B. The mapping Π satisfies conditions (2.2) and (2.3) concerning the functional $\mathcal{L}$ (cf. the proof of Theorem 2.2). As 4.1 (v) is assumed, the condition (2.4) is valid, too. Hence the proof is completed by Theorem 2.1. $\square$

The convergence of the discrete solutions is established by the following theorem.

Theorem 4.2. *Let assumptions* 4.1 *be satisfied and let* $\{(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$ *be a sequence of solutions of the discretized problem* (4.1). *Then there is a subsequence* $\{(\tilde{\varrho}_{h'}, \tilde{u}_{h'})\}_{h'>0} \subset \{(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$ *and elements* $\tilde{\varrho} \in U_{\mathrm{ad}}$ *and* $\tilde{u} \in K$ *such that*

$$\tilde{\varrho}_{h'} \rightharpoonup^* \tilde{\varrho} \quad \text{for } h' \to 0+ \text{ in } L^\infty(\Omega) \quad \text{and} \quad \tilde{u}_{h'} \rightharpoonup \tilde{u} \quad \text{for } h' \to 0+ \text{ in } V.$$

Moreover, the couple $(\tilde{\varrho}, \tilde{u})$ *solves the problem* (3.9), *and*

$$\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \to \Pi(\tilde{\varrho}, \tilde{u}) \quad \text{for } h' \to 0+.$$

P r o o f. *Step 1.* The sequence $\{\tilde{\varrho}\}_{h>0}$ is bounded by the definition of U_{ad} and assumption 4.1 (i); therefore as $L^\infty(\Omega)$ is the dual space of $L^1(\Omega)$, Alaoglu Theorem confirms the existence of a subsequence $\{\tilde{\varrho}_{h'''}\}_{h'''>0} \subset \{\tilde{\varrho}_h\}_{h>0}$ and a mapping $\tilde{\varrho} \in L^\infty(\Omega)$ such that

$$\tilde{\varrho}_{h'''} \rightharpoonup^* \tilde{\varrho} \quad \text{for } h''' \to 0+ \text{ in } L^\infty(\Omega).$$

U_{ad} is weakly* closed, thus $\tilde{\varrho} \in U_{\mathrm{ad}}$.

Step 2. As K is considered non-empty (cf. Sec. 1), assumption 4.1 (iv) guarantees the existence of a bounded sequence $\{\tilde{v}_h\}_{h>0}$, $\tilde{v}_h \in K_h$, $\|\tilde{v}_h\|_V \leqslant C$. Elements of $U_{\mathrm{ad},h}$ are bounded by the definition of U_{ad} (cf. 4.1 (i)), hence there exists a constant C_1 such that for all sufficiently small $h > 0$

$$\Pi(\tilde{\varrho}_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, \tilde{v}_h) < C_1.$$

Step 3. Let the set $\{\tilde{\varrho}_h\}_{h>0}$ satisfy assumption 4.1 (v). Inequality

$$M\|\tilde{u}_h\|_V^\alpha - N \leqslant \Pi(\overline{\varrho}_h, \tilde{u}_h) \leqslant \Pi(\tilde{\varrho}_h, \tilde{u}_h) < C_1,$$

following from the fact that $(\tilde{\varrho}_h, \tilde{u}_h)$ solves the problem (4.1), the assumption 4.1 (v), and step 2 give immediately the boundedness of $\{\tilde{u}_h\}_{h>0}$ in V. The space V is reflexive, hence the sequence $\{\tilde{u}_h\}_{h>0}$ as well as its subsequence $\{\tilde{u}_{h'''}\}_{h'''>0}$, where

h''' means the same choice of indexes as in step 1, is weakly sequentially compact, in the sense that there exists a subsequence $\{\tilde{u}_{h''}\}_{h''>0} \subset \{\tilde{u}_{h'''}\}_{h'''>0}$ and a mapping $\tilde{u} \in V$ such that

$$\tilde{u}_{h''} \rightharpoonup \tilde{u} \quad \text{for } h'' \to 0+ \text{ in } V.$$

Moreover, the assumption 4.1 (iii) ensures that $\tilde{u} \in K$.

Step 4. The inequality presented in step 3 guarantees also boundedness of the potential energy

$$-N \leqslant \Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''}) < C_1,$$

where $\{\Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''})\}_{h''>0}$ is a subsequence of $\{\Pi(\tilde{\varrho}_h, \tilde{u}_h)\}_{h>0}$, where the choice of indexes h'' is the same as in step 3. Then the Bolzano-Weierstrass theorem yields the existence of a real number β and a convergent subsequence $\{\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'})\}_{h'>0} \subset \{\Pi(\tilde{\varrho}_{h''}, \tilde{u}_{h''})\}_{h''>0}$ such that

$$\Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \to \beta \quad \text{for } h' \to 0+.$$

Step 5. Let $(\varrho, u) \in U_{\text{ad}} \times K$ be arbitrary. Then 4.1 (ii) implies the existence of a sequence $\{\varrho_{h'}\}_{h'>0}$, $\varrho_{h'} \in U_{\text{ad},h'}$ such that

$$\varrho_{h'} \to \varrho \quad \text{for } h' \to 0+ \text{ a.e. in } \Omega,$$

and 4.1 (iv) leads to the existence of a sequence $\{u_{h'}\}_{h'>0}$, $u_{h'} \in K_{h'}$ such that

$$u_{h'} \to u \quad \text{for } h' \to 0+ \text{ in } V$$

(indexes h' were chosen coherently with step 4). The fact that $(\tilde{\varrho}_{h'}, \tilde{u}_{h'})$ solves the problem (4.1) gives

$$\Pi(\varrho_{h'}, \tilde{u}_{h'}) \leqslant \Pi(\tilde{\varrho}_{h'}, \tilde{u}_{h'}) \leqslant \Pi(\tilde{\varrho}_{h'}, u_{h'}).$$

Lemma 4.1 and step 1 imply

$$\Pi(\tilde{\varrho}_{h'}, u_{h'}) \to \Pi(\tilde{\varrho}, u) \quad \text{for } h' \to 0+.$$

Lemma 4.2 and step 3 give

$$\Pi(\varrho, \tilde{u}) \leqslant \liminf_{h' \to 0+} \Pi(\varrho_{h'}, \tilde{u}_{h'}).$$

The two above facts together with step 4 imply

$$\Pi(\varrho, \tilde{u}) \leqslant \beta \leqslant \Pi(\tilde{\varrho}, u) \quad \text{for each } (\varrho, u) \in U_{\text{ad}} \times K.$$

The above relation must hold for $u = \tilde{u}$ and also for $\varrho = \tilde{\varrho}$. These two choices yield

$$\Pi(\varrho, \tilde{u}) \leqslant \Pi(\tilde{\varrho}, \tilde{u}) \leqslant \Pi(\tilde{\varrho}, u).$$

This must be also valid when $u = \tilde{u}$ and simultaneously $\varrho = \tilde{\varrho}$. This leads to

$$\Pi(\tilde{\varrho}, \tilde{u}) = \beta.$$

□

5. Example of discretization

The aim of this section is to describe one of the possible discretizations which would satisfy assumptions 4.1. It is constructed for implementing on rectangular domains. The Sobolev space $H^1(\Omega)$ is approximated by Q_1 elements. The discretization is similar to that which is presented in [18]. In that article the use of the discretization is demonstrated on triangles.

5.1. Discretization

Assumption. The domain Ω can be divided into m squares Ω_i which are all of the same size and the sides of which are parallel to the chosen coordinate system.

(The standard iso-parametric concept can be used otherwise; cf. [10].) Let n be the number of distinct vertices (nodes of rectangulation) of Ω_i, $i = 1, \dots, m$. Let A_i, $i = 1, \dots, n_{cn}$ be the contact nodes, i.e. distinct vertices of Ω_i, $i = 1, \dots, m$, lying on $\overline{\Gamma}_c$ (cf. Sec. 1).

Assumption. There exists a partition of $[a, b]$, namely $a = t_1 < t_2 < \dots < t_{n_{cn}} = b$ such that $A_i = (t_i, \psi(t_i))$, $i = 1, \dots, n_{cn}$, in the local coordinate system (ξ_1, ξ_2).

Let $\{\mathcal{R}_h\}_{h>0}$ be a regular family of rectangulations (cf. [10]) of $\overline{\Omega}$ (each of them formed by squares Ω_i). For each $\mathcal{R}_h$ let

$$\begin{aligned} V_h &:= \{v_h \in [C(\overline{\Omega})]^2 \colon\ v_h|_{\Omega_i} \in [Q_1(\Omega_i)]^2 \quad \text{for each } \Omega_i \in \mathcal{R}_h,\ \ v_h \in V\},\\ U_h &:= \{\varrho_h \in L^\infty(\Omega) \colon\ \varrho_h|_{\Omega_i} \in Q_0(\Omega_i) \quad \text{for each } \Omega_i \in \mathcal{R}_h\} \end{aligned}$$

($Q_i(\Omega_i)$ is the space of bilinear polynomials in Ω_i, $Q_0(\Omega_i)$ denotes constant functions in Ω_i, cf. [10]),

$$K_h := \{v_h \in V_h \colon\ \langle v_h(A_i), \xi(A_i)\rangle \leqslant \psi(t_i) - \varphi(t_i),\ \text{ for each } i = 1, \dots, n_{cn}\}$$

(t_i is a suitable ξ_1-coordinate in the local (ξ_1, ξ_2) coordinate system such that $(t_i, \psi(t_i)) = A_i$, $i = 1, \dots, n_{cn}$, cf. Sec. 1 and Sec. 5.1). It is readily seen that in general K_h is an external approximation of K. Further let

$$U_{\mathrm{ad},h} := \left\{ \varrho_h \in U_h \colon\ 0 \leqslant \varrho_h \leqslant \bar{t} \quad \text{a.e. in } \Omega,\ \int_\Omega \varrho_h(x)\,\mathrm{d}x \leqslant \overline{V} \right\} = U_{\mathrm{ad}} \cap U_h$$

(cf. Sec. 3). The discretization of the problem (3.3) is:

$$\begin{aligned} &\text{Find } \tilde{\varrho}_h \in U_{\mathrm{ad},h} \text{ and } \tilde{u}_h \in K_h \text{ such that} \\ (5.1)\qquad &\Pi(\tilde{\varrho}_h, \tilde{u}_h) = \min_{u_h \in K_h} \Pi(\tilde{\varrho}_h, u_h) = \max_{\varrho_h \in U_{\mathrm{ad},h}} \inf_{u_h \in K_h} \Pi(\varrho_h, u_h) \\ &\qquad = \max_{\varrho_h \in U_{\mathrm{ad},h}} \Pi(\varrho_h, \tilde{u}_h) = \min_{u_h \in K_h} \max_{\varrho_h \in U_{\mathrm{ad},h}} \Pi(\varrho_h, u_h). \end{aligned}$$

According to Lemma 2.2 the solution of this discrete problem solves also the discrete problem (4.1).

5.2. Verification of assumptions 4.1

The following lemma is important for the validity of assumption 4.1 (iii).

Lemma 5.1. *Let f be a continuous function defined in $[a, b]$, $a, b \in \mathbb{R}$, $a < b$. Let $D_n\colon\ a \equiv t_0^n < t_1^n < \ldots < t_n^n \equiv b$ be a partition of $[a, b]$ whose norm tends to zero as $n \to +\infty$. Let $\{\tau_n\}_{n=1}^\infty$ be a sequence of piecewise linear continuous functions such that*

$$\tau_n(t_i^n) \geqslant f(t_i^n) \quad \text{for } i = 0, \dots, n, \quad \text{and for all } n.$$

Let

$$\tau_n \to \tau \quad \text{for } n \to \infty \ \text{ a.e. in } [a, b].$$

Then

$$\tau \geqslant f \quad \text{a.e. in } [a, b].$$

The proof can be found in [13]. For the sake of simplicity, let the coordinate system (ξ_1, ξ_2) coincide with the Cartesian system (x_1, x_2). Then $\xi = (0, -1)$, and the non-penetrating condition is simplified to the inequality

$$u_2(x_1, \psi(x_1)) \geqslant \varphi(x_1) - \psi(x_1) \quad \text{(cf. Sec. 1)},$$

and thus

$$\begin{aligned} K &= \{v \in V\colon\ v_2(x_1, \psi(x_1)) \geqslant \varphi(x_1) - \psi(x_1),\ \text{for all } x_1 \in [a, b]\}, \\ K_h &= \{v_h \in V_h\colon\ v_{2,h}(t_i, \psi(t_i)) \geqslant \varphi(t_i) - \psi(t_i),\ \text{for each } i = 1, \dots, n_{cn}\} \end{aligned}$$

(cf. Sec. 1).

The verification of the assumption 4.1 (iv) is based on the next lemma on smooth approximations in K.

Lemma 5.2. *Let the coordinate system (ξ_1, ξ_2) coincide with the Cartesian system (x_1, x_2), as described above. Let a continuous function $\varphi\colon [a,b] \to \mathbb{R}$ have an extension $\tilde{\varphi}\colon \mathbb{R} \to \mathbb{R}$ such that*

(i) $\tilde{\varphi}|_{[a,b]} = \varphi$;

(ii) $\tilde{\varphi}$ *is sufficiently smooth;*

(iii) *there exists a neighborhood $\mathcal{U}_{\Gamma_0} \supset \Gamma_0$ such that $\mathcal{U}_{\Gamma_0} \subset R^C$, where*

$$R^C := \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_2 \geqslant \tilde{\varphi}(x_1),\ \ x_1 \in \mathbb{R}\}.$$

Then for any $v \in K$ there exists a sequence $\{\tilde{v}_\delta\}_{\delta \to 0+}$, $\tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2$ such that

$$\tilde{v}_\delta \to v \quad \text{for } \delta \to 0+ \text{ in } V.$$

P r o o f. The proof can be found in [18].

Proposition 5.1. *Let V_h, U_h, K_h and $U_{\mathrm{ad},h}$ be defined as described in Sec.* 5.1. *Let the assumptions of Lemma* 5.2 *be satisfied. Then the assumptions* 4.1 *are satisfied.*

P r o o f. *Step 1.* The assumption 4.1 (i) is satisfied by the choice of $U_{\mathrm{ad},h}$.

Step 2. For a given $\varrho \in U_{\mathrm{ad}}$ let

$$\varrho_i := \frac{1}{\operatorname{meas}(\Omega_i)} \int_{\Omega_i} \varrho(x)\,\mathrm{d}x \quad i = 1, \ldots, m$$

and

$$\varrho_h := \sum_{i=1}^{m} \varrho_i \chi_{\Omega_i},$$

where χ_{Ω_i} is the characteristic function of a square $\Omega_i \in \mathcal{R}_h$ (cf. Sec. 5.1). It is straightforward from the definition that $\varrho_h \in U_{\mathrm{ad},h}$. The validity of assumption 4.1 (ii) then follows from Lebesgue's point theorem (cf. [12]).

Step 3. Let the sequence $\{v_h\}_{h>0}$, $v_h \in K_h$, converge weakly in V. The trace operator $T\colon V \to [L^2(\Gamma_c)]^2$ is compact; therefore there exists a subsequence $\{v_{h'}\}_{h'>0} \subset \{v_h\}_{h>0}$ such that

$$v_{h'} \to v \quad \text{for } h' \to 0+ \text{ in } [L^2(\Gamma_c)]^2.$$

For every sequence converging in $[L^2(\Gamma_c)]^2$ there exists a subsequence which converges a.e. in Γ_c. Thus there is a subsequence $\{v_{h''}\}_{h''>0} \subset \{v_{h'}\}_{h'>0}$ such that in the local coordinate system (ξ_1, ξ_2)

$$v_{h''}([\xi_1, \psi(\xi_1)]) \to v([\xi_1, \psi(\xi_1)]) \quad \text{for } h'' \to 0+ \text{ a.e. in } [a, b].$$

Further, continuity of the scalar product gives that

$$\langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi\rangle \to \langle v([\xi_1, \psi(\xi_1)]), \xi\rangle \quad \text{for } h'' \to 0+ \text{ a.e. in } [a, b].$$

Because $v_{h''} \in K_{h''}$, the mapping $\xi_1 \mapsto \langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi\rangle$ is piecewise linear in $[a, b]$. Lemma 5.1 used for $\{-\langle v_{h''}([\xi_1, \psi(\xi_1)]), \xi\rangle\}_{h''>0}$ as the sequence $\{\tau_n\}_{n=1}^{\infty}$ and for $\varphi - \psi$ as the continuous function f (cf. Sec. 1) completes successfully the verification of assumption 4.1 (iii).

Step 4. Let v be an arbitrary element of K. According to Lemma 5.2 there exists a sequence $\{\tilde{v}_\delta\}_{\delta\to 0+}$, $\tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2$ such that

$$\tilde{v}_\delta \to v \quad \text{for } \delta \to 0+ \text{ in } V.$$

Classical approximation properties of V_h supply a piecewise bilinear interpolant $\pi_h \tilde{v}_\delta \in K_h$ such that

$$\pi_h \tilde{v}_\delta \to \tilde{v}_\delta \quad \text{in } V \text{ for } h \to 0+ \text{ for all } \tilde{v}_\delta \in K \cap [C^\infty(\overline{\Omega})]^2.$$

Moreover, thanks to the boundedness of the sequence $\{\tilde{v}_\delta\}_{\delta\to 0+}$, the convergence of $\{\{\pi_h \tilde{v}_\delta\}_{h>0}\}_{\delta\to 0+}$ is uniform in h. Then the sequence $\{v_h\}_{h>0}$, $v_h := \pi_h \tilde{v}_h$ satisfies assumption 4.1 (iv).

Step 5. Let

$$\varrho_h := \min\left\{\frac{\overline{V}}{\operatorname{meas}(\Omega)}, \hat{t}\right\}, \quad \text{in } \Omega \text{ for each } h > 0.$$

The set $\{\varrho_h\}_{h>0}$ together with constants

$$\alpha = 2, \quad N = \frac{1}{2\varepsilon}\|f\|^2_{[L^2(\Gamma_f)]^2} \quad \text{and } M = \frac{1}{2}\left(C_{\mathcal{K}} \min\left\{\frac{\overline{V}}{\operatorname{meas}(\Omega)}, \bar{t}\right\} - \varepsilon C_T\right)$$

verifies assumption 4.1 (v) (cf. Sec. 1.1 for definitions of $\overline{V}$ and $\bar{t}$). $C_{\mathcal{K}}$ comes from Korn's inequality (2.1). C_T is the norm of the trace operator

$$T\colon V \to [L^2(\Gamma_f)]^2, \quad Tu = u|_{\Gamma_f}.$$

And ε is an arbitrary real number such that

$$0 < \varepsilon < \frac{C_{\mathcal{K}}}{C_T} \min\left\{ \frac{\overline{V}}{\operatorname{meas}(\Omega)}, \bar{t} \right\}.$$

Correctness of the estimate in assumption 4.1 (v) is based on Korn's inequality and Young's inequality. Derivation of the estimate involves similar ideas to those used in the proof of Theorem 2.2. □

6. Numerical solution

Numerical solution of the problem (5.1) leads to a so-called semi-definite program (cf. [15]). It can be solved by modern interior point polynomial time methods (cf. [15]), as well as by penalty/barrier multipliers method (cf. [2]).

An example computed by penalty/barrier multipliers method is presented in Figs. 2–4. A similar example can be found in [14, p. 325], where variable sheet thickness approach is used, and also in [2], where free material optimization approach is used.

The domain Ω is a square fixed at the top. It is pulled at the bottom by a constant downward lineload traction. It is supported by a rigid foundation consisting of two parts located symmetrically around the vertical axis of symmetry. The function that determines the boundary of the left part of the obstacle is defined as $\varphi_1(x_1) = -6.4\, x_1^4$ in the Cartesian system with the origin at the lower left corner of the domain Ω. The function that determines the boundary of the right part of the obstacle is defined symmetrically (cf. Fig. 2).

The calculated values of the function ϱ_h (cf. Sec. 5.1) for the mesh 30×30 are shown in Fig. 3. The values of ϱ_h are depicted by gradations of grey, i.e. full black corresponds to high values of ϱ_h, white corresponds to ϱ_h equal zero, which can be interpreted as void material (cf. Sec. 1.1). Fig. 4 presents directions and magnitudes of principal stresses in the finite elements.

The interested reader can find more on numerical realization of the free material optimization in [2], [15], [20].

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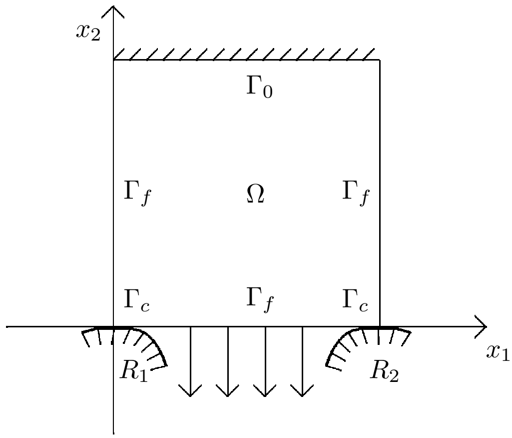

Figure 2. Boundary conditions.

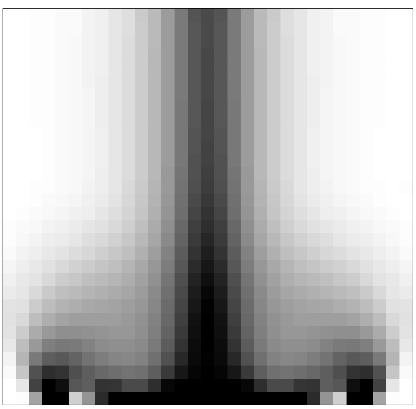

Figure 3. Values of the function ϱ_h.

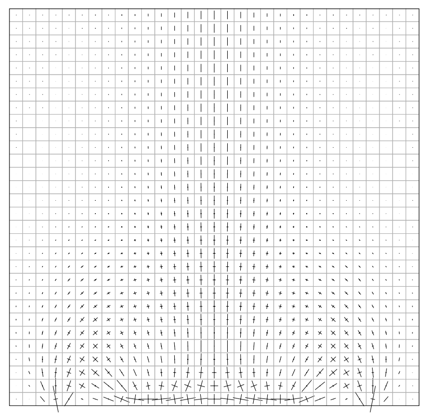

Figure 4. Principal stress directions and magnitudes.

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