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APPROXIMATION OF PERIODIC SOLUTIONS OF A SYSTEM OF PERIODIC LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

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Abstract. The present paper does not introduce a new approximation but it modifies a certain known method. This method for obtaining a periodic approximation of a periodic solution of a linear nonhomogeneous differential equation with periodic coefficients and periodic right-hand side is used in technical practice. However, the conditions ensuring the existence of a periodic solution may be violated and therefore the purpose of this paper is to modify the method in order that these conditions remain valid.

Keywords: oscillation problem, periodic differential equation, periodic solution, ω-periodic solution, trigonometric polynomial, trigonometric approximation, Gram's determinant

MSC 2000: 34C25, 42A10

1. INTRODUCTION

1.1. Preliminaries

Many oscillation problems lead to a periodic differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t), \tag{1.1}$$

where n is a positive integer, $y^{(k)} = \mathrm{d}^k y/\mathrm{d}t^k$, $k = 1, \ldots, n$, and $a_1, \ldots, a_n$, g are periodic continuous complex functions of the real variable $t \in \mathbb{R} = (-\infty, \infty)$ with a common positive period ω, or to a system of periodic linear differential equations (in matrix form)

$$\dot{x} = A(t)x + f(t), \tag{1.2}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, A is an ω-periodic continuous complex square matrix function of nth order and f is an ω-periodic continuous complex column matrix function of the real variable $t \in \mathbb{R}$ with n elements.

1.2. Technical practice

In technical practice given problems are simplified and the right-hand side in (1.1) or (1.2) is replaced by a few first terms of its Fourier expansion. Terms which are small in some sense are neglected. However, these simplifications can destroy conditions warranting the existence of a periodic solution of (1.1) or (1.2). The aim of this paper is not to introduce a new approximation method but to define conditions and an approach ensuring the well-posedness of the simplified problem. The justification of this method has not been examined in literature. Here we will show a constructive method based on an approximation of the right-hand side by a periodic trigonometric polynomial such that the necessary and sufficient condition for the existence of a periodic solution is preserved while the periodic solution of the simplified problem approximates uniformly on $\mathbb{R}$ the corresponding periodic solution of the original problem with arbitrary accuracy given in advance.

1.3. Some notions, notation and assertions

The set of all real numbers is denoted by $\mathbb{R}$. The symbol 0 denotes the number zero or the zero matrix the type of which is evident from the context. If C is a complex matrix then C^T is its transposed and C^* its transposed and complex conjugated matrix. If C is a square matrix then $\det C$ is its deteminant. The unit matrix of nth order is denoted by I_n or I. If r, m are two positive integers then by $\mathbb{C}^{r\times m}$ we denote the space of all (constant) complex matrices of the type $r \times m$, i.e. with r rows and m columns. The space $\mathbb{C}^{r\times m}$ becomes a Banach space (B-space) if it is equipped by one from the following norms $|\cdot|_1$, $|\cdot|_2$, $|\cdot|$ given for any matrix $A = (\alpha_{ij}) \in \mathbb{C}^{r\times m}$:

$$\begin{aligned}
|A|_1 &= \max\Big\{\sum_{j=1}^{m} |\alpha_{ij}|\colon\ i = 1,\dots,r\Big\},\\
|A|_2 &= \max\Big\{\sum_{i=1}^{r} |\alpha_{ij}|\colon\ j = 1,\dots,m\Big\},\\
|A| &= \max\{|A|_1, |A|_2\}.
\end{aligned}$$

In the sequel we will use the norm $|\cdot|$ for which $|A^*| = |A|$ and $|I| = 1$ is true. Here and in what follows let ω be a given positive number. A periodic function defined on $\mathbb{R}$ with the period ω is called ω-periodic.

If a complex matrix function φ is defined and continuous on the closed interval $[0,\omega]$ then the nonnegative number $|\varphi| = \sup|\varphi(t)| = \sup\{|\varphi(t)|\colon\ t \in [0,\omega]\}$ is the norm of the function φ. If the function φ has a continuous derivative $\dot{\varphi}$ on the closed interval $[0,\omega]$ then we define another norm $\|\varphi\| = \max\{|\varphi|, |\dot{\varphi}|\}$ of the function φ.

If k is a nonnegative integer and r, m positive integers then we denote by $C^kP_\omega^{r\times m}$ the space of all complex ω-periodic matrix functions with r rows and m columns which are defined and have ω-periodic continuous derivatives on $\mathbb{R}$ up to the kth order. For $k = 0$ or $m = 1$ or $r = 1$ we use the notation $CP_\omega^{r\times m} = C^0P_\omega^{r\times m}$, $C^kP_\omega^r = C^kP_\omega^{r\times 1}$, $CP_\omega^r = C^0P_\omega^r$, $CP_\omega = CP_\omega^1$. $CP_\omega^{r\times m}$ and $C^1P_\omega^{r\times m}$ are B-spaces with norms $|\cdot|$ and $\|\cdot\|$, respectively. In the space CP_ω^r we define an inner product by the formula $\langle u,v\rangle = \frac{1}{\omega}\int_0^\omega v^*(t)u(t)\,\mathrm{d}t$ and obtain the corresponding norm $|\!|u|\!| = \sqrt{\langle u,u\rangle}$. It is evident that the inequalities $|\!|u|\!| \leqslant |u|$ and $\langle u,v\rangle \leqslant |\!|u|\!|\cdot|\!|v|\!| \leqslant |u|\,|v|$ hold. In the case of constant functions u, v from CP_ω^r their inner product is $\langle u,v\rangle = v^*u$.

For brevity of notation in some operations with two arbitrary matrix functions g of a type $l\times k$ and h of a type $l\times m$ continuous on the interval $[0,\omega]$, where k, l, m are positive integers, we introduce their "inner product" $\langle g,h\rangle$ by the formula $\langle g,h\rangle = \frac{1}{\omega}\int_0^\omega h^*(t)g(t)\,\mathrm{d}t$. It is evident that $\langle g,h\rangle \in \mathbb{C}^{m\times k}$ and for constant f, g we have $\langle g,h\rangle = h^*g$. If $h = g$ holds then $\langle g,g\rangle \in \mathbb{C}^{k\times k}$ and $\det\langle g,g\rangle$ is a nonnegative number, which is Gram's determinant for the columns of g. If the columns of g are linearly independent then $\det\langle g,g\rangle$ is a positive number and the inverse matrix $\langle g,g\rangle^{-1}$ to $\langle g,g\rangle$ exists. Analogously the inequality $|\langle g,h\rangle| \leqslant |g|\,|h|$ is valid.

Let $\mathbf{E}$ be a B-space and k a positive integer. Given k mutually distinct real numbers $\lambda_1,\ldots,\lambda_k$ and k non-zero elements $b_1,\ldots,b_k$ from $\mathbf{E}$, the function Q defined by the formula $Q(t) = b_1\exp(\mathrm{i}\lambda_1 t) + \ldots + b_k\exp(\mathrm{i}\lambda_k t)$, $t\in\mathbb{R}$, is called an $\mathbf{E}$-trigonometric or simply a trigonometric polynomial ($\exp(s)$ denotes the usual exponential function e^s).

2. Approximation theorems

2.1. Weierstrass approximation theorem

Since we deal with uniform approximations of continuous ω-periodic functions by ω-periodic trigonometric polynomials, we will recall the following approximation theorems.

Theorem 2.1. *For every function $f\in CP_\omega$ and every positive number ε there exists a trigonometric polynomial $Q\in CP_\omega$ such that $|f-Q|\leqslant\varepsilon$ holds.*

Corollary 2.2. *For every matrix function f from $CP_\omega^{r\times m}$ and any positive number ε there exists a trigonometric polynomial $Q\in CP_\omega^{r\times m}$ such that $|f-Q|\leqslant\varepsilon$ holds.*

3.1. Homogeneous and conjugated equations

In what follows we shall treat only (1.2) since (1.1) can be modified to the form of (1.2) which induces the homogeneous equation

$$\dot{y} = A(t)y, \quad t \in \mathbb{R}, \tag{3.1}$$

and the conjugated equation with (3.1)

$$\dot{z} = -A^*(t)z, \quad t \in \mathbb{R}. \tag{3.2}$$

The conditions given above in 1.1 ensure the existence and uniqueness of a solution of the Cauchy problem for the equations (1.1), (1.2), (3.1), (3.2).

Let $Y = Y(t)$, $t \in \mathbb{R}$, be a fundamental matrix (of solutions) of (3.1) and let $Z = Z(t)$, $t \in \mathbb{R}$, be a fundamental matrix of (3.2). The validity of the following relations can be easily verified: $Y(t) = (Z^*(t))^{-1}Z^*(0)Y(0)$, $Y^{-1}(t) = Y^{-1}(0)(Z^*(0))^{-1}Z^*(t)$, $t \in \mathbb{R}$.

3.2. Periodic solutions

If x is an ω-periodic solution of (1.2) then the equality

$$x(\omega) - x(0) = 0 \tag{3.3}$$

holds. This equality is not only a necessary but also a sufficient condition for the ω-periodicity of the solution x of (1.2).

If H is a fundamental matrix of (3.1) then the general solution x of (2.1) can be written in the form

$$x = x(t) = H(t)H^{-1}(0)x_0 + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R} \ \ (x(0) = x_0). \tag{3.4}$$

Remark 3.1. Note that owing to the validity of the relations $\tilde{H}(t)\tilde{H}^{-1}(s) = H(t)H^{-1}(s)$ for any real numbers s, t and arbitrary fundamental matrices H, $\tilde{H}$ of (3.1) the integral part $H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, of a solution of (1.2) stays unchanged for every given f from CP_ω^n and for an arbitrary fundamental matrix H of (3.1).

The condition (3.3) has now the form

$$0 = x(\omega) - x(0) = (H(\omega) - H(0))H^{-1}(0)x_0 + H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s,$$

i.e.

$$(H(\omega) - H(0))H^{-1}(0)x_0 = -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s \tag{3.5}$$

with the unchanged right-hand side for every given f and for an arbitrary fundamental matrix H of (3.1), owing to Remark 3.1. The system (3.5) has n linear algebraic equations with n unknowns which are elements of the initial matrix-column $x_0 \in \mathbb{C}^{n\times 1}$ and it can be arranged into the system

$$(3.5^*) \qquad (H(\omega) - H(0))u_0 = -(Z^*(\omega))^{-1} \langle f, Z\rangle \, \omega$$

with the unknown matrix-column $u_0 = H^{-1}(0)x_0 \in \mathbb{C}^{n\times 1}$ ($x_0 = H(0)u_0$) and an arbitrary fundamental matrix Z of (3.2) (with unchanged right-hand side independent of Z).

Definition 3.2.

a) We denote by B_1 the space of all functions $f \in CP^n_\omega$ for which Equation (1.2) has an ω-periodic solution and denote by B_2 the space $C^1P^n_\omega$.

b) For a given function f from B_1 we denote by V_f the space of all ω-periodic solutions of Equations (1.2). For $f = 0$ we have the space V_0 of all ω-periodic solutions of Equation (3.1).

c) We denote by V_0^* the space of all ω-periodic solutions of Equation (3.2).

Remark 3.3. The spaces B_1, V_0 and V_0^* are B-spaces.

If (3.5) or (3.5*) has a solution x_0 then (3.3) is for this x_0 fulfilled and (1.2) has an ω-periodic solution (3.4).

It is known the two following valid theorems.

Theorem 3.4. *The necessary and sufficient condition for the existence of an ω-periodic solution of Equation* (1.2) *is the validity of the equality*

$$(3.6) \qquad \langle f, z\rangle = \frac{1}{\omega}\int_0^\omega z^*(t)f(t)\,\mathrm{d}t = 0$$

for every ω-periodic solution z of Equation (3.2), *i.e. for every z from V_0^*.*

Theorem 3.5. *The* B-*space V_0 has the same finite dimension as the* B-*space V_0^*.*

4. Transformations

4.1. B-spaces

Recall that the spaces CP^n_ω and B_1 with the norm $|\cdot|$ and the space $B_2 = C^1P^n_\omega$ with the norm $\|\cdot\|$ are B-spaces. The space B_2 can be viewed as the space of all ω-periodic solutions of (1.2) with any right-hand sides from B_1. Namely, if x is from B_2 then x is an ω-periodic solution of (1.2) with the right-hand side $f = \dot{x} - Ax$.

4.2. Fundamental matrix $H = (H', H'')$

Definition 4.1. The set of all fundamental matrices H of Equation (3.1) whose first k columns create a basis of V_0 with $k = \dim V_0$ is denoted by W_0. For every H from W_0 we introduce its decomposition $H = (H', H'')$ where H' is from $C^1 P_\omega^{n\times k}$. The set of all fundamental matrices Z of Equation (3.2) whose first k columns create a basis of V_0^* is denoted by W_0^*.

4.3. Periodic solutions

For any f from B_1 and an arbitrary $H = (H', H'')$ from W_0 every ω-periodic solution x from V_f can be expressed in the form

$$x = x(t) = H'(t)u_0' + H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1}$$

where $x_0 = x(0) = x(\omega)$, $u_0' \in \mathbb{C}^{k\times 1}$, $u_0'' \in \mathbb{C}^{(n-k)\times 1}$, $u_0 = (u_0'^T, u_0''^T)^T = H^{-1}(0)x_0$, $x_0 = H(0)u_0$.

In this section and further we use a fixed given fundamental matrix $H = (H', H'')$ from W_0, unless stated otherwise. Let a function f from B_1 be given. If we have a particular ω-periodic solution

$$u = u(t) = H'(t)u_0' + H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1'}$$

with fixed u_0', u_0'' then we decompose this solution into a "homogeneous part" $u_H = H'u_0'$ and a "particular part"

$$u_f = u_f(t) = H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

So we have $u = u_H + u_f$ where $u_H \in V_0$ and $u_f \in V_f$. The part u_f is uniquely determined by $f \in B_1$. Indeed, if we have another particular ω-periodic solution

$$\tilde{u} = \tilde{u}(t) = H'(t)\tilde{u}_0' + H''(t)\tilde{u}_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1''}$$

with fixed $\tilde{u}_0' \in \mathbb{C}^{k\times 1}$, $\tilde{u}_0'' \in \mathbb{C}^{(n-k)\times 1}$, then evidently the ω-periodic solution $u - \tilde{u} - u_H + \tilde{u}_H = H''(u_0'' - \tilde{u}_0'')$ is from V_0. Since the columns of H' create a basis of V_0 the equality $\tilde{u}_0'' = u_0''$ necessarily holds. This uniqueness of u_0'' we record by $u_0'' = u_0''(f)$. So we get:

Lemma 4.2. *For every function f from B_1 all solutions from V_f have the same "particular part".*

Now we complete the solution of (3.5*) for u_0. Since $H'(\omega) - H'(0) = 0$ we get from (3.5*) the equivalent system

$$(H''(\omega) - H''(0))u_0'' = -(Z^*(\omega))^{-1} \langle f, Z \rangle \omega \tag{3.5**}$$

in which u_0' does not occur, i.e. the k elements of u_0' play here the role of k arbitrary complex parameters independent of $f \in B_1$. (If we have $u_0'' = u_0''(f)$ then it follows from (3.5**) that $x_0 = H(0)u_0 = H'u_0' + H''u_0''$ remains unchanged for all $u_0' \in \mathbb{C}^{k\times 1}$.) For shortness of the record we denote by κ the matrix $H''(\omega) - H''(0)$. The $n-k$ columns of κ are linearly independent. This follows besides other from the uniqueness of $u_0''(f)$. So Gram's determinant $\det \langle \kappa, \kappa \rangle = \det(\kappa^* \kappa)$ is a positive number and the inverse matrix $\langle \kappa, \kappa \rangle^{-1}$ to $\langle \kappa, \kappa \rangle = \kappa^* \kappa$ exists. If we multiply (3.5**) by the matrix $\langle \kappa, \kappa \rangle^{-1} \kappa^*$ from the left then we get

$$u_0'' = u_0''(f) = -\langle \kappa, \kappa \rangle^{-1} \kappa^* (Z^*(\omega))^{-1} \langle f, Z \rangle \omega \tag{4.2}$$

for $Z \in V_0^*$ and the estimates

$$|u_0''(f)| \leqslant 2\omega |H|\,|Z|\,|Z^{-1}|\,\langle \kappa, \kappa \rangle^{-1}\,|\,|f|, \tag{4.3}$$

$$|u_0''(f)| \leqslant 2\omega |H|^2 |H^{-1}|\,|\,\langle \kappa, \kappa \rangle^{-1}\,|\,|f| \tag{4.3'}$$

hold since $|\kappa| = |\kappa^*| \leqslant 2|H|$. The estimate (4.3′) follows from

$$(H''(\omega) - H''(0))u_0'' = H(\omega) \int_0^\omega H^{-1}(s) f(s)\, \mathrm{d}s \tag{3.5'}$$

in the analogous way as (4.3) from (3.5**).

4.4. Transformations

On the basis of the above results we obtain the following assertions.

Lemma 4.3. *The map $\mathcal{U}\colon B_1 \to \mathbb{C}^{(n-k)\times 1}$ given by the formula $\mathcal{U} f = u_0''(f)$ for any f from B_1 is a linear and bounded operator.*

Proof. The linearity of $\mathcal{U}$ is evident from (4.2). The boundedness of $\mathcal{U}$ follows for a fixed given Z from (4.3) and from (4.3′) by the choice of the positive constant

$$K_0 = 2\omega |H|\,|\,\langle \kappa, \kappa \rangle^{-1}\,|\max\{|Z|\,|Z^{-1}|,\ |H|\,|H^{-1}|\} \tag{4.4}$$

for the estimate

$$|\mathcal{U} f| \leqslant K_0 |f| \tag{4.5}$$

for every f from B_1, where K_0 does not depend on f. □

Theorem 4.4. *If a map $\mathcal{A}\colon B_1 \to V_f$ is defined so that for every f from B_1 its image $\mathcal{A}f$ is equal to the common "particular part" of any solution u from V_f, i.e.*

$$(\mathcal{A}f)(t) = H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.6}$$

then the operator $\mathcal{A}$ is linear and bounded.

Proof. The linearity of $\mathcal{A}$ is evident from Lemma 4.3 and from (4.6). Further, an estimate

$$|\mathcal{A}f| \leqslant K_1|f| \tag{4.7}$$

holds owing to (4.5) and (4.6) for $K_1 = (K_0 + \omega|H^{-1}|)|H|$, where K_0 is from (4.4). The constant K_1 does not depend on f. □

Remark 4.5. If $H = (H', H'')$ and $\tilde{H} = (\tilde{H}', \tilde{H}'')$ are two arbitrary fundamental matrices from W_0 then by virtue of the properties of the matrices H', H'' and $\tilde{H}'$, $\tilde{H}''$ there exist regular matrices $C_1 \in \mathbb{C}^{k\times k}$, $C_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ and $C = \operatorname{diag}(C_1, C_2)$ such that $\tilde{H}' = H'C_1$, $\tilde{H}'' = H''C_2$ and $\tilde{H} = HC$.

Theorem 4.6. *The operator $\mathcal{A}$ from Theorem 4.4 does not depend on the choice of a fundamental matrix H from W_0.*

Proof. Let f be a given function from B_1. Let two arbitrary fundamental matrices $H = (H', H'')$ and $\tilde{H} = (\tilde{H}', \tilde{H}'')$ from W_0 be given. Assume that besides a particular ω-periodic solution u from (4.1′) belonging to V_f there exists another particular ω-periodic solution $\tilde{u} = \tilde{u}(t) = \tilde{H}'(t)\tilde{u}_0' + \tilde{H}''(t)\tilde{u}_0'' + \tilde{H}(t)\int_0^t \tilde{H}^{-1}(s)f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, from V_f. Owing to Remark 3.1 and Remark 4.5 the relations $\tilde{H}(t)\int_0^t \tilde{H}^{-1}(s)f(s)\,\mathrm{d}s = H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s$ are true for any $t \in \mathbb{R}$, so that the solution $u - \tilde{u} - u_H + \tilde{u}_H = H''u_0'' - \tilde{H}''\tilde{u}_0'' = H''(u_0'' - C_2\tilde{u}_0'')$ belongs to V_0. But this is possible only for $u_0'' = C_2\tilde{u}''$, i.e. $\tilde{u}_0'' = C_2^{-1}u_0''$, $\tilde{H}''\tilde{u}_0'' = H''C_2C_2^{-1}u_0'' = H''u_0''$ so that $\tilde{u}_f = u_f$. □

Corollary 4.7. *For every function f from B_1 the space V_f has the representation $V_f = \mathcal{A}f + V_0$, i.e. V_f is a k-dimensional manifold in $B_2 = C^1P_\omega^n$ and $B_2 = \bigcup\limits_{f\in B_1} V_f$.*

Consider now a map $\mathcal{B} = \mathcal{B}(\cdot,\cdot)$ defined on $B_1 \times B_1$ the values of which are one-to-one operators $\mathcal{B}(f,g)\colon V_f \to V_g$ for any two functions f, g from B_1 and which satisfies $\mathcal{B}^{-1}(f,g) = \mathcal{B}(g,f)\colon V_g \to V_f$. If two functions f, g from B_1 are given then with regard to our intention to approximate elements $u = u_H + \mathcal{A}f$ from V_f by their images $\mathcal{B}(f,g)u = \tilde{u} = \tilde{u}_H + \mathcal{A}g$ from V_g we choose the simplest way and define the approximation by the formula

$$\mathcal{B}(f,g)u = \tilde{u} = u_H + \mathcal{A}g \in V_g \quad \text{for } u \in V_f \tag{4.8}$$

(i.e. $\tilde{u}_H = u_H$). For another definition of $\mathcal{B}(f,g)$ we could solve optimalization problems. However, for our purposes we can always approximate u by $\mathcal{B}(f,g)u$ more precisely so that the approximation of f by g will be also more precise as we shall see in the sequel.

Owing to (4.7) and (1.2) we get the estimates

$$|u - \mathcal{B}(f,g)u| = |\mathcal{A}(f-g)| \leqslant K_1|f-g|, \tag{4.9}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(u - \mathcal{B}(f,g)u)\right| = |A(u - \mathcal{B}(f,g)u) + f - g| \leqslant (|A|K_1 + 1)|f-g|, \tag{4.10}$$

$$\|(u - \mathcal{B}(f,g)u\| \leqslant K|f-g| \tag{4.11}$$

for any u from V_f with the positive constant $K = \max\{K_1, 1 + |A|K_1\}$ independent of f and g from B_1. ($\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}f = A\mathcal{A}f + f$ and $u - \mathcal{B}(f,g)u = \mathcal{A}(f-g)$.)

Remark 4.8. From the estimates (4.9) and (4.11) it follows that for g sufficiently close to f in B_1 also $\tilde{u} = \mathcal{B}(f,g)u$ is sufficiently close to u in B_2. The accuracy of such approximation can be chosen arbitrarily small positive. (For any positive number η there exists a positive number ε such that $|f-g| \leqslant \varepsilon$ implies $\|u - \mathcal{B}(f,g)u\| \leqslant K|f-g| \leqslant K\varepsilon \leqslant \eta$ for $\varepsilon \leqslant \eta/K$).

5. Approximation of periodic solution

5.1. Auxiliary assertions

In the sequel we build up the results obtained and the notation and suppose that we have a fixed given fundamental matrix $H = (H', H'')$ from W_0.

Let k be a positive integer. The space $\mathbb{C}^{k\times k}$ equipped with the norm $|\cdot|$ and the usual arithmetic operations, addition and multiplication, for its elements (in addition to the multiplication of these elements by scalars) is a Banach algebra with the unit $I = I_k$ ($|I| = 1$ and $|AB| \leqslant |A|\,|B|$ for any A, B from $\mathbb{C}^{k\times k}$).

The following two assertions are well-known from the theory of Banach algebras.

Lemma 5.1. *If for a matrix B from $\mathbb{C}^{k\times k}$ the inequality $|I-B|<1$ holds then in $\mathbb{C}^{k\times k}$ the inverse matrix B^{-1} to B exists ($B^{-1}=I+\sum\limits_{m=1}^{\infty}(I-B)^m$). Moreover, the estimates*

$$|I-B^{-1}|\leqslant\frac{|I-B|}{1-|I-B|},\quad |B^{-1}|\leqslant\frac{1}{1-|I-B|} \tag{5.1}$$

are valid.

Corollary 5.2. *Let matrices A, B be from $\mathbb{C}^{k\times k}$ and let the inverse matrix A^{-1} to A exist in $\mathbb{C}^{k\times k}$. If the inequality $|A-B|<|A^{-1}|^{-1}$ holds then in $\mathbb{C}^{k\times k}$ the inverse matrix B^{-1} to B exists. Moreover, the estimates*

$$|A^{-1}-B^{-1}|\leqslant|A^{-1}|^2\frac{|A-B|}{1-|A^{-1}|\,|A-B|},\quad |B^{-1}|\leqslant\frac{|A^{-1}|}{1-|A^{-1}|\,|A-B|} \tag{5.2}$$

are valid ($B^{-1}=A^{-1}+\sum\limits_{m=1}^{\infty}(I-A^{-1}B)^mA^{-1}$).

Remark 5.3. If a nonnegative number ϱ is such that for matrices A, B from Corollary 5.2 the equality $|A-B|=\varrho|A^{-1}|^{-1}<|A^{-1}|^{-1}$ holds then the estimates

$$|A^{-1}-B^{-1}|\leqslant\frac{\varrho|A^{-1}|}{1-\varrho},\quad |B^{-1}|\leqslant\frac{|A^{-1}|}{1-\varrho} \tag{5.2$'$}$$

are true. If we require that $\varrho\in\left[0;\frac{1}{2}\right]$ hold, i.e. $|A-B|\leqslant\frac{1}{2}|A^{-1}|^{-1}$, then we get estimates

$$|A^{-1}-B^{-1}|\leqslant|A^{-1}|,\quad |B^{-1}|\leqslant 2|A^{-1}|. \tag{5.2$''$}$$

Lemma 5.4. *Let a positive integer k be given. For a positive number ε and functions $f,z_1,\dots,z_k$ from CP_ω^n there exists a trigonometric polynomial Q_ε from CP_ω^n such that the relations*

$$|f-Q_\varepsilon|\leqslant\varepsilon, \tag{5.3}$$

$$\langle f,z_j\rangle=\langle Q_\varepsilon,z_j\rangle,\quad j=1,\dots,k, \tag{5.4}$$

are valid.

Proof. We can assume that the functions $z_1,\dots,z_k$ are linearly independent and we denote by Z_1 the matrix $(z_1,\dots,z_k)\in CP_\omega^{n\times k}$. The equalities (5.4) have now the form

$$\langle f,Z_1\rangle=\langle Q_\varepsilon,Z_1\rangle. \tag{5.4$'$}$$

Further, we define the matrix function $M = M(\Delta Z) = \langle Z_1 + \Delta Z, Z_1\rangle$ for $\Delta Z \in CP_\omega^{n\times k}$ and denote $M_0 = M(0) = \langle Z_1, Z_1\rangle$. Since $|M_0 - M| = |\langle \Delta Z, Z_1\rangle| \leqslant |\Delta Z|\,|Z_1|$, the estimates $|M_0^{-1} - M^{-1}| \leqslant |M_0^{-1}|$ and $|M^{-1}| \leqslant 2|M_0^{-1}|$ are true by virtue of (5.2″) for $|\Delta Z| \leqslant \frac{1}{2}|Z_1|^{-1}\,|M_0^{-1}|^{-1}$, ($\det M_0 = \det\langle Z_1, Z_1\rangle$ is a positive number). Let ε_0 be a positive number the magnitude of which will be determined later and let ε_1 be a positive number satisfying the inequality $\varepsilon_1 \leqslant \frac{1}{2}|Z_1|^{-1}\,|M_0^{-1}|^{-1}$. Let $Q_0 \in CP_\omega^n$ and $Q_1 \in CP_\omega^{n\times k}$ be trigonometric polynomials such that $|f - Q_0| \leqslant \varepsilon_0$ and $|Z_1 - Q_1| \leqslant \varepsilon_1$. The existence of Q_0 and Q_1 follows from Corollary 2.2. We will find $q \in \mathbb{C}^{k\times 1}$ such that the trigonometric polynomial $Q_\varepsilon = Q_0 + Q_1 q$ from CP_ω^n satisfies the relations (5.3) and (5.4′). That means that $\langle f, Z_1\rangle = \langle Q_\varepsilon, Z_1\rangle = \langle Q_0, Z_1\rangle + \langle Q_1, Z_1\rangle\, q$. Consequently, $\langle Q_1, Z_1\rangle\, q = \langle f - Q_0, Z_1\rangle$. Note that $Q_1 = Z_1 + (Q_1 - Z_1) = Z_1 + \Delta Z$, $\Delta Z = Q_1 - Z_1$, $|\Delta Z| \leqslant \varepsilon_1$. So $\langle Q_1, Z_1\rangle = M = M(Q_1 - Z_1)$ and $|M_0 - M| = |M_0 - M(Q_1 - Z_1)| \leqslant |Q_1 - Z_1|\,|Z_1| \leqslant \frac{1}{2}|M_0^{-1}|^{-1} < |M_0^{-1}|^{-1}$. According to Lemma 5.2 the inverse matrix $M^{-1} = M^{-1}(Q_1 - Z_1)$ to M exists for which owing to Remark 5.3 the estimate $|M^{-1}| \leqslant 2|M_0^{-1}|$ is valid.

So we get $q = M^{-1}\langle f - Q_0, Z_1\rangle$ which ensures the validity of (5.4′). The inequalities $|Q_1| \leqslant |Z_1| + \varepsilon_1 \leqslant |Z_1| + 1/(2|Z_1|\,|M_0^{-1}|) = (2|Z_1|^2|M_0^{-1}| + 1)/(2|Z_1|\,|M_0^{-1}|)$, $|q| \leqslant 2|M_0^{-1}|\,|Z_1|\,|f - Q_0| \leqslant 2|Z_1|\,|M_0^{-1}|\varepsilon_0$ hold. Finally, we have $|f - Q_\varepsilon| \leqslant |f - Q_0| + |Q_1|\,|q| \leqslant 2\varepsilon_0(1 + |Z_1|^2\,|M_0^{-1}|) \leqslant \varepsilon$ if $\varepsilon_0 \leqslant \varepsilon/(2(1 + |Z_1|^2|M_0^{-1}|))$, which ensures the validity of (5.3). This completes the proof. □

5.2. Approximations

We can already start to construct an approximation of an ω-periodic solution of (1.2).

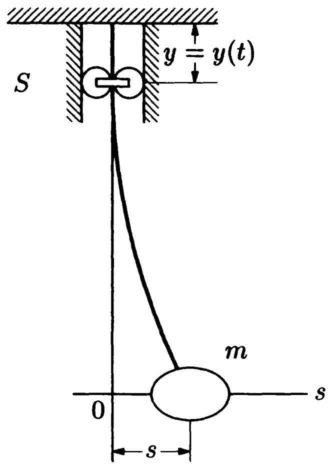
Theorem 5.5. *Let a given function f be from B_1 and let η be a given positive number. There exists a trigonometric polynomial $g = g_\eta$ from B_1 such that for every solution u from V_f its image $\tilde{u}$ under the map $\mathcal{B}(f, g)\colon V_f \to V_g$ from (4.8) satisfies the estimate $\|u - \tilde{u}\| = \|u - \mathcal{B}(f, g)u\| \leqslant \eta$.*

Proof. Let a fixed fundamental matrix $Z = (Z_1, Z_2)$ from W_0^* be given, $Z_1 \in C^1P_\omega^{n\times k}$, $k = \dim V_0 = \dim V_0^*$. According to Lemma 5.4, for any positive number ε there exists a trigonometric polynomial Q_ε from CP_ω^n satisfying (5.3) and (5.4′) which ensures $Q_\varepsilon \in B_1$. By means of the choice $\varepsilon = \eta/K$ we get $\|u - \tilde{u}\| \leqslant \eta$ if we put $g = g_\eta = Q_\varepsilon$ in (4.11), where K is from Remark 4.8. □

Thus we have reached the aim of the present paper to justify the approximation method for periodic solutions used in technical practice.

6.1. Application

As an application of a linear differential equation with periodic coefficients and with a periodic right-hand side we can consider the simple technical device which is described in [8]. This device is shown in the figure where we have a mass m attached to a thin cantilever linear spring, the effective length of which can vary with time t in any prescribed manner ($y = y(t)$) by moving the support S. Thus in discussing lateral vibrations of the mass m, we have a spring characteristic c that varies with time t (controlled by $y(t)$):



We have the equation

$$m\ddot{s} = -c(t)s - a\dot{s} + Q(t) \tag{6.1}$$

with variable coefficients where s is the displacement of the mass m from its central position. For a periodic motion of the support S and for a periodic external force Q with the same positive period ω, if we neglect the resistance of air, i.e. $a = 0$, then we have the ω-periodic equation of motion in the form

$$\ddot{s} + k(t)s = g(t) \tag{6.2}$$

where $k(t) = c(t)/m$, $g(t) = Q(t)/m$, $t \in \mathbb{R}$.

6.2. Illustration

If there exists an ω-periodic nonconstant function $F = F(t)$ with continuous derivatives up to and including the second order on $\mathbb{R}$ which satisfies the nonlinear equation $\dot{F}^2 + \ddot{F} = -k(t)$ then the homogeneous linear equation

$$\ddot{s} + k(t)s = 0 \tag{6.3}$$

has the general solution in the form

$$s = C_1 \exp(F) + C_2 \exp(F) \cdot G \tag{6.4}$$

where $G = G(t) = \int_0^t \exp(-2F(\sigma))\,\mathrm{d}\sigma$, $t \in \mathbb{R}$.

Note that the function G is increasing because its first derivative $\exp(-2F)$ is positive with $G(0) = 0$. Thus G is not periodic and $G(\omega) > 0$. The motion equation of the technical device from the figure can be transformed into the system of linear equations

$$\dot{x} = A(t)x + f(t) \tag{6.5}$$

(in matrix notation) where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -k(t) & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}, \quad f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

With System (6.5) we associate the homogeneous system

$$\dot{y} = A(t)y \tag{6.6}$$

and the adjoint homogeneous system

$$\dot{z} = -A^*(t)z. \tag{6.7}$$

Here y, z are matrix-columns with two elements and $A^*(t) = A^T(t)$ because the matrix $A(t)$ is real for every $t \in \mathbb{R}$.

We choose the matrix

$$H = H(t) = \begin{pmatrix} \exp(F), & \exp(F)G \\ \dot{F}\exp(F), & \dot{F}\exp(F)G + \exp(-F) \end{pmatrix}$$

as the fundamental matrix H of the system (6.6) from Section 4.3. We denote by h_1, h_2 the columns of H, i.e. $H = (h_1, h_2)$. Because $\det H = 1$, the inverse matrix has the form

$$H^{-1} = H^{-1}(t) = \begin{pmatrix} \dot{F}\exp(F)G + \exp(-F), & -\exp(F)G \\ -\dot{F}\exp(F), & \exp(F) \end{pmatrix}.$$

We denote by z_1, z_2 the rows of the inverse matrix H^{-1}, i.e. $H^{-1} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. The matrix H is real and therefore $(H^*)^{-1} = (H^{-1})^T$ which is a fundamental matrix of solutions of System (6.7).

The columns h_1, z_2^T are ω-periodic solutions of Systems (6.6), (6.7), respectively. Since the function G is not periodic the columns h_2, z_1^T are non-periodic solutions of Systems (6.6), (6.7), respectively. (h_1, z_2^T form bases of the sets of all ω-periodic solutions of Systems (6.6), (6.7), respectively.) Hence, if $\int_0^\omega \exp(F(t))g(t)\,\mathrm{d}t = 0$, i.e. $\langle f, z_2^T \rangle = 0$, then the condition (3.6) is fulfilled and therefore the existence of an ω-periodic solution of System (6.5) is ensured. According to the previous reasoning the function

$$\begin{aligned} \mathcal{A}f(t) &= H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s \\ &= h_2(t)u_{02} + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \end{aligned}$$

where $u_0'' = u_{02}$, satisfies the condition $\mathcal{A}f(\omega) - \mathcal{A}f(0) = 0$. Hence the linear algebraic system

$$[h_2(\omega) - h_2(0)]u_{02} = -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s \tag{6.8}$$

is fulfilled because the equalities

$$\begin{aligned} h_2(\omega) - h_2(0) &= \begin{pmatrix} 1 \\ \dot{F}(\omega) \end{pmatrix} \exp(F(\omega))G(\omega), \\ -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s &= H(\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s \\ &= \begin{pmatrix} 1 \\ \dot{F}(\omega) \end{pmatrix} \exp(F(\omega))\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s \end{aligned}$$

(due to $\langle f, z_2^T \rangle = 0$) are valid and the linear algebraic system (6.8) has the solution $u_{02} + G(\omega)^{-1}\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s$.

By virtue of $|H| = |H^{-1}|$ and $|f| = |g|$ we get the inequalities

$$\begin{aligned} |u_{02}| &\leqslant |g|\int_0^\omega \exp(F(s))\,\mathrm{d}s, \\ |\mathcal{A}f| &\leqslant \tilde{K}|g| = \tilde{K}|f|, \\ \|\mathcal{A}F\| &\leqslant \tilde{K}|g| = \tilde{K}|f|, \end{aligned}$$

where $\tilde{K} = |h_2|\int_0^\omega \exp(F(s))\,\mathrm{d}s$ and $K = \max\{\tilde{K}, |A|\tilde{K} + 1\}$.

6.3. Example

Now we consider the case $F(t) = 2\cos t$, $g(t) = a_0 + \cos t + \varepsilon\cos(qt)$, where a_0 is a real number, q is a positive integer greater than one, ε is a real non-zero number with a very small absolute value and $\omega = 2\pi$. First we construct the Fourier series of the function $\exp(F)$. Notice that $F(t) = 2\cos t = \exp(it) + \exp(-it)$, hence

$$\begin{aligned}
\exp(F(t)) &= \exp(\exp(it) + \exp(-it)) = \sum_{p=0}^{\infty} \frac{1}{p!}(\exp(it) + \exp(-it))^p \\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{\exp(i(p-2k)t)}{(p-k)!\,k!} = \mathrm{Re}(\exp(F)) = \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{\cos(p-2k)t}{(p-k)!\,k!} \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{p=0}^{\infty}\left[\sum_{k=0}^{p} \frac{\cos(2p-2k)t}{(2p-k)!\,k!} + \sum_{k=0}^{p} \frac{\cos(2p+1-2k)t}{(2p+1-k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{k=0}^{\infty}\left[\sum_{p=k}^{\infty} \frac{\cos(2p-2k)t}{(2p-k)!\,k!} + \sum_{p=k}^{\infty} \frac{\cos(2p+1-2k)t}{(2p+1-k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty} \frac{\cos 2pt}{(2p+k)!\,k!} + \sum_{p=0}^{\infty} \frac{\cos(2p+1)t}{(2p+1+k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{p=0}^{\infty}\left[\left\{\sum_{k=0}^{\infty} \frac{1}{(2p+k)!\,k!}\right\}\cos 2pt\right. \\
&\quad \left. + \left\{\sum_{k=0}^{\infty} \frac{1}{(2p+1+k)!\,k!}\right\}\cos(2p+1)t\right] \\
&= -\alpha_0 + 2\sum_{p=1}^{\infty} \alpha_p \cos pt
\end{aligned}$$

(by virtue of the absolute convergence of this series for every $t \in \mathbb{R}$), where

$$\alpha_{2p} = \sum_{k=0}^{\infty} \frac{1}{(2p+k)!\,k!}, \quad \alpha_{2p+1} = \sum_{k=0}^{\infty} \frac{1}{(2p+k+1)!\,k!}, \quad p = 0, 1, 2, \ldots.$$

Let there exist an ω-periodic solution of System (6.5). Then the equality $a_0 = -(\alpha_1 + 2\alpha_q)/\alpha_0$ follows from the validity of the condition (3.6). In a technical practice the "very small" term $\varepsilon\cos qt$ of the right-hand side g could be neglected and g could be replaced by the trigonometric polynomial $a_0 + \cos t$. This procedure, however, destroys the former valid necessary and sufficient condition for the existence of a 2π-periodic solution of System (1.2). However, if g is replaced by the trigonometric polynomial $g_\eta(t) = \tilde{a}_0 + \cos t$ with $\tilde{a}_0 = -(\alpha_1/\alpha_0)a_0$, which is equally simple as $a_0 + \cos t$, then the existence of a 2π-periodic solution is preserved, while $|g - g_\eta| \leqslant$

$\eta \leqslant |\varepsilon|(1+\alpha_q/\alpha_0)$, $\|\mathcal{A}g - \mathcal{A}g_\eta\| \leqslant K|g - g_\eta| \leqslant K|\varepsilon|(1+\alpha_q/\alpha_0) < 2|\varepsilon|K$, because $2 < \alpha_0$, $2 \leqslant q$, $0 < \alpha_q < \mathrm{e}/q! < 2$. (Here e is the Euler's number.)

6.4. Conclusion

Recently, approximation methods for periodic solutions have concerned nonlinear differential equations. In the papers [9], [10] numerical methods supported by computers are presented. [5], [7] deal with Fourier approximations of periodic solutions of nonlinear differential equations. Finally, the paper [1] generalizes this approximation problem to Banach spaces.

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APPROXIMATION OF PERIODIC SOLUTIONS OF A SYSTEM OF PERIODIC LINEAR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

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Abstract. The present paper does not introduce a new approximation but it modifies a certain known method. This method for obtaining a periodic approximation of a periodic solution of a linear nonhomogeneous differential equation with periodic coefficients and periodic right-hand side is used in technical practice. However, the conditions ensuring the existence of a periodic solution may be violated and therefore the purpose of this paper is to modify the method in order that these conditions remain valid.

Keywords: oscillation problem, periodic differential equation, periodic solution, ω-periodic solution, trigonometric polynomial, trigonometric approximation, Gram's determinant

MSC 2000: 34C25, 42A10

1. Introduction

1.1. Preliminaries

Many oscillation problems lead to a periodic differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y = g(t), \tag{1.1}$$

where n is a positive integer, $y^{(k)} = \mathrm{d}^k y/\mathrm{d}t^k$, $k = 1, \ldots, n$, and $a_1, \ldots, a_n$, g are periodic continuous complex functions of the real variable $t \in \mathbb{R} = (-\infty, \infty)$ with a common positive period ω, or to a system of periodic linear differential equations (in matrix form)

$$\dot{x} = A(t)x + f(t), \tag{1.2}$$

where $\dot{x} = \mathrm{d}x/\mathrm{d}t$, A is an ω-periodic continuous complex square matrix function of nth order and f is an ω-periodic continuous complex column matrix function of the real variable $t \in \mathbb{R}$ with n elements.

1.2. Technical practice

In technical practice given problems are simplified and the right-hand side in (1.1) or (1.2) is replaced by a few first terms of its Fourier expansion. Terms which are small in some sense are neglected. However, these simplifications can destroy conditions warranting the existence of a periodic solution of (1.1) or (1.2). The aim of this paper is not to introduce a new approximation method but to define conditions and an approach ensuring the well-posedness of the simplified problem. The justification of this method has not been examined in literature. Here we will show a constructive method based on an approximation of the right-hand side by a periodic trigonometric polynomial such that the necessary and sufficient condition for the existence of a periodic solution is preserved while the periodic solution of the simplified problem approximates uniformly on $\mathbb{R}$ the corresponding periodic solution of the original problem with arbitrary accuracy given in advance.

1.3. Some notions, notation and assertions

The set of all real numbers is denoted by $\mathbb{R}$. The symbol 0 denotes the number zero or the zero matrix the type of which is evident from the context. If C is a complex matrix then C^T is its transposed and C^* its transposed and complex conjugated matrix. If C is a square matrix then $\det C$ is its deteminant. The unit matrix of nth order is denoted by I_n or I. If r, m are two positive integers then by $\mathbb{C}^{r\times m}$ we denote the space of all (constant) complex matrices of the type $r \times m$, i.e. with r rows and m columns. The space $\mathbb{C}^{r\times m}$ becomes a Banach space (B-space) if it is equipped by one from the following norms $|\cdot|_1$, $|\cdot|_2$, $|\cdot|$ given for any matrix $A = (\alpha_{ij}) \in \mathbb{C}^{r\times m}$:

$$\begin{aligned}
|A|_1 &= \max\Big\{\sum_{j=1}^{m} |\alpha_{ij}|\colon\ i = 1, \dots, r\Big\},\\
|A|_2 &= \max\Big\{\sum_{i=1}^{r} |\alpha_{ij}|\colon\ j = 1, \dots, m\Big\},\\
|A| &= \max\{|A|_1, |A|_2\}.
\end{aligned}$$

In the sequel we will use the norm $|\cdot|$ for which $|A^*| = |A|$ and $|I| = 1$ is true. Here and in what follows let ω be a given positive number. A periodic function defined on $\mathbb{R}$ with the period ω is called ω-periodic.

If a complex matrix function φ is defined and continuous on the closed interval $[0, \omega]$ then the nonnegative number $|\varphi| = \sup|\varphi(t)| = \sup\{|\varphi(t)|\colon\ t \in [0, \omega]\}$ is the norm of the function φ. If the function φ has a continuous derivative $\dot\varphi$ on the closed interval $[0,\omega]$ then we define another norm $\|\varphi\| = \max\{|\varphi|, |\dot\varphi|\}$ of the function φ.

If k is a nonnegative integer and r, m positive integers then we denote by $C^kP_\omega^{r\times m}$ the space of all complex ω-periodic matrix functions with r rows and m columns which are defined and have ω-periodic continuous derivatives on $\mathbb{R}$ up to the kth order. For $k=0$ or $m=1$ or $r=1$ we use the notation $CP_\omega^{r\times m} = C^0P_\omega^{r\times m}$, $C^kP_\omega^r = C^kP_\omega^{r\times 1}$, $CP_\omega^r = C^0P_\omega^r$, $CP_\omega = CP_\omega^1$. $CP_\omega^{r\times m}$ and $C^1P_\omega^{r\times m}$ are B-spaces with norms $|\cdot|$ and $\|\cdot\|$, respectively. In the space CP_ω^r we define an inner product by the formula $\langle u,v\rangle = \frac{1}{\omega}\int_0^\omega v^*(t)u(t)\,\mathrm{d}t$ and obtain the corresponding norm $\mid u\mid = \sqrt{\langle u,u\rangle}$. It is evident that the inequalities $\mid u\mid \leqslant |u|$ and $\langle u,v\rangle \leqslant \mid u\mid\cdot\mid v\mid \leqslant |u|\,|v|$ hold. In the case of constant functions u, v from CP_ω^r their inner product is $\langle u,v\rangle = v^*u$.

For brevity of notation in some operations with two arbitrary matrix functions g of a type $l\times k$ and h of a type $l\times m$ continuous on the interval $[0,\omega]$, where k, l, m are positive integers, we introduce their "inner product" $\langle g,h\rangle$ by the formula $\langle g,h\rangle = \frac{1}{\omega}\int_0^\omega h^*(t)g(t)\,\mathrm{d}t$. It is evident that $\langle g,h\rangle \in \mathbb{C}^{m\times k}$ and for constant f, g we have $\langle g,h\rangle = h^*g$. If $h=g$ holds then $\langle g,g\rangle \in \mathbb{C}^{k\times k}$ and $\det\langle g,g\rangle$ is a nonnegative number, which is Gram's determinant for the columns of g. If the columns of g are linearly independent then $\det\langle g,g\rangle$ is a positive number and the inverse matrix $\langle g,g\rangle^{-1}$ to $\langle g,g\rangle$ exists. Analogously the inequality $|\langle g,h\rangle| \leqslant |g|\,|h|$ is valid.

Let $\mathbf{E}$ be a B-space and k a positive integer. Given k mutually distinct real numbers $\lambda_1,\ldots,\lambda_k$ and k non-zero elements $b_1,\ldots,b_k$ from $\mathbf{E}$, the function Q defined by the formula $Q(t) = b_1\exp(\mathrm{i}\lambda_1 t) + \ldots + b_k\exp(\mathrm{i}\lambda_k t)$, $t\in\mathbb{R}$, is called an $\mathbf{E}$-trigonometric or simply a trigonometric polynomial ($\exp(s)$ denotes the usual exponential function e^s).

2. Approximation theorems

2.1. Weierstrass approximation theorem

Since we deal with uniform approximations of continuous ω-periodic functions by ω-periodic trigonometric polynomials, we will recall the following approximation theorems.

Theorem 2.1. *For every function $f\in CP_\omega$ and every positive number ε there exists a trigonometric polynomial $Q\in CP_\omega$ such that $|f-Q|\leqslant\varepsilon$ holds.*

Corollary 2.2. *For every matrix function f from $CP_\omega^{r\times m}$ and any positive number ε there exists a trigonometric polynomial $Q\in CP_\omega^{r\times m}$ such that $|f-Q|\leqslant\varepsilon$ holds.*

3. Existence of a periodic solution

3.1. Homogeneous and conjugated equations

In what follows we shall treat only (1.2) since (1.1) can be modified to the form of (1.2) which induces the homogeneous equation

$$\dot{y} = A(t)y, \quad t \in \mathbb{R}, \tag{3.1}$$

and the conjugated equation with (3.1)

$$\dot{z} = -A^*(t)z, \quad t \in \mathbb{R}. \tag{3.2}$$

The conditions given above in 1.1 ensure the existence and uniqueness of a solution of the Cauchy problem for the equations (1.1), (1.2), (3.1), (3.2).

Let $Y = Y(t)$, $t \in \mathbb{R}$, be a fundamental matrix (of solutions) of (3.1) and let $Z = Z(t)$, $t \in \mathbb{R}$, be a fundamental matrix of (3.2). The validity of the following relations can be easily verified: $Y(t) = (Z^*(t))^{-1}Z^*(0)Y(0)$, $Y^{-1}(t) = Y^{-1}(0)(Z^*(0))^{-1}Z^*(t)$, $t \in \mathbb{R}$.

3.2. Periodic solutions

If x is an ω-periodic solution of (1.2) then the equality

$$x(\omega) - x(0) = 0 \tag{3.3}$$

holds. This equality is not only a necessary but also a sufficient condition for the ω-periodicity of the solution x of (1.2).

If H is a fundamental matrix of (3.1) then the general solution x of (2.1) can be written in the form

$$x = x(t) = H(t)H^{-1}(0)x_0 + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R} \ \ (x(0) = x_0). \tag{3.4}$$

R e m a r k 3.1. Note that owing to the validity of the relations $\tilde{H}(t)\tilde{H}^{-1}(s) = H(t)H^{-1}(s)$ for any real numbers s, t and arbitrary fundamental matrices H, $\tilde{H}$ of (3.1) the integral part $H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, of a solution of (1.2) stays unchanged for every given f from CP^n_ω and for an arbitrary fundamental matrix H of (3.1).

The condition (3.3) has now the form

$$0 = x(\omega) - x(0) = (H(\omega) - H(0))H^{-1}(0)x_0 + H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s,$$

i.e.

$$(H(\omega) - H(0))H^{-1}(0)x_0 = -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s \tag{3.5}$$

with the unchanged right-hand side for every given f and for an arbitrary fundamental matrix H of (3.1), owing to Remark 3.1. The system (3.5) has n linear algebraic equations with n unknowns which are elements of the initial matrix-column $x_0 \in \mathbb{C}^{n\times 1}$ and it can be arranged into the system

$$(H(\omega) - H(0))u_0 = -(Z^*(\omega))^{-1} \langle f, Z\rangle\, \omega \tag{3.5*}$$

with the unknown matrix-column $u_0 = H^{-1}(0)x_0 \in \mathbb{C}^{n\times 1}$ ($x_0 = H(0)u_0$) and an arbitrary fundamental matrix Z of (3.2) (with unchanged right-hand side independent of Z).

Definition 3.2.

a) We denote by B_1 the space of all functions $f \in CP^n_\omega$ for which Equation (1.2) has an ω-periodic solution and denote by B_2 the space $C^1P^n_\omega$.

b) For a given function f from B_1 we denote by V_f the space of all ω-periodic solutions of Equations (1.2). For $f = 0$ we have the space V_0 of all ω-periodic solutions of Equation (3.1).

c) We denote by V_0^* the space of all ω-periodic solutions of Equation (3.2).

Remark 3.3. The spaces B_1, V_0 and V_0^* are B-spaces.

If (3.5) or (3.5*) has a solution x_0 then (3.3) is for this x_0 fulfilled and (1.2) has an ω-periodic solution (3.4).

It is known the two following valid theorems.

Theorem 3.4. *The necessary and sufficient condition for the existence of an ω-periodic solution of Equation* (1.2) *is the validity of the equality*

$$\langle f, z\rangle = \frac{1}{\omega}\int_0^\omega z^*(t)f(t)\,\mathrm{d}t = 0 \tag{3.6}$$

for every ω-periodic solution z of Equation (3.2), *i.e. for every z from V_0^*.*

Theorem 3.5. *The* B*-space V_0 has the same finite dimension as the* B*-space V_0^*.*

4. Transformations

4.1. B-spaces

Recall that the spaces CP^n_ω and B_1 with the norm $|\cdot|$ and the space $B_2 = C^1P^n_\omega$ with the norm $\|\cdot\|$ are B-spaces. The space B_2 can be viewed as the space of all ω-periodic solutions of (1.2) with any right-hand sides from B_1. Namely, if x is from B_2 then x is an ω-periodic solution of (1.2) with the right-hand side $f = \dot{x} - Ax$.

4.2. Fundamental matrix $H = (H', H'')$

Definition 4.1. The set of all fundamental matrices H of Equation (3.1) whose first k columns create a basis of V_0 with $k = \dim V_0$ is denoted by W_0. For every H from W_0 we introduce its decomposition $H = (H', H'')$ where H' is from $C^1P_\omega^{n\times k}$. The set of all fundamental matrices Z of Equation (3.2) whose first k columns create a basis of V_0^* is denoted by W_0^*.

4.3. Periodic solutions

For any f from B_1 and an arbitrary $H = (H', H'')$ from W_0 every ω-periodic solution x from V_f can be expressed in the form

$$x = x(t) = H'(t)u_0' + H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1}$$

where $x_0 = x(0) = x(\omega)$, $u_0' \in \mathbb{C}^{k\times 1}$, $u_0'' \in \mathbb{C}^{(n-k)\times 1}$, $u_0 = (u_0'^T, u_0''^T)^T = H^{-1}(0)x_0$, $x_0 = H(0)u_0$.

In this section and further we use a fixed given fundamental matrix $H = (H', H'')$ from W_0, unless stated otherwise. Let a function f from B_1 be given. If we have a particular ω-periodic solution

$$u = u(t) = H'(t)u_0' + H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1$'$}$$

with fixed u_0', u_0'' then we decompose this solution into a "homogeneous part" $u_H = H'u_0'$ and a "particular part"

$$u_f = u_f(t) = H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

So we have $u = u_H + u_f$ where $u_H \in V_0$ and $u_f \in V_f$. The part u_f is uniquely determined by $f \in B_1$. Indeed, if we have another particular ω-periodic solution

$$\tilde{u} = \tilde{u}(t) = H'(t)\tilde{u}_0' + H''(t)\tilde{u}_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.1$''$}$$

with fixed $\tilde{u}_0' \in \mathbb{C}^{k\times 1}$, $\tilde{u}_0'' \in \mathbb{C}^{(n-k)\times 1}$, then evidently the ω-periodic solution $u - \tilde{u} - u_H + \tilde{u}_H = H''(u_0'' - \tilde{u}_0'')$ is from V_0. Since the columns of H' create a basis of V_0 the equality $\tilde{u}_0'' = u_0''$ necessarily holds. This uniqueness of u_0'' we record by $u_0'' = u_0''(f)$. So we get:

Lemma 4.2. *For every function f from B_1 all solutions from V_f have the same "particular part".*

Now we complete the solution of (3.5*) for u_0. Since $H'(\omega) - H'(0) = 0$ we get from (3.5*) the equivalent system

$$(H''(\omega) - H''(0))u_0'' = -(Z^*(\omega))^{-1} \langle f, Z\rangle \omega \tag{3.5**}$$

in which u_0' does not occur, i.e. the k elements of u_0' play here the role of k arbitrary complex parameters independent of $f \in B_1$. (If we have $u_0'' = u_0''(f)$ then it follows from (3.5**) that $x_0 = H(0)u_0 = H'u_0' + H''u_0''$ remains unchanged for all $u_0' \in \mathbb{C}^{k\times 1}$.) For shortness of the record we denote by κ the matrix $H''(\omega) - H''(0)$. The $n-k$ columns of κ are linearly independent. This follows besides other from the uniqueness of $u_0''(f)$. So Gram's determinant $\det\langle\kappa,\kappa\rangle = \det(\kappa^*\kappa)$ is a positive number and the inverse matrix $\langle\kappa,\kappa\rangle^{-1}$ to $\langle\kappa,\kappa\rangle = \kappa^*\kappa$ exists. If we multiply (3.5**) by the matrix $\langle\kappa,\kappa\rangle^{-1}\kappa^*$ from the left then we get

$$u_0'' = u_0''(f) = -\langle\kappa,\kappa\rangle^{-1}\kappa^*(Z^*(\omega))^{-1}\langle f, Z\rangle\omega \tag{4.2}$$

for $Z \in V_0^*$ and the estimates

$$|u_0''(f)| \leqslant 2\omega|H|\,|Z|\,|Z^{-1}|\,\langle\kappa,\kappa\rangle^{-1}|\,|f|, \tag{4.3}$$

$$|u_0''(f)| \leqslant 2\omega|H|^2|H^{-1}|\,|\langle\kappa,\kappa\rangle^{-1}|\,|f| \tag{4.3'}$$

hold since $|\kappa| = |\kappa^*| \leqslant 2|H|$. The estimate (4.3′) follows from

$$(H''(\omega) - H''(0))u_0'' = H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s \tag{3.5'}$$

in the analogous way as (4.3) from (3.5**).

4.4. Transformations

On the basis of the above results we obtain the following assertions.

Lemma 4.3. *The map $\mathcal{U}\colon B_1 \to \mathbb{C}^{(n-k)\times 1}$ given by the formula $\mathcal{U}f = u_0''(f)$ for any f from B_1 is a linear and bounded operator.*

P r o o f. The linearity of $\mathcal{U}$ is evident from (4.2). The boundedness of $\mathcal{U}$ follows for a fixed given Z from (4.3) and from (4.3′) by the choice of the positive constant

$$K_0 = 2\omega|H|\,|\langle\kappa,\kappa\rangle^{-1}|\max\{|Z|\,|Z^{-1}|,\ |H|\,|H^{-1}|\} \tag{4.4}$$

for the estimate

$$|\mathcal{U}f| \leqslant K_0|f| \tag{4.5}$$

for every f from B_1, where K_0 does not depend on f. □

Theorem 4.4. *If a map $\mathcal{A}\colon B_1 \to V_f$ is defined so that for every f from B_1 its image $\mathcal{A}f$ is equal to the common "particular part" of any solution u from V_f, i.e.*

$$(\mathcal{A}f)(t) = H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{4.6}$$

then the operator $\mathcal{A}$ is linear and bounded.

Proof. The linearity of $\mathcal{A}$ is evident from Lemma 4.3 and from (4.6). Further, an estimate

$$|\mathcal{A}f| \leqslant K_1|f| \tag{4.7}$$

holds owing to (4.5) and (4.6) for $K_1 = (K_0 + \omega|H^{-1}|)|H|$, where K_0 is from (4.4). The constant K_1 does not depend on f. □

Remark 4.5. If $H = (H', H'')$ and $\tilde{H} = (\tilde{H}', \tilde{H}'')$ are two arbitrary fundamental matrices from W_0 then by virtue of the properties of the matrices H', H'' and $\tilde{H}'$, $\tilde{H}''$ there exist regular matrices $C_1 \in \mathbb{C}^{k\times k}$, $C_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ and $C = \operatorname{diag}(C_1, C_2)$ such that $\tilde{H}' = H'C_1$, $\tilde{H}'' = H''C_2$ and $\tilde{H} = HC$.

Theorem 4.6. *The operator $\mathcal{A}$ from Theorem 4.4 does not depend on the choice of a fundamental matrix H from W_0.*

Proof. Let f be a given function from B_1. Let two arbitrary fundamental matrices $H = (H', H'')$ and $\tilde{H} = (\tilde{H}', \tilde{H}'')$ from W_0 be given. Assume that besides a particular ω-periodic solution u from (4.1′) belonging to V_f there exists another particular ω-periodic solution $\tilde{u} = \tilde{u}(t) = \tilde{H}'(t)\tilde{u}_0' + \tilde{H}''(t)\tilde{u}_0'' + \tilde{H}(t)\int_0^t \tilde{H}^{-1}(s)f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, from V_f. Owing to Remark 3.1 and Remark 4.5 the relations $\tilde{H}(t)\int_0^t \tilde{H}^{-1}(s)f(s)\,\mathrm{d}s = H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s$ are true for any $t \in \mathbb{R}$, so that the solution $u - \tilde{u} - u_H + \tilde{u}_H = H''u_0'' - \tilde{H}''\tilde{u}_0'' = H''(u_0'' - C_2\tilde{u}_0'')$ belongs to V_0. But this is possible only for $u_0'' = C_2\tilde{u}''$, i.e. $\tilde{u}_0'' = C_2^{-1}u_0''$, $\tilde{H}''\tilde{u}_0'' = H''C_2C_2^{-1}u_0'' = H''u_0''$ so that $\tilde{u}_f = u_f$. □

Corollary 4.7. *For every function f from B_1 the space V_f has the representation $V_f = \mathcal{A}f + V_0$, i.e. V_f is a k-dimensional manifold in $B_2 = C^1P_\omega^n$ and $B_2 = \bigcup_{f\in B_1} V_f$.*

Consider now a map $\mathcal{B} = \mathcal{B}(\cdot,\cdot)$ defined on $B_1 \times B_1$ the values of which are one-to-one operators $\mathcal{B}(f, g)\colon V_f \to V_g$ for any two functions f, g from B_1 and which satisfies $\mathcal{B}^{-1}(f, g) = \mathcal{B}(g, f)\colon V_g \to V_f$. If two functions f, g from B_1 are given then with regard to our intention to approximate elements $u = u_H + \mathcal{A}f$ from V_f by their images $\mathcal{B}(f,g)u = \tilde{u} = \tilde{u}_H + \mathcal{A}g$ from V_g we choose the simplest way and define the approximation by the formula

$$\mathcal{B}(f,g)u = \tilde{u} = u_H + \mathcal{A}g \in V_g \quad \text{for } u \in V_f \tag{4.8}$$

(i.e. $\tilde{u}_H = u_H$). For another definition of $\mathcal{B}(f,g)$ we could solve optimalization problems. However, for our purposes we can always approximate u by $\mathcal{B}(f,g)u$ more precisely so that the approximation of f by g will be also more precise as we shall see in the sequel.

Owing to (4.7) and (1.2) we get the estimates

$$|u - \mathcal{B}(f,g)u| = |\mathcal{A}(f-g)| \leqslant K_1|f-g|, \tag{4.9}$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(u - \mathcal{B}(f,g)u)\right| = |A(u - \mathcal{B}(f,g)u) + f - g| \leqslant (|A|K_1 + 1)|f-g|, \tag{4.10}$$

$$\|(u - \mathcal{B}(f,g)u\| \leqslant K|f-g| \tag{4.11}$$

for any u from V_f with the positive constant $K = \max\{K_1, 1 + |A|K_1\}$ independent of f and g from B_1. ($\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}f = A\mathcal{A}f + f$ and $u - \mathcal{B}(f,g)u = \mathcal{A}(f-g)$.)

Remark 4.8. From the estimates (4.9) and (4.11) it follows that for g sufficiently close to f in B_1 also $\tilde{u} = \mathcal{B}(f,g)u$ is sufficiently close to u in B_2. The accuracy of such approximation can be chosen arbitrarily small positive. (For any positive number η there exists a positive number ε such that $|f-g| \leqslant \varepsilon$ implies $\|u - \mathcal{B}(f,g)u\| \leqslant K|f-g| \leqslant K\varepsilon \leqslant \eta$ for $\varepsilon \leqslant \eta/K$).

5. Approximation of periodic solution

5.1. Auxiliary assertions

In the sequel we build up the results obtained and the notation and suppose that we have a fixed given fundamental matrix $H = (H', H'')$ from W_0.

Let k be a positive integer. The space $\mathbb{C}^{k\times k}$ equipped with the norm $|\cdot|$ and the usual arithmetic operations, addition and multiplication, for its elements (in addition to the multiplication of these elements by scalars) is a Banach algebra with the unit $I = I_k$ ($|I| = 1$ and $|AB| \leqslant |A|\,|B|$ for any A, B from $\mathbb{C}^{k\times k}$).

The following two assertions are well-known from the theory of Banach algebras.

Lemma 5.1. *If for a matrix B from $\mathbb{C}^{k\times k}$ the inequality $|I-B|<1$ holds then in $\mathbb{C}^{k\times k}$ the inverse matrix B^{-1} to B exists ($B^{-1}=I+\sum\limits_{m=1}^{\infty}(I-B)^m$). Moreover, the estimates*

$$|I-B^{-1}|\leqslant\frac{|I-B|}{1-|I-B|},\quad |B^{-1}|\leqslant\frac{1}{1-|I-B|}\tag{5.1}$$

are valid.

Corollary 5.2. *Let matrices A, B be from $\mathbb{C}^{k\times k}$ and let the inverse matrix A^{-1} to A exist in $\mathbb{C}^{k\times k}$. If the inequality $|A-B|<|A^{-1}|^{-1}$ holds then in $\mathbb{C}^{k\times k}$ the inverse matrix B^{-1} to B exists. Moreover, the estimates*

$$|A^{-1}-B^{-1}|\leqslant|A^{-1}|^2\frac{|A-B|}{1-|A^{-1}|\,|A-B|},\quad |B^{-1}|\leqslant\frac{|A^{-1}|}{1-|A^{-1}|\,|A-B|}\tag{5.2}$$

are valid ($B^{-1}=A^{-1}+\sum\limits_{m=1}^{\infty}(I-A^{-1}B)^mA^{-1}$).

R e m a r k 5.3. If a nonnegative number ϱ is such that for matrices A, B from Corollary 5.2 the equality $|A-B|=\varrho|A^{-1}|^{-1}<|A^{-1}|^{-1}$ holds then the estimates

$$|A^{-1}-B^{-1}|\leqslant\frac{\varrho|A^{-1}|}{1-\varrho},\quad |B^{-1}|\leqslant\frac{|A^{-1}|}{1-\varrho}\tag{5.2$'$}$$

are true. If we require that $\varrho\in\left[0;\frac{1}{2}\right]$ hold, i.e. $|A-B|\leqslant\frac{1}{2}|A^{-1}|^{-1}$, then we get estimates

$$|A^{-1}-B^{-1}|\leqslant|A^{-1}|,\quad |B^{-1}|\leqslant 2|A^{-1}|.\tag{5.2$''$}$$

Lemma 5.4. *Let a positive integer k be given. For a positive number ε and functions $f,z_1,\dots,z_k$ from CP^n_ω there exists a trigonometric polynomial Q_ε from CP^n_ω such that the relations*

$$|f-Q_\varepsilon|\leqslant\varepsilon,\tag{5.3}$$

$$\langle f,z_j\rangle=\langle Q_\varepsilon,z_j\rangle\,,\quad j=1,\dots,k,\tag{5.4}$$

are valid.

P r o o f. We can assume that the functions $z_1,\dots,z_k$ are linearly independent and we denote by Z_1 the matrix $(z_1,\dots,z_k)\in CP^{n\times k}_\omega$. The equalities (5.4) have now the form

$$\langle f,Z_1\rangle=\langle Q_\varepsilon,Z_1\rangle\,.\tag{5.4$'$}$$

Further, we define the matrix function $M = M(\Delta Z) = \langle Z_1 + \Delta Z, Z_1\rangle$ for $\Delta Z \in CP_\omega^{n\times k}$ and denote $M_0 = M(0) = \langle Z_1, Z_1\rangle$. Since $|M_0 - M| = |\langle \Delta Z, Z_1\rangle| \leqslant |\Delta Z|\,|Z_1|$, the estimates $|M_0^{-1} - M^{-1}| \leqslant |M_0^{-1}|$ and $|M^{-1}| \leqslant 2|M_0^{-1}|$ are true by virtue of (5.2″) for $|\Delta Z| \leqslant \frac{1}{2}|Z_1|^{-1}\,|M_0^{-1}|^{-1}$, ($\det M_0 = \det\langle Z_1, Z_1\rangle$ is a positive number). Let ε_0 be a positive number the magnitude of which will be determined later and let ε_1 be a positive number satisfying the inequality $\varepsilon_1 \leqslant \frac{1}{2}|Z_1|^{-1}\,|M_0^{-1}|^{-1}$. Let $Q_0 \in CP_\omega^n$ and $Q_1 \in CP_\omega^{n\times k}$ be trigonometric polynomials such that $|f - Q_0| \leqslant \varepsilon_0$ and $|Z_1 - Q_1| \leqslant \varepsilon_1$. The existence of Q_0 and Q_1 follows from Corollary 2.2. We will find $q \in \mathbb{C}^{k\times 1}$ such that the trigonometric polynomial $Q_\varepsilon = Q_0 + Q_1 q$ from CP_ω^n satisfies the relations (5.3) and (5.4′). That means that $\langle f, Z_1\rangle = \langle Q_\varepsilon, Z_1\rangle = \langle Q_0, Z_1\rangle + \langle Q_1, Z_1\rangle\, q$. Consequently, $\langle Q_1, Z_1\rangle\, q = \langle f - Q_0, Z_1\rangle$. Note that $Q_1 = Z_1 + (Q_1 - Z_1) = Z_1 + \Delta Z$, $\Delta Z = Q_1 - Z_1$, $|\Delta Z| \leqslant \varepsilon_1$. So $\langle Q_1, Z_1\rangle = M = M(Q_1 - Z_1)$ and $|M_0 - M| = |M_0 - M(Q_1 - Z_1)| \leqslant |Q_1 - Z_1|\,|Z_1| \leqslant \frac{1}{2}|M_0^{-1}|^{-1} < |M_0^{-1}|^{-1}$. According to Lemma 5.2 the inverse matrix $M^{-1} = M^{-1}(Q_1 - Z_1)$ to M exists for which owing to Remark 5.3 the estimate $|M^{-1}| \leqslant 2|M_0^{-1}|$ is valid.

So we get $q = M^{-1}\langle f - Q_0, Z_1\rangle$ which ensures the validity of (5.4′). The inequalities $|Q_1| \leqslant |Z_1| + \varepsilon_1 \leqslant |Z_1| + 1/(2|Z_1|\,|M_0^{-1}|) = (2|Z_1|^2|M_0^{-1}| + 1)/(2|Z_1|\,|M_0^{-1}),$ $|q| \leqslant 2|M_0^{-1}|\,|Z_1|\,|f - Q_0| \leqslant 2|Z_1|\,|M_0^{-1}|\varepsilon_0$ hold. Finally, we have $|f - Q_\varepsilon| \leqslant |f - Q_0| + |Q_1|\,|q| \leqslant 2\varepsilon_0(1 + |Z_1|^2\,|M_0^{-1}|) \leqslant \varepsilon$ if $\varepsilon_0 \leqslant \varepsilon/(2(1 + |Z_1|^2|M_0^{-1}|))$, which ensures the validity of (5.3). This completes the proof. □

5.2. Approximations

We can already start to construct an approximation of an ω-periodic solution of (1.2).

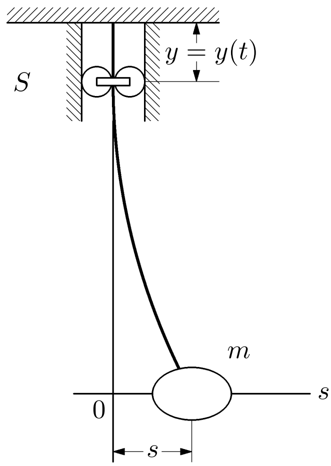
Theorem 5.5. *Let a given function f be from B_1 and let η be a given positive number. There exists a trigonometric polynomial $g = g_\eta$ from B_1 such that for every solution u from V_f its image $\tilde u$ under the map $\mathcal{B}(f, g)\colon V_f \to V_g$ from* (4.8) *satisfies the estimate $\|u - \tilde u\| = \|u - \mathcal{B}(f, g)u\| \leqslant \eta$.*

P r o o f. Let a fixed fundamental matrix $Z = (Z_1, Z_2)$ from W_0^* be given, $Z_1 \in C^1P_\omega^{n\times k}$, $k = \dim V_0 = \dim V_0^*$. According to Lemma 5.4, for any positive number ε there exists a trigonometric polynomial Q_ε from CP_ω^n satisfying (5.3) and (5.4′) which ensures $Q_\varepsilon \in B_1$. By means of the choice $\varepsilon = \eta/K$ we get $\|u - \tilde u\| \leqslant \eta$ if we put $g = g_\eta = Q_\varepsilon$ in (4.11), where K is from Remark 4.8. □

Thus we have reached the aim of the present paper to justify the approximation method for periodic solutions used in technical practice.

6.1. Application

As an application of a linear differential equation with periodic coefficients and with a periodic right-hand side we can consider the simple technical device which is described in [8]. This device is shown in the figure where we have a mass m attached to a thin cantilever linear spring, the effective length of which can vary with time t in any prescribed manner ($y = y(t)$) by moving the support S. Thus in discussing lateral vibrations of the mass m, we have a spring characteristic c that varies with time t (controlled by $y(t)$):



We have the equation

$$m\ddot{s} = -c(t)s - a\dot{s} + Q(t) \tag{6.1}$$

with variable coefficients where s is the displacement of the mass m from its central position. For a periodic motion of the support S and for a periodic external force Q with the same positive period ω, if we neglect the resistance of air, i.e. $a = 0$, then we have the ω-periodic equation of motion in the form

$$\ddot{s} + k(t)s = g(t) \tag{6.2}$$

where $k(t) = c(t)/m$, $g(t) = Q(t)/m$, $t \in \mathbb{R}$.

6.2. Illustration

If there exists an ω-periodic nonconstant function $F = F(t)$ with continuous derivatives up to and including the second order on $\mathbb{R}$ which satisfies the nonlinear equation $\dot{F}^2 + \ddot{F} = -k(t)$ then the homogeneous linear equation

$$\ddot{s} + k(t)s = 0 \tag{6.3}$$

has the general solution in the form

$$s = C_1 \exp(F) + C_2 \exp(F) \cdot G \tag{6.4}$$

where $G = G(t) = \int_0^t \exp(-2F(\sigma))\,\mathrm{d}\sigma$, $t \in \mathbb{R}$.

Note that the function G is increasing because its first derivative $\exp(-2F)$ is positive with $G(0) = 0$. Thus G is not periodic and $G(\omega) > 0$. The motion equation of the technical device from the figure can be transformed into the system of linear equations

$$\dot{x} = A(t)x + f(t) \tag{6.5}$$

(in matrix notation) where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -k(t) & 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} s \\ \dot{s} \end{pmatrix}, \quad f(t) = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

With System (6.5) we associate the homogeneous system

$$\dot{y} = A(t)y \tag{6.6}$$

and the adjoint homogeneous system

$$\dot{z} = -A^*(t)z. \tag{6.7}$$

Here y, z are matrix-columns with two elements and $A^*(t) = A^T(t)$ because the matrix $A(t)$ is real for every $t \in \mathbb{R}$.

We choose the matrix

$$H = H(t) = \begin{pmatrix} \exp(F), & \exp(F)G \\ \dot{F}\exp(F), & \dot{F}\exp(F)G + \exp(-F) \end{pmatrix}$$

as the fundamental matrix H of the system (6.6) from Section 4.3. We denote by h_1, h_2 the columns of H, i.e. $H = (h_1, h_2)$. Because $\det H = 1$, the inverse matrix has the form

$$H^{-1} = H^{-1}(t) = \begin{pmatrix} \dot{F}\exp(F)G + \exp(-F), & -\exp(F)G \\ -\dot{F}\exp(F), & \exp(F) \end{pmatrix}.$$

We denote by z_1, z_2 the rows of the inverse matrix H^{-1}, i.e. $H^{-1} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. The matrix H is real and therefore $(H^*)^{-1} = (H^{-1})^T$ which is a fundamental matrix of solutions of System (6.7).

The columns h_1, z_2^T are ω-periodic solutions of Systems (6.6), (6.7), respectively. Since the function G is not periodic the columns h_2, z_1^T are non-periodic solutions of Systems (6.6), (6.7), respectively. (h_1, z_2^T form bases of the sets of all ω-periodic solutions of Systems (6.6), (6.7), respectively.) Hence, if $\int_0^\omega \exp(F(t))g(t)\,\mathrm{d}t = 0$, i.e. $\langle f, z_2^T \rangle = 0$, then the condition (3.6) is fulfilled and therefore the existence of an ω-periodic solution of System (6.5) is ensured. According to the previous reasoning the function

$$\begin{aligned} \mathcal{A}f(t) &= H''(t)u_0'' + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s \\ &= h_2(t)u_{02} + H(t)\int_0^t H^{-1}(s)f(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \end{aligned}$$

where $u_0'' = u_{02}$, satisfies the condition $\mathcal{A}f(\omega) - \mathcal{A}f(0) = 0$. Hence the linear algebraic system

$$[h_2(\omega) - h_2(0)]u_{02} = -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s \tag{6.8}$$

is fulfilled because the equalities

$$\begin{aligned} h_2(\omega) - h_2(0) &= \begin{pmatrix} 1 \\ \dot{F}(\omega) \end{pmatrix} \exp(F(\omega))G(\omega), \\ -H(\omega)\int_0^\omega H^{-1}(s)f(s)\,\mathrm{d}s &= H(\omega)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s \\ &= \begin{pmatrix} 1 \\ \dot{F}(\omega) \end{pmatrix} \exp(F(\omega))\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s \end{aligned}$$

(due to $\langle f, z_2^T \rangle = 0$) are valid and the linear algebraic system (6.8) has the solution $u_{02} + G(\omega)^{-1}\int_0^\omega G(s)\exp(F(s))g(s)\,\mathrm{d}s$.

By virtue of $|H| = |H^{-1}|$ and $|f| = |g|$ we get the inequalities

$$\begin{aligned} |u_{02}| &\leqslant |g|\int_0^\omega \exp(F(s))\,\mathrm{d}s, \\ |\mathcal{A}f| &\leqslant \tilde{K}|g| = \tilde{K}|f|, \\ \|\mathcal{A}F\| &\leqslant \tilde{K}|g| = \tilde{K}|f|, \end{aligned}$$

where $\tilde{K} = |h_2|\int_0^\omega \exp(F(s))\,\mathrm{d}s$ and $K = \max\{\tilde{K}, |A|\tilde{K} + 1\}$.

6.3. Example

Now we consider the case $F(t) = 2\cos t$, $g(t) = a_0 + \cos t + \varepsilon \cos(qt)$, where a_0 is a real number, q is a positive integer greater than one, ε is a real non-zero number with a very small absolute value and $\omega = 2\pi$. First we construct the Fourier series of the function $\exp(F)$. Notice that $F(t) = 2\cos t = \exp(\mathrm{i}t) + \exp(-\mathrm{i}t)$, hence

$$\begin{aligned}
\exp(F(t)) &= \exp(\exp(\mathrm{i}t) + \exp(-\mathrm{i}t)) = \sum_{p=0}^{\infty} \frac{1}{p!}(\exp(\mathrm{i}t) + \exp(-\mathrm{i}t))^p \\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{\exp(\mathrm{i}(p-2k)t)}{(p-k)!\,k!} = \mathrm{Re}(\exp(F)) = \sum_{p=0}^{\infty}\sum_{k=0}^{p} \frac{\cos(p-2k)t}{(p-k)!\,k!} \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{p=0}^{\infty}\left[\sum_{k=0}^{p} \frac{\cos(2p-2k)t}{(2p-k)!\,k!} + \sum_{k=0}^{p} \frac{\cos(2p+1-2k)t}{(2p+1-k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{k=0}^{\infty}\left[\sum_{p=k}^{\infty} \frac{\cos(2p-2k)t}{(2p-k)!\,k!} + \sum_{p=k}^{\infty} \frac{\cos(2p+1-2k)t}{(2p+1-k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{k=0}^{\infty}\left[\sum_{p=0}^{\infty} \frac{\cos 2pt}{(2p+k)!\,k!} + \sum_{p=0}^{\infty} \frac{\cos(2p+1)t}{(2p+1+k)!\,k!}\right] \\
&= -\sum_{p=0}^{\infty} \frac{1}{(p!)^2} + 2\sum_{p=0}^{\infty}\left[\left\{\sum_{k=0}^{\infty} \frac{1}{(2p+k)!\,k!}\right\}\cos 2pt \right.\\
&\qquad \left. + \left\{\sum_{k=0}^{\infty} \frac{1}{(2p+1+k)!\,k!}\right\}\cos(2p+1)t \right] \\
&= -\alpha_0 + 2\sum_{p=1}^{\infty} \alpha_p \cos pt
\end{aligned}$$

(by virtue of the absolute convergence of this series for every $t \in \mathbb{R}$), where

$$\alpha_{2p} = \sum_{k=0}^{\infty} \frac{1}{(2p+k)!\,k!}, \quad \alpha_{2p+1} = \sum_{k=0}^{\infty} \frac{1}{(2p+k+1)!\,k!}, \quad p = 0, 1, 2, \dots .$$

Let there exist an ω-periodic solution of System (6.5). Then the equality $a_0 = -(\alpha_1 + 2\alpha_q)/\alpha_0$ follows from the validity of the condition (3.6). In a technical practice the "very small" term $\varepsilon \cos qt$ of the right-hand side g could be neglected and g could be replaced by the trigonometric polynomial $a_0 + \cos t$. This procedure, however, destroys the former valid necessary and sufficient condition for the existence of a 2π-periodic solution of System (1.2). However, if g is replaced by the trigonometric polynomial $g_\eta(t) = \tilde{a_0} + \cos t$ with $\tilde{a_0} = -(\alpha_1/\alpha_0)a_0$, which is equally simple as $a_0 + \cos t$, then the existence of a 2π-periodic solution is preserved, while $|g - g_\eta| \leqslant$

$\eta \leqslant |\varepsilon|(1 + \alpha_q/\alpha_0)$, $\|\mathcal{A}g - \mathcal{A}g_\eta\| \leqslant K|g - g_\eta| \leqslant K|\varepsilon|(1 + \alpha_q/\alpha_0) < 2|\varepsilon|K$, because $2 < \alpha_0$, $2 \leqslant q$, $0 < \alpha_q < \mathrm{e}/q! < 2$. (Here e is the Euler's number.)

6.4. Conclusion

Recently, approximation methods for periodic solutions have concerned nonlinear differential equations. In the papers [9], [10] numerical methods supported by computers are presented. [5], [7] deal with Fourier approximations of periodic solutions of nonlinear differential equations. Finally, the paper [1] generalizes this approximation problem to Banach spaces.

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