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Applications of Mathematics, Vol. 49 (2004), No. 3, 185–199

Persistent URL: http://dml.cz/dmlcz/134566

49 (2004) APPLICATIONS OF MATHEMATICS No. 3, 185–199

STUDY OF SOME NONCOOPERATIVE LINEAR ELLIPTIC SYSTEMS[1]

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(Received November 21, 2001)

Abstract. Using an approximation method, we show the existence of solutions for some noncooperative elliptic systems defined on an unbounded domain.

Keywords: Schrödinger's operators, weighted Sobolev spaces, maximum principle, min-max formula, noncooperative systems

MSC 2000: 35P15

1. INTRODUCTION

We study here some noncooperative elliptic systems defined on a connected and unbounded open set $\Omega \subseteq \mathbb{R}^N$ ($N \geqslant 3$) of the form

$$\text{(S)} \qquad \begin{cases} -\Delta u + q_1 u = a\varrho_1 u + b\varrho_2 v + f & \text{in } \Omega, \\ -\Delta v + q_2 v = c\varrho_3 u + d\varrho_4 v + g & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega; \quad u, v \to 0 \text{ for } |x| \to +\infty \end{cases}$$

where ϱ_1, ϱ_2, ϱ_3, ϱ_4 (sometimes referred to as *weight functions*), q_1, q_2, are positive functions; f and g are measurable functions; a, b, c and d are real numbers. u and v are unknown real-valued functions defined in Ω and belonging to appropriate function spaces. The system (S) is noncooperative since b and c are not necessarily positive. Under appropriate assumptions on the coefficients, we show the existence of non-trivial solutions.

[1] This work was supported by a grant from ANDRU under No. CU39904.

Generally, in order to study a nonlinear problem, we consider the linear approximation, which is easy enough to resolve of course. Here we have explored the inverse process. In other words, we show that the linear system (S) can be taken as the limit of a sequence of nonlinear systems.

The paper is organized as follows. In Section 1, we establish a Maximum Principle result in the scalar case. We choose decreasing weight functions which lead to a gain of compactness, (see [7] and [8]). In Section 2, we obtain an existence and uniqueness theorem for system (S). In order to prove it, we apply a nonlinear method introduced in [3] and [4]. The main tool used here is Schauder's fixed point theorem. The Maximum Principle guarantees the invariance of subsets under operators. So, we can prove the existence of solution for nonlinear systems. We observe that there exists a vast literature on the use of nonlinear methods to the study of noncooperative elliptic systems. We point out that an interesting version of the Maximum Principle for noncooperative systems was given in [13]. We refer the reader to the works in [5], [6] and references therein.

2. Notation

We denote

(1) $u^{\pm} = \max(\pm u, 0)$, so $u = u^+ - u^-$.

(2) $\mathrm{mes}(\cdot)$ is the Lebesgue N-dimensional measure.

(3) For $\varepsilon \in]0,1[$ fixed: $B_\varepsilon = \Omega \cap B(0, 1/\varepsilon) = \{x \in \Omega / |x| < 1/\varepsilon\}$.
1_{B_ε} is the characteristic function of B_ε.

(4) $p_\alpha(x) = (1 + |x|^2)^{-\alpha}$, $\alpha \in \mathbb{R}_+^*$; $\mathbb{R}_+^*$ is the set of positive real numbers.

(5) Let $D(\Omega)$ be the set of all infinitely differentiable functions with compact support in Ω. We denote by $D'(\Omega)$ the dual space of $D(\Omega)$. Let us define $V(\mathbb{R}^N) = \left\{u \in D'(\mathbb{R}^N) / \int_{\mathbb{R}^N} (|\nabla u|^2 + p_1|u|^2)\,\mathrm{d}x < +\infty\right\}$.
The space V is the closure of $D(\Omega)$ with respect to the norm

$$\|u\|_V = \left(\int_\Omega (|\nabla u|^2 + p_1|u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

equivalent (under some conditions) to the norms

$$\|u\|_i = \left(\int_\Omega (|\nabla u|^2 + (q_i + m\varrho_j)|u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}} \quad (i = 1,2;\ j = 1,4;\ m \in \mathbb{R}_+^*).$$

$V(\mathbb{R}^N)$ is a Hilbert space, $\forall N \geqslant 3$ (see [17], p. 230).

(6) $L_p^2(\Omega) = \{u \in D'(\Omega)/p^{\frac{1}{2}}u \in L^2(\Omega)\}$ where $p(x)$ is a positive function defined on Ω.

$H^m(\Omega) = \{u \in D'(\Omega)/D^\alpha u \in L^2(\Omega),\ 0 \leqslant |\alpha| \leqslant m\}$.

(7) $\varrho = \max_{x\in\Omega}(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$.

3. Hypotheses

We suppose that

(8) $\exists \alpha, \beta \in \mathbb{R}$, $\alpha > N/2$, $\beta \geqslant 1$; $\exists k_i > 0$ $(i = 1,2,3,4)$, $\exists c_i > 0$ $(i = 1,2)$ such that

$$\begin{aligned} 0 \leqslant &\varrho_i(x) \leqslant k_i p_\alpha(x), \quad i = 1,2,3,4,\\ 0 \leqslant &q_i(x) \leqslant c_i p_\beta(x), \quad i = 1,2. \end{aligned}$$

(9) $\varrho_2(x) \leqslant \sqrt{\varrho_1(x)\varrho_4(x)}$, $\quad \forall x \in \Omega$, $\varrho_3(x) \leqslant \sqrt{\varrho_1(x)\varrho_4(x)} \quad \forall x \in \Omega$.

(10) $f, g \in L^2_{p_1^{-1}}(\Omega) = \{u \in D'(\Omega)/ \int_\Omega (1 + |x|^2)|u|^2\, dx < +\infty\}$.

4. Existence of solutions

4.1. Remarks on the scalar case

We consider the problem

$$\begin{cases} -\Delta u + q_1 u = \lambda \varrho_1 u & \text{in } \Omega,\\ u = 0 \text{ on } \partial\Omega; & u \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{11}$$

By [7], [8], Problem (11) possesses an increasing infinite sequence of positive eigenvalues.

Moreover, the first eigenvalue $\lambda(q_1, \varrho_1)$ is principal ([9]) and is characterized variationally by

$$\lambda(q_1, \varrho_1) \int_\Omega \varrho_1 |u|^2\, dx \leqslant \int_\Omega \left(|\nabla u|^2 + q_1 |u|^2\right) dx; \quad \forall u \in V. \tag{12}$$

We consider now the problem

$$\begin{cases} -\Delta u + q_1 u = \lambda \varrho_1 u + h & \text{in } \Omega,\\ u = 0 \text{ on } \partial\Omega; & u \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{13}$$

We claim that the Maximum Principle holds for Problem (13) with condition $h \geqslant 0$, if all solutions of Problem (13) are nonnegative.

Theorem 1. *Assume that* (8) *holds and* $h \in L^2_{p_1^{-1}}(\Omega)$. *Problem* (13) *satisfies the Maximum Principle if and only if* $\lambda(q_1, \varrho_1) > \lambda$.

Proof. Let φ be an eigenfunction associated with $\lambda(q_1, \varrho_1)$, then φ does not change sign. Multiplying the equation in (13) by φ and integrating over Ω, we obtain

$$(\lambda(q_1, \varrho_1) - \lambda) \int_\Omega \varrho_1 u \varphi \, dx = \int_\Omega h \varphi \, dx.$$

If the Maximum Principle holds then $\int_\Omega h\varphi \, dx > 0$ implies $\int_\Omega \varrho_1 u \varphi \, dx > 0$.

Consequently $\lambda(q_1, \varrho_1) > \lambda$.

Conversely, if $\lambda(q_1, \varrho_1) > \lambda$ then $u \geqslant 0$. Indeed, observe that $u = u^+ - u^-$. Now, multiplying the equation of Problem (13) by u^- and integrating over Ω, we get

$$\int_\Omega (\nabla u \nabla u^- + q_1 u u^-) \, dx = \lambda \int_\Omega \varrho_1 u u^- \, dx + \int_\Omega h u^- \, dx.$$

Since

$$\int_\Omega \nabla u^+ \nabla u^- \, dx = \int_\Omega q_1 u^+ u^- \, dx = \int_\Omega \varrho_1 u^+ u^- \, dx = 0,$$

we obtain

$$\int_\Omega (|\nabla u^-|^2 + q_1 |u^-|^2) \, dx = \lambda \int_\Omega \varrho_1 |u^-|^2 \, dx - \int_\Omega h u^- \, dx.$$

By (12), we have

$$\lambda(q_1, \varrho_1) \int_\Omega \varrho_1 |u^-|^2 \, dx \leqslant \lambda \int_\Omega \varrho_1 |u^-|^2 \, dx - \int_\Omega h u^- \, dx,$$
$$0 < (\lambda(q_1, \varrho_1) - \lambda) \int_\Omega \varrho_1 |u^-|^2 \, dx \leqslant - \int_\Omega h u^- \, dx \leqslant 0$$

Hence $u \geqslant 0$. □

Theorem 2. *Under the assumptions of Theorem* 1, *Problem* (13) *with condition* $h \geqslant 0$ *has one nonnegative solution in* V *if and only if* $\lambda(q_1, \varrho_1) > \lambda$.

Proof. In view of Theorem 1, the necessary condition holds.

Let us consider the sufficient condition. The differential form

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + q_1 u v) \, dx - \lambda \int_\Omega \varrho_1 u v \, dx$$

is continuous and coercive on $V \times V$. Indeed, let $m \in \mathbb{R}_+^*$ be such that $\lambda + m > 0$. Taking into account the variational characterization of $\lambda(q_1, \varrho_1)$ we have

$$
\begin{aligned}
a(u,u) &= \int_\Omega \left[|\nabla u|^2 + (q_1 + m\varrho_1)|u|^2\right] \mathrm{d}x - (\lambda + m) \int_\Omega \varrho_1 |u|^2 \,\mathrm{d}x \\
&\geqslant \Big(1 - \frac{\lambda + m}{\lambda(q_1, \varrho_1) + m}\Big) \|u\|_1^2
\end{aligned}
$$

where $\|\cdot\|_1$ is a norm equivalent to $\|\cdot\|_V$ by virtue of (8) and Hardy's inequality

$$
\Big(\forall N \geqslant 3,\ \exists \gamma > 0\colon \forall u \in D(\mathbb{R}^N),\ \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \,\mathrm{d}x \leqslant \gamma \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x\Big).
$$

Since $f \in L^2_{p_1^{-1}}(\Omega)$, the application $v \mapsto \int_\Omega hv \,\mathrm{d}x$ is a continuous linear form on V. Hence by the Lax-Milgram lemma, Problem (13) possesses one solution u. By Theorem 1, u is nonnegative. □

4.2. Vectorial case

Let us consider the system (S) now. In what follows, we will find a sufficient condition for the system (S) to have a unique solution.

Theorem 3. *Let* (8), (9) *and* (10) *be satisfied, as well as*

$$
\lambda(q_1, \varrho_1) > a; \quad (\lambda(q_1, \varrho_1) - a)(\lambda(q_2, \varrho_4) - d) > |bc|. \tag{14}
$$

Then the system (S) *has one (weak) solution in* $V \times V$.

First, we state the following lemma.

Lemma 1. *Assume that the hypothesis of Theorem* 3 *holds. Let* $(z, w) \in V \times V$ *be a solution of*

$$
\begin{cases}
(-\Delta + q_1)z = a\varrho_1 z + b\varrho_2 w & \text{in } \Omega, \\
(-\Delta + q_2)w = c\varrho_3 z + d\varrho_4 w & \text{in } \Omega, \\
z = w = 0 \text{ on } \partial\Omega; \quad z, w \to 0 \text{ for } |x| \to +\infty.
\end{cases} \tag{15}
$$

Then $(z, w) = (0, 0)$.

Proof. We multiply the first equation of the system (15) by z and integrate over Ω:

$$
\int_\Omega (|\nabla z|^2 + q_1 |z|^2) \,\mathrm{d}x = a \int_\Omega \varrho_1 |z|^2 \,\mathrm{d}x + b \int_\Omega \varrho_2 z w \,\mathrm{d}x.
$$

Using the variational characterization of $\lambda(q_1, \varrho_1)$, we have

$$0 < (\lambda(q_1, \varrho_1) - a) \int_\Omega \varrho_1 |z|^2 \, dx \leqslant |b| \left| \int_\Omega \varrho_2 z w \, dx \right|.$$

By hypothesis (9) and the Cauchy-Schwarz inequality we obtain

$$(\lambda(q_1, \varrho_1) - a) \int_\Omega \varrho_1 |z|^2 \, dx \leqslant |b| \left(\int_\Omega \varrho_1 |z|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \varrho_4 |w|^2 \, dx \right)^{\frac{1}{2}}. \tag{16}$$

Similarly, multiplying the second equation of system (15) by w and integrating over Ω, we get

$$(\lambda(q_2, \varrho_4) - d) \int_\Omega \varrho_4 |w|^2 \, dx \leqslant |c| \left(\int_\Omega \varrho_1 |z|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega \varrho_4 |w|^2 \, dx \right)^{\frac{1}{2}}. \tag{17}$$

Combining (16) with (17), we obtain

$$0 < [(\lambda(q_1; \varrho_1) - a)(\lambda(q_2, \varrho_4) - d) - |bc|] \int_\Omega \varrho_1 |z|^2 \, dx \cdot \int_\Omega \varrho_4 |w|^2 \, dx \leqslant 0,$$

which implies that $z = w = 0$ a.e. □

Proof of Theorem 3. Let $m \in \mathbb{R}_+^*$ be such that $a + m > 0$ and $d + m > 0$. We define an operator

$$T \colon L^2_\varrho(\Omega) \times L^2_\varrho(\Omega) \to V \times V, \quad (\xi, \eta) \mapsto T(\xi, \eta) = (\omega, v)$$

such that (ω, v) verifies the system

$$\begin{cases} (-\Delta + q_1 + m\varrho_1)\omega = (a+m)\dfrac{\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + b \dfrac{\varrho_2 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon} + f & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)v = c\dfrac{\varrho_3 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + (d+m) \dfrac{\varrho_4 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon} + g & \text{in } \Omega, \\ \omega = v = 0 \text{ on } \partial\Omega, \quad \omega, v \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{18}$$

(i) First we verify that T is well defined.
Let $(\xi, \eta) \in L^2_\varrho(\Omega) \times L^2_\varrho(\Omega)$, we put

$$\begin{cases} \Psi_1(\xi, \eta) = (a+m)\dfrac{\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + b \dfrac{\varrho_2 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon}, \\ \Psi_2(\xi, \eta) = c\dfrac{\varrho_3 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + (d+m) \dfrac{\varrho_4 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon}. \end{cases} \tag{19}$$

We have

$$\int_\Omega (1+|x|^2)\left|\frac{\varrho_i\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\right|^2 \mathrm{d}x \leqslant \int_\Omega (1+|x|^2)|\varrho_i\xi|^2 \mathrm{d}x$$
$$\leqslant k_i \int_\Omega \varrho|\xi|^2 \mathrm{d}x, \quad (i=1,3),$$

and

$$\int_\Omega (1+|x|^2)\left|\frac{\varrho_i\eta}{1+\varepsilon|\eta|}1_{B_\varepsilon}\right|^2 \mathrm{d}x \leqslant \int_\Omega (1+|x|^2)|\varrho_i\eta|^2 \mathrm{d}x$$
$$\leqslant k_i \int_\Omega \varrho|\eta|^2 \mathrm{d}x, \quad (i=2,4).$$

So $\Psi_1(\xi,\eta)$, $\Psi_2(\xi,\eta) \in L^2_{p_1^{-1}}(\Omega)$.

In view of Theorem 2, the system

$$(20) \qquad \begin{cases} (-\Delta+q_1)\omega = -m\varrho_1\omega + (\Psi_1(\xi,\eta)+f) \text{ in } \Omega;\ \omega=0 \text{ on } \partial\Omega; \\ \hfill \omega\to 0 \text{ for } |x|\to+\infty, \\ (-\Delta+q_2)v = -m\varrho_4 v + (\Psi_2(\xi,\eta)+g) \text{ in } \Omega;\ v=0 \text{ on } \partial\Omega; \\ \hfill v\to 0 \text{ for } |x|\to+\infty \end{cases}$$

has one solution (ω, v) in $V\times V$, since

$$(\Psi_1(\xi,\eta)+f) \in L^2_{p_1^{-1}}(\Omega), \quad (\Psi_2(\xi,\eta)+g) \in L^2_{p_1^{-1}}(\Omega),$$
$$\lambda(q_1,\varrho_1) > -m \quad \text{and} \quad \lambda(q_2,\varrho_4) > -m.$$

(ii) For all $(\xi,\eta) \in L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$, we have

$$\left|\frac{\varrho_i\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\right| = \frac{1}{\varepsilon}\frac{\varepsilon\varrho_i|\xi|}{1+\varepsilon|\xi|}1_{B_\varepsilon} \leqslant \frac{k_i}{\varepsilon}1_{B_\varepsilon} \text{ a.e. in } \Omega, \quad (i=1,3), \quad 1_{B_\varepsilon} \in L^2_{p_1^{-1}}(\Omega).$$

Similarly

$$\left|\frac{\varrho_i\eta}{1+\varepsilon|\eta|}1_{B_\varepsilon}\right| \leqslant \frac{k_i}{\varepsilon}1_{B_\varepsilon} \text{ a.e. in } \Omega, \quad (i=2,4).$$

Then

$$(21) \qquad \begin{cases} |\Psi_1(\xi,\eta)| \leqslant 2\max(a+m,|b|)\dfrac{k}{\varepsilon}1_{B_\varepsilon}, \\ |\Psi_2(\xi,\eta)| \leqslant 2\max(|c|,d+m)\dfrac{k}{\varepsilon}1_{B_\varepsilon} \end{cases}$$

with $k=\max(k_1,k_2,k_3,k_4)$.

We put $h = 2\max(a+m, |b|, |c|, d+m)\frac{k}{\varepsilon}1_{B_\varepsilon}$.

We have

$$h \in L^2_{p_1^{-1}}(\Omega) \text{ and } |\Psi_1(\xi,\eta)| \leqslant h;\ |\Psi_2(\xi,\eta)| \leqslant h;\ \forall(\xi,\eta) \in L^2_\varrho(\Omega) \times L^2_\varrho(\Omega).$$

According to Theorem 2, the problem

$$P_1 := (-\Delta + q_1 + m\varrho_1)u = h + f \text{ in } \Omega;\ u = 0 \text{ on } \partial\Omega;\ u \to 0 \text{ for } |x| \to +\infty$$

(or with P_1' defined by replacing h by $-h$ in P_1) possesses one solution ξ^0 (ξ_0) in V.

Similarly, the problem

$$P_2 := (-\Delta + q_2 + m\varrho_4)v = h + g \text{ in } \Omega;\ v = 0 \text{ on } \partial\Omega;\ v \to 0 \text{ for } |x| \to +\infty$$

(or with P_2' obtained by replacing h by $-h$ in P_2) has one solution η^0 (η_0) in V.

Observe that $\xi_0 \leqslant \xi^0$. Indeed, we have

$$(-\Delta + q_1 + m\varrho_1)(\xi^0 - \xi_0) = 2h \text{ in } \Omega,\quad \xi^0 - \xi_0 = 0 \text{ on } \partial\Omega,$$
$$(\xi^0 - \xi_0) \to 0 \text{ for } |x| \to +\infty.$$

According to Theorem 1, $\xi^0 - \xi_0 \geqslant 0$. Similarly $\eta_0 - \eta^0 \leqslant 0$.

We consider now the restriction of T, denoted again by T, to the rectangle $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$.

We show that T admits a fixed point using Schauder's Theorem.

(iii) We prove first that the closed convex $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ is invariant by T.

Let $(\xi, \eta) \in [\xi_0, \xi^0] \times [\eta_0, \eta^0]$. We show that $\omega \in [\xi_0, \xi^0]$ and $v \in [\eta_0, \eta^0]$. Combining (20) with P_1, we get

$$(-\Delta + q_1 + m\varrho_1)(\xi^0 - \omega) = h - \Psi_1(\xi,\eta) \text{ in } \Omega,\quad \xi^0 - \omega = 0 \text{ on } \partial\Omega,$$
$$(\xi^0 - \omega) \to 0 \text{ for } |x| \to +\infty.$$

Since $h - \Psi_1(\xi,\eta) \geqslant 0$ then $\omega \leqslant \xi^0$. In the same way, we obtain $v \leqslant \eta^0$. On the other hand, we have

$$(-\Delta + q_1 + m\varrho_1)(\omega - \xi_0) = \Psi_1(\xi,\eta) + h \text{ in } \Omega,\quad \omega - \xi_0 = 0 \text{ on } \partial\Omega,$$
$$(\omega - \xi_0) \to 0 \text{ for } |x| \to +\infty.$$

Since $\Psi_1(\xi,\eta) + h \geqslant 0$ then $\xi_0 \leqslant \omega$. Analogously, we obtain $\eta_0 \leqslant v$.

Consequently $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ is invariant by T.

(iv) We show now that T is continuous. Let (ξ_n, η_n) be a sequence of $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ convergent to (ξ, η) in $L^2_\varrho(\Omega) \times L^2_\varrho(\Omega)$.

We put $T(\xi_n, \eta_n) = (\omega_n, v_n)$, $\quad T(\xi, \eta) = (\omega, v)$.

From (18) and (19), we get

$$(22) \qquad \begin{cases} (-\Delta + q_1 + m\varrho_1)(\omega_n - \omega) = \Psi_1(\xi_n, \eta_n) - \Psi_1(\xi, \eta) \quad \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)(v_n - v) = \Psi_2(\xi_n, \eta_n) - \Psi_2(\xi, \eta) \quad \text{in } \Omega, \\ \omega_n = \omega = v_n = v = 0 \text{ on } \partial\Omega; \quad \omega_n, \omega, v_n, v \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

Multiplying both the first and the second equation of the system (22) by $(\omega_n - \omega)$ and $(v_n - v)$ respectively, later integrating over Ω, we obtain by virtue of Hardy's inequality

$$\|\omega_n - \omega\|_1 \leqslant \sqrt{\gamma}\|\Psi_1(\xi_n, \eta_n) - \Psi_1(\xi, \eta)\|_{L^2_{p_1^{-1}}(\Omega)},$$
$$\|v_n - v\|_2 \leqslant \sqrt{\gamma}\|\Psi_2(\xi_n, \eta_n) - \Psi_2(\xi, \eta)\|_{L^2_{p_1^{-1}}(\Omega)}.$$

Then, to prove $(\omega_n, v_n) \to (\omega, v)$ in $V \times V$, it suffices to show that

$$\Psi_1(\xi_n, \eta_n) \to \Psi_1(\xi, \eta) \quad \text{and} \quad \Psi_2(\xi_n, \eta_n) \to \Psi_2(\xi, \eta) \text{ in } L^2_{p_1^{-1}}(\Omega).$$

We have

$$\left\| \frac{\varrho_1 \xi_n}{1 + \varepsilon|\xi_n|} 1_{B_\varepsilon} - \frac{\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} \right\|_{L^2_{p_1^{-1}}(\Omega)} = \frac{1}{\varepsilon} \left\| \frac{\varepsilon\varrho_1 \xi_n}{1 + \varepsilon|\xi_n|} 1_{B_\varepsilon} - \frac{\varepsilon\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} \right\|_{L^2_{p_1^{-1}}(\Omega)}.$$

The function $l(x) = x(1 + |x|)^{-1}$ is Lipschitzian on $\mathbb{R}$ and verifies

$$(23) \qquad \forall x, y \in \mathbb{R}, \quad |l(x) - l(y)| \leqslant |x - y|.$$

Consequently,

$$\begin{aligned} \frac{1}{\varepsilon} \left\| \frac{\varepsilon\varrho_1 \xi_n}{1 + \varepsilon|\xi_n|} 1_{B_\varepsilon} - \frac{\varepsilon\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} \right\|_{L^2_{p_1^{-1}}(\Omega)} &\leqslant \frac{1}{\varepsilon} \|\varepsilon\varrho_1\xi_n - \varepsilon\varrho_1\xi\|_{L^2_{p_1^{-1}}(\Omega)} \\ &\leqslant \|\varrho_1\xi_n - \varrho_1\xi\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } n \to +\infty, \end{aligned}$$

since

$$\begin{aligned} \int_\Omega (1 + |x|^2)\varrho_1^2 |\xi_n - \xi|^2 \,\mathrm{d}x &\leqslant \int_\Omega (1 + |x|^2) \frac{k_1}{1 + |x|^2} \varrho |\xi_n - \xi|^2 \,\mathrm{d}x \\ &\leqslant k_1 \|\xi_n - \xi\|^2_{L^2_\varrho(\Omega)} \to 0 \text{ for } n \to +\infty. \end{aligned}$$

Similarly, we show that $\varrho_2\eta_n(1+\varepsilon|\eta_n|)^{-1}1_{B_\varepsilon} \to \varrho_2\eta(1+\varepsilon|\eta|)^{-1}1_{B_\varepsilon}$ for $\varepsilon \to 0$ in $L^2_{p_1^{-1}}(\Omega)$.

Then $\Psi_1(\xi_n,\eta_n) \to \Psi_1(\xi,\eta)$ in $L^2_{p_1^{-1}}(\Omega)$ and therefore $\omega_n \to \omega$ in V.

In the same way, $v_n \to v$ in V. Consequently, T is continuous.

(v) Next we show that $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega) \to L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$ is compact.

We have the compact imbedding $V \subset L^2_\varrho(\Omega)$.

Indeed, let $(u_n)_n$ be a bounded sequence in V. $(u_n)_n$ is bounded in $H^1(B_\varepsilon)$. Hence the imbedding $H^1(B_\varepsilon)$ into $L^2(B_\varepsilon)$ is compact; there exists a subsequence denoted again by $(u_n)_n$, such that

$$\int_{B_\varepsilon} \varrho|u_p-u_q|^2\,\mathrm{d}x \leqslant \int_{B_\varepsilon}|u_p-u_q|^2\,\mathrm{d}x \to 0 \quad \text{when } p,q\to+\infty.$$

On the other hand, since the weight ϱ tends to 0 at infinity, we have

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon} \varrho|u_p-u_q|^2\,\mathrm{d}x &= \int_{\Omega\setminus B_\varepsilon}(1+|x|^2)\varrho\frac{1}{(1+|x|^2)}|u_p-u_q|^2\,\mathrm{d}x\\ &\leqslant k\sup_{|x|>1/\varepsilon}\frac{1}{(1+|x|^2)^{\alpha-1}}\|u_p-u_q\|^2_V \to 0 \text{ for } \varepsilon\to 0.\end{aligned}$$

Consequently, $(u_n)_n$ is a Cauchy sequence in $L^2_\varrho(\Omega)$.

Since $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)\to V\times V$ is continuous, $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)\to L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$ is compact. According to (iii) and (v), we can apply Schauder's fixed point theorem. Then there exists $(\xi,\eta)\in[\xi_0,\xi^0]\times[\eta_0,\eta^0]$ such that $T(\xi,\eta)=(\xi,\eta)$.

Since ξ and η depend on ε, we denote $\xi=u_\varepsilon$ and $\eta=v_\varepsilon$.

So $u_\varepsilon, v_\varepsilon$ verify the system

$$(24)\quad \begin{cases}(-\Delta+q_1+m\varrho_1)u_\varepsilon=(a+m)\dfrac{\varrho_1u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon}+b\dfrac{\varrho_2v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon}+f \text{ in } \Omega,\\[2mm] (-\Delta+q_2+m\varrho_4)v_\varepsilon=c\dfrac{\varrho_3u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon}+(d+m)\dfrac{\varrho_4v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon}+g \text{ in } \Omega,\\ \qquad u_\varepsilon=v_\varepsilon=0 \text{ on } \partial\Omega;\quad u_\varepsilon,v_\varepsilon\to 0 \text{ for } |x|\to+\infty.\end{cases}$$

(vi) We show that $(\varepsilon u_\varepsilon)_\varepsilon$ (as well as $(\varepsilon v_\varepsilon)_\varepsilon$) is a bounded sequence in V.

We multiply the first equation of the system (24) by $\varepsilon^2u_\varepsilon$ and integrate over Ω, obtaining

$$\begin{aligned}\int_\Omega(|\nabla(\varepsilon u_\varepsilon)|^2+(q_1+m\varrho_1)|\varepsilon u_\varepsilon|^2)\,\mathrm{d}x &= (a+m)\int_\Omega\frac{\varepsilon\varrho_1u_\varepsilon}{1+\varepsilon|u_\varepsilon|}\varepsilon u_\varepsilon1_{B_\varepsilon}\,\mathrm{d}x\\ &\quad+b\int_\Omega\frac{\varepsilon\varrho_2v_\varepsilon}{1+\varepsilon|v_\varepsilon|}\varepsilon u_\varepsilon1_{B_\varepsilon}\,\mathrm{d}x+\int_\Omega\varepsilon f\varepsilon u_\varepsilon\,\mathrm{d}x.\end{aligned}$$

Since $\varepsilon|u_\varepsilon|(1+\varepsilon|u_\varepsilon|)^{-1} < 1$, $\varepsilon|v_\varepsilon|(1+\varepsilon|v_\varepsilon|)^{-1} < 1$, $0 < \varepsilon < 1$, $f \in L^2_{p_1^{-1}}(\Omega)$ and $\alpha > N/2$, there exists a constant $M > 0$ such that $\|\varepsilon u_\varepsilon\|_1 \leqslant M$.

Similarly, there exists $M' > 0$ such that $\|\varepsilon v_\varepsilon\|_2 \leqslant M'$.

(vii) We show that $\varepsilon u_\varepsilon \to 0$ (as well as $\varepsilon v_\varepsilon \to 0$) when $\varepsilon \to 0$ in V.

Since $(\varepsilon u_\varepsilon)_\varepsilon$ is bounded in V, there exists a subsequence denoted again by $(\varepsilon u_\varepsilon)_\varepsilon$ (we denote it by writing for example $\varepsilon = \frac{1}{n}$, $n > 1$) weakly convergent to u^* in V and hence strongly convergent to u^* in $L^2_\varrho(\Omega)$.

We multiply the first equation of the system (24) by ε:

$$(-\Delta + (q_1 + m\varrho_1))(\varepsilon u_\varepsilon) = (a+m)\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + b\frac{\varepsilon\varrho_2 v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \varepsilon f \quad \text{in } \Omega.$$

Then $\forall\varphi \in D(\Omega)$,

$$\int_\Omega [\nabla(\varepsilon u_\varepsilon)\cdot\nabla\varphi + (q_1 + m\varrho_1)\varepsilon u_\varepsilon\varphi]\,\mathrm{d}x \int_\Omega [\nabla u^*\cdot\nabla\varphi + (q_1 + m\varrho_1)u^*\varphi]\,\mathrm{d}x \ \text{ for } \varepsilon \to 0.$$

Moreover $\forall\varphi \in D(\Omega)$, $\int_\Omega \varepsilon f\varphi\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$.

On the other hand,

$$\int_\Omega (1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x = \int_{B_\varepsilon}(1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x$$
$$+ \int_{\Omega\setminus B_\varepsilon}(1+|x|^2)\left|\frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x,$$

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon}(1+|x|^2)\left|\frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x &\leqslant \int_{\Omega\setminus B_\varepsilon}(1+|x|^2)|\varrho_1 u^*|^2\,\mathrm{d}x \\ &\leqslant k_1\int_{\Omega\setminus B_\varepsilon}\varrho|u^*|^2\,\mathrm{d}x \to 0 \ \text{ for } \varepsilon \to 0.\end{aligned}$$

Taking (23) into account, we have

$$\begin{aligned}\int_{B_\varepsilon}(1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x &\leqslant \int_\Omega (1+|x|^2)|\varepsilon\varrho_1 u_\varepsilon - \varrho_1 u^*|^2\,\mathrm{d}x \\ &\leqslant k_1\int_\Omega \varrho|\varepsilon u_\varepsilon - u^*|^2\,\mathrm{d}x \to 0 \ \text{ for } \varepsilon \to 0.\end{aligned}$$

Similarly, we show that $\varepsilon\varrho_2 v_\varepsilon(1+\varepsilon|v_\varepsilon|)^{-1}1_{B_\varepsilon} \to \varrho_2 v^*(1+|v^*|)^{-1}$ for $\varepsilon \to 0$ in $L^2_{p_1^{-1}}(\Omega)$.

In the same manner we can establish that

$$\varepsilon\Psi_2(u_\varepsilon, v_\varepsilon) \to c\frac{\varrho_3 u^*}{1+|u^*|} + (d+m)\frac{\varrho_4 v^*}{1+|v^*|} \ \text{ for } \varepsilon \to 0 \text{ in } L^2_{p_1^{-1}}(\Omega).$$

We get, when $\varepsilon \to 0$,

$$(25)\qquad \begin{cases} (-\Delta + q_1 + m\varrho_1)u^* = (a+m)\dfrac{\varrho_1 u^*}{1+|u^*|} + b\dfrac{\varrho_2 v^*}{1+|v^*|} & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)v^* = c\dfrac{\varrho_3 u^*}{1+|u^*|} + (d+m)\dfrac{\varrho_4 v^*}{1+|v^*|} & \text{in } \Omega, \\ \qquad u^* = v^* = 0 \quad \text{on } \partial\Omega; \quad u^*, v^* \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

We show now that $u^* = v^* = 0$.

We multiply the first equation of the system (25) by u^* and integrate over Ω:

$$\begin{aligned} \int_\Omega \left(|\nabla u^*|^2 + (q_1 + m\varrho_1)|u^*|^2\right) \mathrm{d}x &= (a+m)\int_\Omega \frac{\varrho_1 |u^*|^2}{1+|u^*|} = \mathrm{d}x + b\int_\Omega \frac{\varrho_2 u^* v^*}{1+|v^*|}\,\mathrm{d}x \\ &\leqslant (a+m)\int_\Omega \varrho_1 |u^*|^2\,\mathrm{d}x + |b|\int_\Omega \varrho_2 |u^* v^*|\,\mathrm{d}x. \end{aligned}$$

By virtue of the variational characterization of $\lambda(q_1, \varrho_1)$ and using (10), we get

$$(26)\qquad (\lambda(q_1,\varrho_1) - a)\int_\Omega \varrho_1 |u^*|^2\,\mathrm{d}x \leqslant |b|\Big(\int_\Omega \varrho_1 |u^*|^2\,\mathrm{d}x\Big)^{\frac12} \cdot \Big(\int_\Omega \varrho_4 |v^*|^2\,\mathrm{d}x\Big)^{\frac12}.$$

In the same way, we prove that

$$(27)\qquad (\lambda(q_2,\varrho_4) - d)\int_\Omega \varrho_4 |v^*|^2\,\mathrm{d}x \leqslant |c|\Big(\int_\Omega \varrho_1 |u^*|^2\,\mathrm{d}x\Big)^{\frac12} \cdot \Big(\int_\Omega \varrho_4 |v^*|^2\,\mathrm{d}x\Big)^{\frac12}.$$

Combining (26), (27) with (14), we have $u^* = v^* = 0$.

(viii) We show that the sequence $(u_\varepsilon)_\varepsilon$ (as well as $(v_\varepsilon)_\varepsilon$) is bounded in V. We suppose that $\|u_\varepsilon\|_V \to +\infty$ for $\varepsilon \to 0$ and $\|v_\varepsilon\|_V \to +\infty$ for $\varepsilon \to 0$ and define

$$\begin{aligned} t_\varepsilon &= \max(\|u_\varepsilon\|_V, \|v_\varepsilon t\|_V), \\ z_\varepsilon &= \frac{1}{t_\varepsilon} u_\varepsilon \quad \text{then } \|z_\varepsilon\|_V \leqslant 1, \\ w_\varepsilon &= \frac{1}{t_\varepsilon} v_\varepsilon \quad \text{then } \|w_\varepsilon\|_V \leqslant 1. \end{aligned}$$

Since $(z_\varepsilon)_\varepsilon$ is a bounded sequence in V, there exists a subsequence denoted again by $(z_\varepsilon)_\varepsilon$, weakly convergent to z in V and hence strongly convergent to z in $L^2_\varrho(\Omega)$.

Similarly, $(w_\varepsilon)_\varepsilon$ converges to w , weakly in V and strongly in $L^2_\varrho(\Omega)$.

Taking (24) into account, we get

$$(28)\qquad \begin{cases} (-\Delta + q_1 + m\varrho_1)z_\varepsilon = (a+m)\dfrac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + b\dfrac{\varrho_2 w_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \dfrac{1}{t_\varepsilon}f & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)w_\varepsilon = c\dfrac{\varrho_3 z_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + (d+m)\dfrac{\varrho_4 w_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \dfrac{1}{t_\varepsilon}g & \text{in } \Omega, \\ \qquad z_\varepsilon = w_\varepsilon = 0 \quad \text{on } \partial\Omega; \quad z_\varepsilon, w_\varepsilon \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

We know that $\forall \varphi \in D(\Omega)$, $\int_\Omega (\nabla z_\varepsilon \cdot \nabla\varphi + (q_1 + m\varrho_1) z_\varepsilon \varphi)\,\mathrm{d}x \to \int_\Omega (\nabla z \cdot \nabla\varphi + (q_1 + m\varrho_1) z\varphi)\,\mathrm{d}x$ for $\varepsilon \to 0$ and $\int_\Omega \frac{1}{t_\varepsilon} f\varphi\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$.

On the other hand,

$$\left\| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_1 z \right\|^2_{L^2_{p_1^{-1}}(\Omega)} = \int_{B_\varepsilon} (1+|x|^2) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x + \int_{\Omega\setminus B_\varepsilon} (1+|x|^2) |\varrho_1 z|^2\,\mathrm{d}x.$$

We have $\int_{\Omega\setminus B_\varepsilon} (1+|x|^2)|\varrho_1 z|^2\,\mathrm{d}x \leqslant k_1 \int_{\Omega\setminus B_\varepsilon} \varrho |z|^2\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$, and

$$\begin{aligned}
\int_{B_\varepsilon} (1+|x|^2) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x &\leqslant \int_\Omega (1+|x|^2) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x \\
&\leqslant \int_\Omega (1+|x|^2) \left| \frac{\varrho_1 (z_\varepsilon - z) - \varepsilon\varrho_1 z |u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x \\
&\leqslant 2 \int_\Omega (1+|x|^2) \left(\left| \frac{\varrho_1 (z_\varepsilon - z)}{1+\varepsilon|u_\varepsilon|} \right|^2 + \left| \frac{\varepsilon\varrho_1 z |u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \right) \mathrm{d}x.
\end{aligned}$$

So

$$\begin{aligned}
\int_\Omega (1+|x|^2) \left| \frac{\varrho_1 (z_\varepsilon - z)}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x &\leqslant \int_\Omega (1+|x|^2) |\varrho_1 (z_\varepsilon - z)|^2\,\mathrm{d}x \\
&\leqslant k_1 \int_\Omega \varrho |z_\varepsilon - z|^2\,\mathrm{d}x \to 0 \text{ for } \varepsilon \to 0.
\end{aligned}$$

Moreover,

$$\left| \frac{\varepsilon_1 z |u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \to 0 \text{ for } \varepsilon \to 0 \quad \text{a.e. in } \Omega,$$

$$(1+|x|^2) \left| \frac{\varepsilon\varrho_1 z |u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \leqslant (1+|x|^2)|\varrho_1 z|^2 \leqslant k_1 \varrho |z|^2.$$

Since $z \in L^2_\varrho(\Omega)$, by virtue of the Lebesgue dominated convergence theorem we deduce that

$$\int_\Omega (1+|x|^2) \left| \frac{\varepsilon\varrho_1 z |u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x \to 0 \text{ for } \varepsilon \to 0.$$

Hence

$$\left\| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_1 z \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0.$$

Analogously, we obtain

$$\left\| \frac{\varrho_3 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_3 z \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0$$

and

$$\left\| \frac{\varrho_i w_\varepsilon}{1+\varepsilon|v_\varepsilon|} 1_{B_\varepsilon} - \varrho_i w \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0 \quad (i=2,4).$$

When $\varepsilon \to 0$, we get

$$\text{(29)} \qquad \begin{cases} (-\Delta + q_1)z = a\varrho_1 z + b\varrho_2 w & \text{in } \Omega, \\ (-\Delta + q_2)w = c\varrho_3 z + d\varrho_4 w & \text{in } \Omega, \\ z = w = 0 \text{ on } \partial\Omega; \quad z, w \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

By virtue of Lemma 1, we have $z = w = 0$, which is in contradiction with the fact that at least one sequence ((z_ε) or (w_ε)) has the norm equal to one. The sequence (u_ε) (as well as (v_ε)) is bounded in V.

(ix) We extract a sequence denoted again by (u_ε) ((v_ε)) weakly convergent to u^0 (or v^0) in V and hence strongly convergent to the same limit in $L^2_\varrho(\Omega)$.

We have $\forall \varphi \in D(\Omega)$,

$$\int_\Omega (\nabla u_\varepsilon \cdot \nabla\varphi + (q_1 + m\varrho_1)u_\varepsilon\varphi)\,\mathrm{d}x \to \int_\Omega (\nabla u^0 \cdot \nabla\varphi + (q_1 + m\varrho_1)u^0\varphi)\,\mathrm{d}x, \text{ for } \varepsilon \to 0$$
$$\int_\Omega (\nabla v_\varepsilon \cdot \nabla\varphi + (q_2 + m\varrho_4)v_\varepsilon\varphi)\,\mathrm{d}x \to \int_\Omega (\nabla v^0 \cdot \nabla\varphi + (q_2 + m\varrho_4)v^0\varphi)\,\mathrm{d}x \text{ for } \varepsilon \to 0.$$

In the same way as in (viii), we show that

$$\left\| \frac{\varrho_i u_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_i u^0 \right\|^2_{L_{p_1^{-1}}}(\Omega) \to 0 \text{ for } \varepsilon \to 0 \quad (i=1,3),$$
$$\left\| \frac{\varrho_i v_\varepsilon}{1+\varepsilon|v_\varepsilon|} 1_{B_\varepsilon} - \varrho_i v^0 \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0 \quad (i=2,4).$$

When $\varepsilon \to 0$, we get

$$\text{(S)} \qquad \begin{cases} (-\Delta + q_1)u^0 = a\varrho_1 u^0 + b\varrho_2 v^0 + f & \text{in } \Omega, \\ (-\Delta + q_2)v^0 = c\varrho_3 u^0 + d\varrho_4 v^0 + g & \text{in } \Omega, \\ u^0 = v^0 = 0 \text{ on } \partial\Omega; \quad u^0, v^0 \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

We conclude that (u^0, v^0) is a solution of (S). □

The uniqueness of the solution follows from Lemma 1. Indeed, let (u^1, v^1) be another solution of the system (S). We put $w = u^0 - u^1$ and $z = v^0 - v^1$, then (w, z) is a solution of the system (15). Lemma 1 gives $(w, z) = (0, 0)$.

Acknowledgment. We wish to thank the referee for helpful comments.

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49 (2004) APPLICATIONS OF MATHEMATICS No. 3, 185–199

STUDY OF SOME NONCOOPERATIVE LINEAR ELLIPTIC SYSTEMS[1]

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(Received November 21, 2001)

Abstract. Using an approximation method, we show the existence of solutions for some noncooperative elliptic systems defined on an unbounded domain.

Keywords: Schrödinger's operators, weighted Sobolev spaces, maximum principle, min-max formula, noncooperative systems

MSC 2000: 35P15

1. Introduction

We study here some noncooperative elliptic systems defined on a connected and unbounded open set $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 3)$ of the form

$$
\text{(S)} \qquad \begin{cases} -\Delta u + q_1 u = a\varrho_1 u + b\varrho_2 v + f & \text{in } \Omega, \\ -\Delta v + q_2 v = c\varrho_3 u + d\varrho_4 v + g & \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega; \quad u, v \to 0 \ \text{for } |x| \to +\infty \end{cases}
$$

where ϱ_1, ϱ_2, ϱ_3, ϱ_4 (sometimes referred to as *weight functions*), q_1, q_2, are positive functions; f and g are measurable functions; a, b, c and d are real numbers. u and v are unknown real-valued functions defined in Ω and belonging to appropriate function spaces. The system (S) is noncooperative since b and c are not necessarily positive. Under appropriate assumptions on the coefficients, we show the existence of non-trivial solutions.

[1] This work was supported by a grant from ANDRU under No. CU39904.

Generally, in order to study a nonlinear problem, we consider the linear approximation, which is easy enough to resolve of course. Here we have explored the inverse process. In other words, we show that the linear system (S) can be taken as the limit of a sequence of nonlinear systems.

The paper is organized as follows. In Section 1, we establish a Maximum Principle result in the scalar case. We choose decreasing weight functions which lead to a gain of compactness, (see [7] and [8]). In Section 2, we obtain an existence and uniqueness theorem for system (S). In order to prove it, we apply a nonlinear method introduced in [3] and [4]. The main tool used here is Schauder's fixed point theorem. The Maximum Principle guarantees the invariance of subsets under operators. So, we can prove the existence of solution for nonlinear systems. We observe that there exists a vast literature on the use of nonlinear methods to the study of noncooperative elliptic systems. We point out that an interesting version of the Maximum Principle for noncooperative systems was given in [13]. We refer the reader to the works in [5], [6] and references therein.

2. Notation

We denote

(1) $u^{\pm} = \max(\pm u, 0)$, so $u = u^+ - u^-$.

(2) $\mathrm{mes}(\cdot)$ is the Lebesgue N-dimensional measure.

(3) For $\varepsilon \in \,]0,1[$ fixed: $B_\varepsilon = \Omega \cap B(0, 1/\varepsilon) = \{x \in \Omega / |x| < 1/\varepsilon\}$.
1_{B_ε} is the characteristic function of B_ε.

(4) $p_\alpha(x) = (1+|x|^2)^{-\alpha}$, $\alpha \in \mathbb{R}_+^*$; $\mathbb{R}_+^*$ is the set of positive real numbers.

(5) Let $D(\Omega)$ be the set of all infinitely differentiable functions with compact support in Ω. We denote by $D'(\Omega)$ the dual space of $D(\Omega)$. Let us define $V(\mathbb{R}^N) = \Big\{u \in D'(\mathbb{R}^N) / \int_{\mathbb{R}^N} (|\nabla u|^2 + p_1|u|^2)\,\mathrm{d}x < +\infty\Big\}$.
The space V is the closure of $D(\Omega)$ with respect to the norm

$$\|u\|_V = \left(\int_\Omega (|\nabla u|^2 + p_1|u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

equivalent (under some conditions) to the norms

$$\|u\|_i = \left(\int_\Omega (|\nabla u|^2 + (q_i + m\varrho_j)|u|^2)\,\mathrm{d}x\right)^{\frac{1}{2}} \quad (i = 1,2;\ \ j = 1,4;\ \ m \in \mathbb{R}_+^*).$$

$V(\mathbb{R}^N)$ is a Hilbert space, $\forall N \geqslant 3$ (see [17], p. 230).

(6) $L_p^2(\Omega) = \big\{u \in D'(\Omega)/p^{\frac{1}{2}}u \in L^2(\Omega)\big\}$ where $p(x)$ is a positive function defined on Ω.

$H^m(\Omega) = \{u \in D'(\Omega)/D^\alpha u \in L^2(\Omega),\ 0 \leqslant |\alpha| \leqslant m\}.$

(7) $\varrho = \max_{x\in\Omega}(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$.

3. Hypotheses

We suppose that

(8) $\exists \alpha, \beta \in \mathbb{R},\ \alpha > N/2,\ \beta \geqslant 1;\ \exists k_i > 0\ (i = 1,2,3,4),\ \exists c_i > 0\ (i = 1,2)$ such that

$$\begin{aligned} 0 \leqslant &\varrho_i(x) \leqslant k_i p_\alpha(x), \quad i = 1,2,3,4,\\ 0 \leqslant &q_i(x) \leqslant c_i p_\beta(x), \quad i = 1,2. \end{aligned}$$

(9) $\varrho_2(x) \leqslant \sqrt{\varrho_1(x)\varrho_4(x)},\quad \forall x \in \Omega,\ \varrho_3(x) \leqslant \sqrt{\varrho_1(x)\varrho_4(x)}\quad \forall x \in \Omega.$

(10) $f, g \in L^2_{p_1^{-1}}(\Omega) = \bigl\{u \in D'(\Omega)/ \int_\Omega (1+|x|^2)|u|^2\,\mathrm{d}x < +\infty\bigr\}.$

4. Existence of solutions

4.1. Remarks on the scalar case

We consider the problem

$$\begin{cases} -\Delta u + q_1 u = \lambda \varrho_1 u \quad \text{in } \Omega,\\ u = 0 \ \text{ on } \partial\Omega; \quad u \to 0 \ \text{ for } |x| \to +\infty. \end{cases} \tag{11}$$

By [7], [8], Problem (11) possesses an increasing infinite sequence of positive eigenvalues.

Moreover, the first eigenvalue $\lambda(q_1, \varrho_1)$ is principal ([9]) and is characterized variationally by

$$\lambda(q_1,\varrho_1)\int_\Omega \varrho_1|u|^2\,\mathrm{d}x \leqslant \int_\Omega \bigl(|\nabla u|^2 + q_1|u|^2\bigr)\,\mathrm{d}x; \quad \forall u \in V. \tag{12}$$

We consider now the problem

$$\begin{cases} -\Delta u + q_1 u = \lambda \varrho_1 u + h \quad \text{in } \Omega,\\ u = 0 \ \text{ on } \partial\Omega; \quad u \to 0 \ \text{ for } |x| \to +\infty. \end{cases} \tag{13}$$

We claim that the Maximum Principle holds for Problem (13) with condition $h \geqslant 0$, if all solutions of Problem (13) are nonnegative.

Theorem 1. *Assume that* (8) *holds and* $h \in L^2_{p_1^{-1}}(\Omega)$. *Problem* (13) *satisfies the Maximum Principle if and only if* $\lambda(q_1, \varrho_1) > \lambda$.

Proof. Let φ be an eigenfunction associated with $\lambda(q_1, \varrho_1)$, then φ does not change sign. Multiplying the equation in (13) by φ and integrating over Ω, we obtain

$$(\lambda(q_1, \varrho_1) - \lambda) \int_\Omega \varrho_1 u \varphi \,\mathrm{d}x = \int_\Omega h\varphi \,\mathrm{d}x.$$

If the Maximum Principle holds then $\int_\Omega h\varphi \,\mathrm{d}x > 0$ implies $\int_\Omega \varrho_1 u\varphi \,\mathrm{d}x > 0$.

Consequently $\lambda(q_1, \varrho_1) > \lambda$.

Conversely, if $\lambda(q_1, \varrho_1) > \lambda$ then $u \geqslant 0$. Indeed, observe that $u = u^+ - u^-$. Now, multiplying the equation of Problem (13) by u^- and integrating over Ω, we get

$$\int_\Omega (\nabla u \nabla u^- + q_1 u u^-) \,\mathrm{d}x = \lambda \int_\Omega \varrho_1 u u^- \,\mathrm{d}x + \int_\Omega h u^- \,\mathrm{d}x.$$

Since

$$\int_\Omega \nabla u^+ \nabla u^- \,\mathrm{d}x = \int_\Omega q_1 u^+ u^- \,\mathrm{d}x = \int_\Omega \varrho_1 u^+ u^- \,\mathrm{d}x = 0,$$

we obtain

$$\int_\Omega \big(|\nabla u^-|^2 + q_1 |u^-|^2\big) \,\mathrm{d}x = \lambda \int_\Omega \varrho_1 |u^-|^2 \,\mathrm{d}x - \int_\Omega h u^- \,\mathrm{d}x.$$

By (12), we have

$$\lambda(q_1, \varrho_1) \int_\Omega \varrho_1 |u^-|^2 \,\mathrm{d}x \leqslant \lambda \int_\Omega \varrho_1 |u^-|^2 \,\mathrm{d}x - \int_\Omega h u^- \,\mathrm{d}x,$$

$$0 < (\lambda(q_1, \varrho_1) - \lambda) \int_\Omega \varrho_1 |u^-|^2 \,\mathrm{d}x \leqslant - \int_\Omega h u^- \,\mathrm{d}x \leqslant 0$$

Hence $u \geqslant 0$. □

Theorem 2. *Under the assumptions of Theorem* 1, *Problem* (13) *with condition* $h \geqslant 0$ *has one nonnegative solution in* V *if and only if* $\lambda(q_1, \varrho_1) > \lambda$.

Proof. In view of Theorem 1, the necessary condition holds.

Let us consider the sufficient condition. The differential form

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + q_1 u v) \,\mathrm{d}x - \lambda \int_\Omega \varrho_1 u v \,\mathrm{d}x$$

is continuous and coercive on $V \times V$. Indeed, let $m \in \mathbb{R}_+^*$ be such that $\lambda + m > 0$. Taking into account the variational characterization of $\lambda(q_1, \varrho_1)$ we have

$$\begin{aligned} a(u,u) &= \int_\Omega \big[|\nabla u|^2 + (q_1 + m\varrho_1)|u|^2\big]\,\mathrm{d}x - (\lambda + m)\int_\Omega \varrho_1 |u|^2\,\mathrm{d}x \\ &\geqslant \Big(1 - \frac{\lambda + m}{\lambda(q_1,\varrho_1) + m}\Big)\|u\|_1^2 \end{aligned}$$

where $\|\cdot\|_1$ is a norm equivalent to $\|\cdot\|_V$ by virtue of (8) and Hardy's inequality

$$\Big(\forall N \geqslant 3,\ \ \exists \gamma > 0\colon\ \forall u \in D(\mathbb{R}^N),\ \ \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leqslant \gamma \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x\Big).$$

Since $f \in L^2_{p_1^{-1}}(\Omega)$, the application $v \mapsto \int_\Omega hv\,\mathrm{d}x$ is a continuous linear form on V. Hence by the Lax-Milgram lemma, Problem (13) possesses one solution u. By Theorem 1, u is nonnegative. $\square$

4.2. Vectorial case

Let us consider the system (S) now. In what follows, we will find a sufficient condition for the system (S) to have a unique solution.

Theorem 3. *Let* (8), (9) *and* (10) *be satisfied, as well as*

$$\lambda(q_1,\varrho_1) > a;\ \ (\lambda(q_1,\varrho_1) - a)(\lambda(q_2,\varrho_4) - d) > |bc|. \tag{14}$$

Then the system (S) *has one (weak) solution in* $V \times V$.

First, we state the following lemma.

Lemma 1. *Assume that the hypothesis of Theorem* 3 *holds. Let* $(z,w) \in V \times V$ *be a solution of*

$$\begin{cases} (-\Delta + q_1)z = a\varrho_1 z + b\varrho_2 w & \text{in } \Omega, \\ (-\Delta + q_2)w = c\varrho_3 z + d\varrho_4 w & \text{in } \Omega, \\ z = w = 0 \ \text{on } \partial\Omega; \quad z, w \to 0 \ \text{for } |x| \to +\infty. \end{cases} \tag{15}$$

Then $(z,w) = (0,0)$.

P r o o f. We multiply the first equation of the system (15) by z and integrate over Ω:

$$\int_\Omega (|\nabla z|^2 + q_1|z|^2)\,\mathrm{d}x = a\int_\Omega \varrho_1 |z|^2\,\mathrm{d}x + b\int_\Omega \varrho_2 zw\,\mathrm{d}x.$$

Using the variational characterization of $\lambda(q_1, \varrho_1)$, we have

$$0 < (\lambda(q_1, \varrho_1) - a) \int_\Omega \varrho_1 |z|^2 \, \mathrm{d}x \leqslant |b| \left| \int_\Omega \varrho_2 z w \, \mathrm{d}x \right|.$$

By hypothesis (9) and the Cauchy-Schwarz inequality we obtain

$$(\lambda(q_1, \varrho_1) - a) \int_\Omega \varrho_1 |z|^2 \, \mathrm{d}x \leqslant |b| \Big(\int_\Omega \varrho_1 |z|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_\Omega \varrho_4 |w|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}. \tag{16}$$

Similarly, multiplying the second equation of system (15) by w and integrating over Ω, we get

$$(\lambda(q_2, \varrho_4) - d) \int_\Omega \varrho_4 |w|^2 \, \mathrm{d}x \leqslant |c| \Big(\int_\Omega \varrho_1 |z|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int_\Omega \varrho_4 |w|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}}. \tag{17}$$

Combining (16) with (17), we obtain

$$0 < [(\lambda(q_1, \varrho_1) - a)(\lambda(q_2, \varrho_4) - d) - |bc|] \int_\Omega \varrho_1 |z|^2 \, \mathrm{d}x \cdot \int_\Omega \varrho_4 |w|^2 \, \mathrm{d}x \leqslant 0,$$

which implies that $z = w = 0$ a.e. □

Proof of Theorem 3. Let $m \in \mathbb{R}_+^*$ be such that $a + m > 0$ and $d + m > 0$. We define an operator

$$T \colon L^2_\varrho(\Omega) \times L^2_\varrho(\Omega) \to V \times V, \quad (\xi, \eta) \mapsto T(\xi, \eta) = (\omega, \upsilon)$$

such that (ω, υ) verifies the system

$$\begin{cases} (-\Delta + q_1 + m\varrho_1)\omega = (a+m)\dfrac{\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + b \dfrac{\varrho_2 \eta}{1 + \varepsilon |\eta|} 1_{B_\varepsilon} + f \quad \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)\upsilon = c \dfrac{\varrho_3 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + (d+m) \dfrac{\varrho_4 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon} + g \quad \text{in } \Omega, \\ \omega = \upsilon = 0 \text{ on } \partial\Omega, \quad \omega, \upsilon \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{18}$$

(i) First we verify that T is well defined.
Let $(\xi, \eta) \in L^2_\varrho(\Omega) \times L^2_\varrho(\Omega)$, we put

$$\begin{cases} \Psi_1(\xi, \eta) = (a+m)\dfrac{\varrho_1 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + b \dfrac{\varrho_2 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon}, \\ \Psi_2(\xi, \eta) = c \dfrac{\varrho_3 \xi}{1 + \varepsilon|\xi|} 1_{B_\varepsilon} + (d+m) \dfrac{\varrho_4 \eta}{1 + \varepsilon|\eta|} 1_{B_\varepsilon}. \end{cases} \tag{19}$$

We have

$$\int_\Omega \big(1+|x|^2\big)\Big|\frac{\varrho_i\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\Big|^2\,\mathrm{d}x \leqslant \int_\Omega \big(1+|x|^2\big)|\varrho_i\xi|^2\,\mathrm{d}x$$
$$\leqslant k_i\int_\Omega \varrho|\xi|^2\,\mathrm{d}x, \quad (i=1,3),$$

and

$$\int_\Omega \big(1+|x|^2\big)\Big|\frac{\varrho_i\eta}{1+\varepsilon|\eta|}1_{B_\varepsilon}\Big|^2\,\mathrm{d}x \leqslant \int_\Omega \big(1+|x|^2\big)|\varrho_i\eta|^2\,\mathrm{d}x$$
$$\leqslant k_i\int_\Omega \varrho|\eta|^2\,\mathrm{d}x, \quad (i=2,4).$$

So $\Psi_1(\xi,\eta),\ \Psi_2(\xi,\eta)\in L^2_{p_1^{-1}}(\Omega)$.

In view of Theorem 2, the system

$$\text{(20)}\qquad \begin{cases} (-\Delta+q_1)\omega=-m\varrho_1\omega+\big(\Psi_1(\xi,\eta)+f\big)\ \text{ in }\Omega;\ \ \omega=0\ \text{ on }\partial\Omega;\\ \hfill \omega\to 0\ \text{ for }|x|\to+\infty,\\ (-\Delta+q_2)\upsilon=-m\varrho_4\upsilon+\big(\Psi_2(\xi,\eta)+g\big)\ \text{ in }\Omega;\ \ \upsilon=0\ \text{ on }\partial\Omega;\\ \hfill \upsilon\to 0\ \text{ for }|x|\to+\infty \end{cases}$$

has one solution (ω,υ) in $V\times V$, since

$$\big(\Psi_1(\xi,\eta)+f\big)\in L^2_{p_1^{-1}}(\Omega),\quad \big(\Psi_2(\xi,\eta)+g\big)\in L^2_{p_1^{-1}}(\Omega),$$
$$\lambda(q_1,\varrho_1)>-m\quad\text{and}\quad \lambda(q_2,\varrho_4)>-m.$$

(ii) For all $(\xi,\eta)\in L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$, we have

$$\Big|\frac{\varrho_i\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\Big| = \frac{1}{\varepsilon}\frac{\varepsilon\varrho_i|\xi|}{1+\varepsilon|\xi|}1_{B_\varepsilon}\leqslant\frac{k_i}{\varepsilon}1_{B_\varepsilon}\ \text{ a.e. in }\Omega,\quad (i=1,3),\quad 1_{B_\varepsilon}\in L^2_{p_1^{-1}}(\Omega).$$

Similarly

$$\Big|\frac{\varrho_i\eta}{1+\varepsilon|\eta|}1_{B_\varepsilon}\Big|\leqslant\frac{k_i}{\varepsilon}1_{B_\varepsilon}\ \text{ a.e. in }\Omega,\quad (i=2,4).$$

Then

$$\text{(21)}\qquad \begin{cases} |\Psi_1(\xi,\eta)|\leqslant 2\max(a+m,|b|)\dfrac{k}{\varepsilon}1_{B_\varepsilon},\\[2ex] |\Psi_2(\xi,\eta)|\leqslant 2\max(|c|,d+m)\dfrac{k}{\varepsilon}1_{B_\varepsilon} \end{cases}$$

with $k=\max(k_1,k_2,k_3,k_4)$.

We put $h = 2\max(a+m, |b|, |c|, d+m)\frac{k}{\varepsilon}1_{B_\varepsilon}$.

We have

$$h \in L^2_{p_1^{-1}}(\Omega) \text{ and } |\Psi_1(\xi,\eta)| \leqslant h; \ \ |\Psi_2(\xi,\eta)| \leqslant h; \ \ \forall(\xi,\eta) \in L^2_\varrho(\Omega) \times L^2_\varrho(\Omega).$$

According to Theorem 2, the problem

$$P_1 := (-\Delta + q_1 + m\varrho_1)u = h + f \text{ in } \Omega; \ \ u = 0 \text{ on } \partial\Omega; \ \ u \to 0 \text{ for } |x| \to +\infty$$

(or with P_1' defined by replacing h by $-h$ in P_1) possesses one solution ξ^0 (ξ_0) in V.

Similarly, the problem

$$P_2 := (-\Delta + q_2 + m\varrho_4)v = h + g \text{ in } \Omega; \ \ v = 0 \text{ on } \partial\Omega; \ \ v \to 0 \text{ for } |x| \to +\infty$$

(or with P_2' obtained by replacing h by $-h$ in P_2) has one solution η^0 (η_0) in V.

Observe that $\xi_0 \leqslant \xi^0$. Indeed, we have

$$(-\Delta + q_1 + m\varrho_1)(\xi^0 - \xi_0) = 2h \text{ in } \Omega, \quad \xi^0 - \xi_0 = 0 \text{ on } \partial\Omega,$$
$$(\xi^0 - \xi_0) \to 0 \text{ for } |x| \to +\infty.$$

According to Theorem 1, $\xi^0 - \xi_0 \geqslant 0$. Similarly $\eta_0 - \eta^0 \leqslant 0$.

We consider now the restriction of T, denoted again by T, to the rectangle $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$.

We show that T admits a fixed point using Schauder's Theorem.

(iii) We prove first that the closed convex $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ is invariant by T.

Let $(\xi, \eta) \in [\xi_0, \xi^0] \times [\eta_0, \eta^0]$. We show that $\omega \in [\xi_0, \xi^0]$ and $v \in [\eta_0, \eta^0]$. Combining (20) with P_1, we get

$$(-\Delta + q_1 + m\varrho_1)(\xi^0 - \omega) = h - \Psi_1(\xi,\eta) \text{ in } \Omega, \quad \xi^0 - \omega = 0 \text{ on } \partial\Omega,$$
$$(\xi^0 - \omega) \to 0 \text{ for } |x| \to +\infty.$$

Since $h - \Psi_1(\xi,\eta) \geqslant 0$ then $\omega \leqslant \xi^0$. In the same way, we obtain $v \leqslant \eta^0$. On the other hand, we have

$$(-\Delta + q_1 + m\varrho_1)(\omega - \xi_0) = \Psi_1(\xi,\eta) + h \text{ in } \Omega, \quad \omega - \xi_0 = 0 \text{ on } \partial\Omega,$$
$$(\omega - \xi_0) \to 0 \text{ for } |x| \to +\infty.$$

Since $\Psi_1(\xi,\eta) + h \geqslant 0$ then $\xi_0 \leqslant \omega$. Analogously, we obtain $\eta_0 \leqslant v$.

Consequently $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ is invariant by T.

(iv) We show now that T is continuous. Let (ξ_n, η_n) be a sequence of $[\xi_0, \xi^0] \times [\eta_0, \eta^0]$ convergent to (ξ, η) in $L^2_\varrho(\Omega) \times L^2_\varrho(\Omega)$.

We put $T(\xi_n, \eta_n) = (\omega_n, \upsilon_n)$, $\quad T(\xi, \eta) = (\omega, \upsilon)$.

From (18) and (19), we get

$$\tag{22} \begin{cases} (-\Delta + q_1 + m\varrho_1)(\omega_n - \omega) = \Psi_1(\xi_n, \eta_n) - \Psi_1(\xi, \eta) & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)(\upsilon_n - \upsilon) = \Psi_2(\xi_n, \eta_n) - \Psi_2(\xi, \eta) & \text{in } \Omega, \\ \omega_n = \omega = \upsilon_n = \upsilon = 0 \text{ on } \partial\Omega; \quad \omega_n, \omega, \upsilon_n, \upsilon \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

Multiplying both the first and the second equation of the system (22) by $(\omega_n - \omega)$ and $(\upsilon_n - \upsilon)$ respectively, later integrating over Ω, we obtain by virtue of Hardy's inequality

$$\|\omega_n - \omega\|_1 \leqslant \sqrt{\gamma}\|\Psi_1(\xi_n, \eta_n) - \Psi_1(\xi, \eta)\|_{L^2_{p_1^{-1}}(\Omega)},$$
$$\|\upsilon_n - \upsilon\|_2 \leqslant \sqrt{\gamma}\|\Psi_2(\xi_n, \eta_n) - \Psi_2(\xi, \eta)\|_{L^2_{p_1^{-1}}(\Omega)}.$$

Then, to prove $(\omega_n, \upsilon_n) \to (\omega, \upsilon)$ in $V \times V$, it suffices to show that

$$\Psi_1(\xi_n, \eta_n) \to \Psi_1(\xi, \eta) \quad \text{and} \quad \Psi_2(\xi_n, \eta_n) \to \Psi_2(\xi, \eta) \text{ in } L^2_{p_1^{-1}}(\Omega).$$

We have

$$\Big\|\frac{\varrho_1\xi_n}{1+\varepsilon|\xi_n|}1_{B_\varepsilon} - \frac{\varrho_1\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\Big\|_{L^2_{p_1^{-1}}(\Omega)} = \frac{1}{\varepsilon}\Big\|\frac{\varepsilon\varrho_1\xi_n}{1+\varepsilon|\xi_n|}1_{B_\varepsilon} - \frac{\varepsilon\varrho_1\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\Big\|_{L^2_{p_1^{-1}}(\Omega)}.$$

The function $l(x) = x(1+|x|)^{-1}$ is Lipschitzian on $\mathbb{R}$ and verifies

$$\tag{23} \forall x, y \in \mathbb{R}, \quad |l(x) - l(y)| \leqslant |x - y|.$$

Consequently,

$$\begin{aligned} \frac{1}{\varepsilon}\Big\|\frac{\varepsilon\varrho_1\xi_n}{1+\varepsilon|\xi_n|}1_{B_\varepsilon} - \frac{\varepsilon\varrho_1\xi}{1+\varepsilon|\xi|}1_{B_\varepsilon}\Big\|_{L^2_{p_1^{-1}}(\Omega)} &\leqslant \frac{1}{\varepsilon}\|\varepsilon\varrho_1\xi_n - \varepsilon\varrho_1\xi\|_{L^2_{p_1^{-1}}(\Omega)} \\ &\leqslant \|\varrho_1\xi_n - \varrho_1\xi\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } n \to +\infty, \end{aligned}$$

since

$$\begin{aligned} \int_\Omega (1+|x|^2)\varrho_1^2|\xi_n - \xi|^2 \,\mathrm{d}x &\leqslant \int_\Omega (1+|x|^2)\frac{k_1}{1+|x|^2}\varrho|\xi_n - \xi|^2 \,\mathrm{d}x \\ &\leqslant k_1\|\xi_n - \xi\|^2_{L^2_\varrho(\Omega)} \to 0 \text{ for } n \to +\infty. \end{aligned}$$

Similarly, we show that $\varrho_2\eta_n(1+\varepsilon|\eta_n|)^{-1}1_{B_\varepsilon} \to \varrho_2\eta(1+\varepsilon|\eta|)^{-1}1_{B_\varepsilon}$ for $\varepsilon \to 0$ in $L^2_{p_1^{-1}}(\Omega)$.

Then $\Psi_1(\xi_n,\eta_n) \to \Psi_1(\xi,\eta)$ in $L^2_{p_1^{-1}}(\Omega)$ and therefore $\omega_n \to \omega$ in V.

In the same way, $v_n \to v$ in V. Consequently, T is continuous.

(v) Next we show that $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega) \to L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$ is compact.

We have the compact imbedding $V \subset L^2_\varrho(\Omega)$.

Indeed, let $(u_n)_n$ be a bounded sequence in V. $(u_n)_n$ is bounded in $H^1(B_\varepsilon)$. Hence the imbedding $H^1(B_\varepsilon)$ into $L^2(B_\varepsilon)$ is compact; there exists a subsequence denoted again by $(u_n)_n$, such that

$$\int_{B_\varepsilon} \varrho|u_p-u_q|^2\,\mathrm{d}x \leqslant \int_{B_\varepsilon} |u_p-u_q|^2\,\mathrm{d}x \to 0 \quad \text{when } p,q\to+\infty.$$

On the other hand, since the weight ϱ tends to 0 at infinity, we have

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon} \varrho|u_p-u_q|^2\,\mathrm{d}x &= \int_{\Omega\setminus B_\varepsilon} \big(1+|x|^2\big)\varrho\frac{1}{(1+|x|^2)}|u_p-u_q|^2\,\mathrm{d}x\\ &\leqslant k \sup_{|x|>1/\varepsilon}\frac{1}{(1+|x|^2)^{\alpha-1}}\|u_p-u_q\|_V^2 \to 0 \ \text{ for } \varepsilon\to 0.\end{aligned}$$

Consequently, $(u_n)_n$ is a Cauchy sequence in $L^2_\varrho(\Omega)$.

Since $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega) \to V\times V$ is continuous, $T\colon L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)\to L^2_\varrho(\Omega)\times L^2_\varrho(\Omega)$ is compact. According to (iii) and (v), we can apply Schauder's fixed point theorem. Then there exists $(\xi,\eta)\in[\xi_0,\xi^0]\times[\eta_0,\eta^0]$ such that $T(\xi,\eta)=(\xi,\eta)$.

Since ξ and η depend on ε, we denote $\xi=u_\varepsilon$ and $\eta=v_\varepsilon$.

So $u_\varepsilon, v_\varepsilon$ verify the system

$$\text{(24)}\qquad \begin{cases}(-\Delta+q_1+m\varrho_1)u_\varepsilon=(a+m)\dfrac{\varrho_1u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon}+b\dfrac{\varrho_2v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon}+f \ \text{ in } \Omega,\\[2mm] (-\Delta+q_2+m\varrho_4)v_\varepsilon=c\dfrac{\varrho_3u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon}+(d+m)\dfrac{\varrho_4v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon}+g \ \text{ in } \Omega,\\ \qquad u_\varepsilon=v_\varepsilon=0 \ \text{ on } \partial\Omega;\quad u_\varepsilon,v_\varepsilon\to 0 \ \text{ for } |x|\to+\infty.\end{cases}$$

(vi) We show that $(\varepsilon u_\varepsilon)_\varepsilon$ (as well as $(\varepsilon v_\varepsilon)_\varepsilon$) is a bounded sequence in V.

We multiply the first equation of the system (24) by $\varepsilon^2u_\varepsilon$ and integrate over Ω, obtaining

$$\begin{aligned}\int_\Omega \big(|\nabla(\varepsilon u_\varepsilon)|^2+(q_1+m\varrho_1)|\varepsilon u_\varepsilon|^2\big)\,\mathrm{d}x &= (a+m)\int_\Omega \frac{\varepsilon\varrho_1u_\varepsilon}{1+\varepsilon|u_\varepsilon|}\varepsilon u_\varepsilon 1_{B_\varepsilon}\,\mathrm{d}x\\ &\quad + b\int_\Omega\frac{\varepsilon\varrho_2v_\varepsilon}{1+\varepsilon|v_\varepsilon|}\varepsilon u_\varepsilon 1_{B_\varepsilon}\,\mathrm{d}x+\int_\Omega \varepsilon f\varepsilon u_\varepsilon\,\mathrm{d}x.\end{aligned}$$

Since $\varepsilon|u_\varepsilon|(1+\varepsilon|u_\varepsilon|)^{-1} < 1$, $\varepsilon|v_\varepsilon|(1+\varepsilon|v_\varepsilon|)^{-1} < 1$, $0 < \varepsilon < 1$, $f \in L^2_{p_1^{-1}}(\Omega)$ and $\alpha > N/2$, there exists a constant $M > 0$ such that $\|\varepsilon u_\varepsilon\|_1 \leqslant M$.

Similarly, there exists $M' > 0$ such that $\|\varepsilon v_\varepsilon\|_2 \leqslant M'$.

(vii) We show that $\varepsilon u_\varepsilon \to 0$ (as well as $\varepsilon v_\varepsilon \to 0$) when $\varepsilon \to 0$ in V.

Since $(\varepsilon u_\varepsilon)_\varepsilon$ is bounded in V, there exists a subsequence denoted again by $(\varepsilon u_\varepsilon)_\varepsilon$ (we denote it by writing for example $\varepsilon = \frac{1}{n}$, $n > 1$) weakly convergent to u^* in V and hence strongly convergent to u^* in $L^2_\varrho(\Omega)$.

We multiply the first equation of the system (24) by ε:

$$(-\Delta + (q_1 + m\varrho_1))(\varepsilon u_\varepsilon) = (a+m)\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + b\frac{\varepsilon\varrho_2 v_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \varepsilon f \quad \text{in } \Omega.$$

Then $\forall\varphi \in D(\Omega)$,

$$\int_\Omega [\nabla(\varepsilon u_\varepsilon)\cdot\nabla\varphi + (q_1 + m\varrho_1)\varepsilon u_\varepsilon \varphi]\,\mathrm{d}x \int_\Omega [\nabla u^* \cdot \nabla\varphi + (q_1 + m\varrho_1)u^*\varphi]\,\mathrm{d}x \ \text{ for } \ \varepsilon \to 0.$$

Moreover $\forall\varphi \in D(\Omega)$, $\int_\Omega \varepsilon f\varphi\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$.

On the other hand,

$$\int_\Omega (1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x = \int_{B_\varepsilon} (1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x + \int_{\Omega\setminus B_\varepsilon} (1+|x|^2)\left|\frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x,$$

$$\begin{aligned}\int_{\Omega\setminus B_\varepsilon} (1+|x|^2)\left|\frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x &\leqslant \int_{\Omega\setminus B_\varepsilon} (1+|x|^2)|\varrho_1 u^*|^2\,\mathrm{d}x \\ &\leqslant k_1 \int_{\Omega\setminus B_\varepsilon} \varrho|u^*|^2\,\mathrm{d}x \to 0 \ \text{ for } \ \varepsilon \to 0.\end{aligned}$$

Taking (23) into account, we have

$$\begin{aligned}\int_{B_\varepsilon} (1+|x|^2)\left|\frac{\varepsilon\varrho_1 u_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \frac{\varrho_1 u^*}{1+|u^*|}\right|^2 \mathrm{d}x &\leqslant \int_\Omega (1+|x|^2)|\varepsilon\varrho_1 u_\varepsilon - \varrho_1 u^*|^2\,\mathrm{d}x \\ &\leqslant k_1 \int_\Omega \varrho|\varepsilon u_\varepsilon - u^*|^2\,\mathrm{d}x \to 0 \ \text{ for } \ \varepsilon \to 0.\end{aligned}$$

Similarly, we show that $\varepsilon\varrho_2 v_\varepsilon(1+\varepsilon|v_\varepsilon|)^{-1}1_{B_\varepsilon} \to \varrho_2 v^*(1+|v^*|)^{-1}$ for $\varepsilon \to 0$ in $L^2_{p_1^{-1}}(\Omega)$.

In the same manner we can establish that

$$\varepsilon\Psi_2(u_\varepsilon, v_\varepsilon) \to c\frac{\varrho_3 u^*}{1+|u^*|} + (d+m)\frac{\varrho_4 v^*}{1+|v^*|} \ \text{ for } \varepsilon \to 0 \ \text{ in } \ L^2_{p_1^{-1}}(\Omega).$$

We get, when $\varepsilon \to 0$,

$$
(25) \qquad \begin{cases} (-\Delta + q_1 + m\varrho_1)u^* = (a+m)\dfrac{\varrho_1 u^*}{1+|u^*|} + b\dfrac{\varrho_2 v^*}{1+|v^*|} & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)v^* = c\dfrac{\varrho_3 u^*}{1+|u^*|} + (d+m)\dfrac{\varrho_4 v^*}{1+|v^*|} & \text{in } \Omega, \\ \qquad u^* = v^* = 0 \quad \text{on } \partial\Omega; \quad u^*, v^* \to 0 \ \text{for } |x| \to +\infty. \end{cases}
$$

We show now that $u^* = v^* = 0$.

We multiply the first equation of the system (25) by u^* and integrate over Ω:

$$
\begin{aligned}
\int_\Omega \left(|\nabla u^*|^2 + (q_1 + m\varrho_1)|u^*|^2\right) \mathrm{d}x &= (a+m)\int_\Omega \frac{\varrho_1 |u^*|^2}{1+|u^*|} = \mathrm{d}x + b\int_\Omega \frac{\varrho_2 u^* v^*}{1+|v^*|}\,\mathrm{d}x \\
&\leqslant (a+m)\int_\Omega \varrho_1 |u^*|^2 \,\mathrm{d}x + |b|\int_\Omega \varrho_2 |u^* v^*|\,\mathrm{d}x.
\end{aligned}
$$

By virtue of the variational characterization of $\lambda(q_1, \varrho_1)$ and using (10), we get

$$
(26) \qquad (\lambda(q_1,\varrho_1) - a)\int_\Omega \varrho_1 |u^*|^2 \,\mathrm{d}x \leqslant |b| \left(\int_\Omega \varrho_1 |u^*|^2 \,\mathrm{d}x\right)^{\frac12} \cdot \left(\int_\Omega \varrho_4 |v^*|^2 \,\mathrm{d}x\right)^{\frac12}.
$$

In the same way, we prove that

$$
(27) \qquad (\lambda(q_2,\varrho_4) - d)\int_\Omega \varrho_4 |v^*|^2 \,\mathrm{d}x \leqslant |c| \left(\int_\Omega \varrho_1 |u^*|^2 \,\mathrm{d}x\right)^{\frac12} \cdot \left(\int_\Omega \varrho_4 |v^*|^2 \,\mathrm{d}x\right)^{\frac12}.
$$

Combining (26), (27) with (14), we have $u^* = v^* = 0$.

(viii) We show that the sequence $(u_\varepsilon)_\varepsilon$ (as well as $(v_\varepsilon)_\varepsilon$) is bounded in V. We suppose that $\|u_\varepsilon\|_V \to +\infty$ for $\varepsilon \to 0$ and $\|v_\varepsilon\|_V \to +\infty$ for $\varepsilon \to 0$ and define

$$
\begin{aligned}
t_\varepsilon &= \max(\|u_\varepsilon\|_V, \|v_\varepsilon t\|_V), \\
z_\varepsilon &= \frac{1}{t_\varepsilon} u_\varepsilon \quad \text{then } \|z_\varepsilon\|_V \leqslant 1, \\
w_\varepsilon &= \frac{1}{t_\varepsilon} v_\varepsilon \quad \text{then } \|w_\varepsilon\|_V \leqslant 1.
\end{aligned}
$$

Since $(z_\varepsilon)_\varepsilon$ is a bounded sequence in V, there exists a subsequence denoted again by $(z_\varepsilon)_\varepsilon$, weakly convergent to z in V and hence strongly convergent to z in $L^2_\varrho(\Omega)$.

Similarly, $(w_\varepsilon)_\varepsilon$ converges to w , weakly in V and strongly in $L^2_\varrho(\Omega)$.

Taking (24) into account, we get

$$
(28) \qquad \begin{cases} (-\Delta + q_1 + m\varrho_1)z_\varepsilon = (a+m)\dfrac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + b\dfrac{\varrho_2 w_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \dfrac{1}{t_\varepsilon}f & \text{in } \Omega, \\ (-\Delta + q_2 + m\varrho_4)w_\varepsilon = c\dfrac{\varrho_3 z_\varepsilon}{1+\varepsilon|u_\varepsilon|}1_{B_\varepsilon} + (d+m)\dfrac{\varrho_4 w_\varepsilon}{1+\varepsilon|v_\varepsilon|}1_{B_\varepsilon} + \dfrac{1}{t_\varepsilon}g & \text{in } \Omega, \\ \qquad z_\varepsilon = w_\varepsilon = 0 \quad \text{on } \partial\Omega; \quad z_\varepsilon, w_\varepsilon \to 0 \ \text{for } |x| \to +\infty. \end{cases}
$$

We know that $\forall\varphi \in D(\Omega)$, $\int_\Omega (\nabla z_\varepsilon \cdot \nabla\varphi + (q_1 + m\varrho_1) z_\varepsilon \varphi)\,\mathrm{d}x \to \int_\Omega (\nabla z \cdot \nabla\varphi + (q_1 + m\varrho_1) z\varphi)\,\mathrm{d}x$ for $\varepsilon \to 0$ and $\int_\Omega \frac{1}{t_\varepsilon} f\varphi\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$.

On the other hand,

$$\begin{aligned}\left\| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_1 z \right\|^2_{L^2_{p_1^{-1}}(\Omega)} = {} & \int_{B_\varepsilon} \bigl(1+|x|^2\bigr) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x \\ & + \int_{\Omega\setminus B_\varepsilon} \bigl(1+|x|^2\bigr) |\varrho_1 z|^2\,\mathrm{d}x.\end{aligned}$$

We have $\int_{\Omega\setminus B_\varepsilon} \bigl(1+|x|^2\bigr)|\varrho_1 z|^2\,\mathrm{d}x \leqslant k_1 \int_{\Omega\setminus B_\varepsilon} \varrho |z|^2\,\mathrm{d}x \to 0$ for $\varepsilon \to 0$, and

$$\begin{aligned}\int_{B_\varepsilon} \bigl(1+|x|^2\bigr) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x & \leqslant \int_\Omega \bigl(1+|x|^2\bigr) \left| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} - \varrho_1 z \right|^2 \mathrm{d}x \\ & \leqslant \int_\Omega \bigl(1+|x|^2\bigr) \left| \frac{\varrho_1 (z_\varepsilon - z) - \varepsilon\varrho_1 z|u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x \\ & \leqslant 2\int_\Omega \bigl(1+|x|^2\bigr) \left(\left| \frac{\varrho_1(z_\varepsilon - z)}{1+\varepsilon|u_\varepsilon|} \right|^2 + \left| \frac{\varepsilon\varrho_1 z|u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \right) \mathrm{d}x.\end{aligned}$$

So

$$\begin{aligned}\int_\Omega \bigl(1+|x|^2\bigr) \left| \frac{\varrho_1(z_\varepsilon - z)}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x & \leqslant \int_\Omega \bigl(1+|x|^2\bigr) |\varrho_1(z_\varepsilon - z)|^2\,\mathrm{d}x \\ & \leqslant k_1 \int_\Omega \varrho |z_\varepsilon - z|^2\,\mathrm{d}x \to 0 \text{ for } \varepsilon \to 0.\end{aligned}$$

Moreover,

$$\left| \frac{\varepsilon_1 z|u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \to 0 \text{ for } \varepsilon \to 0 \quad \text{a.e. in } \Omega,$$
$$\bigl(1+|x|^2\bigr) \left| \frac{\varepsilon\varrho_1 z|u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \leqslant \bigl(1+|x|^2\bigr)|\varrho_1 z|^2 \leqslant k_1 \varrho |z|^2.$$

Since $z \in L^2_\varrho(\Omega)$, by virtue of the Lebesgue dominated convergence theorem we deduce that

$$\int_\Omega \bigl(1+|x|^2\bigr) \left| \frac{\varepsilon\varrho_1 z|u_\varepsilon|}{1+\varepsilon|u_\varepsilon|} \right|^2 \mathrm{d}x \to 0 \text{ for } \varepsilon \to 0.$$

Hence

$$\left\| \frac{\varrho_1 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_1 z \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0.$$

Analogously, we obtain

$$\left\| \frac{\varrho_3 z_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_3 z \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0$$

and

$$\left\| \frac{\varrho_i w_\varepsilon}{1+\varepsilon|v_\varepsilon|} 1_{B_\varepsilon} - \varrho_i w \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0 \quad (i=2,4).$$

When $\varepsilon \to 0$, we get

(29) $$\begin{cases} (-\Delta + q_1)z = a\varrho_1 z + b\varrho_2 w & \text{in } \Omega, \\ (-\Delta + q_2)w = c\varrho_3 z + d\varrho_4 w & \text{in } \Omega, \\ z = w = 0 \text{ on } \partial\Omega; \quad z, w \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

By virtue of Lemma 1, we have $z = w = 0$, which is in contradiction with the fact that at least one sequence ((z_ε) or (w_ε)) has the norm equal to one. The sequence (u_ε) (as well as (v_ε)) is bounded in V.

(ix) We extract a sequence denoted again by (u_ε) ((v_ε)) weakly convergent to u^0 (or v^0) in V and hence strongly convergent to the same limit in $L^2_\varrho(\Omega)$.

We have $\forall \varphi \in D(\Omega)$,

$$\int_\Omega (\nabla u_\varepsilon \cdot \nabla\varphi + (q_1 + m\varrho_1)u_\varepsilon\varphi)\,\mathrm{d}x \to \int_\Omega (\nabla u^0 \cdot \nabla\varphi + (q_1 + m\varrho_1)u^0\varphi)\,\mathrm{d}x, \text{ for } \varepsilon \to 0$$
$$\int_\Omega (\nabla v_\varepsilon \cdot \nabla\varphi + (q_2 + m\varrho_4)v_\varepsilon\varphi)\,\mathrm{d}x \to \int_\Omega (\nabla v^0 \cdot \nabla\varphi + (q_2 + m\varrho_4)v^0\varphi)\,\mathrm{d}x \text{ for } \varepsilon \to 0.$$

In the same way as in (viii), we show that

$$\left\| \frac{\varrho_i u_\varepsilon}{1+\varepsilon|u_\varepsilon|} 1_{B_\varepsilon} - \varrho_i u^0 \right\|^2_{L_{p_1^{-1}}}(\Omega) \to 0 \text{ for } \varepsilon \to 0 \quad (i=1,3),$$
$$\left\| \frac{\varrho_i v_\varepsilon}{1+\varepsilon|v_\varepsilon|} 1_{B_\varepsilon} - \varrho_i v^0 \right\|_{L^2_{p_1^{-1}}(\Omega)} \to 0 \text{ for } \varepsilon \to 0 \quad (i=2,4).$$

When $\varepsilon \to 0$, we get

(S) $$\begin{cases} (-\Delta + q_1)u^0 = a\varrho_1 u^0 + b\varrho_2 v^0 + f & \text{in } \Omega, \\ (-\Delta + q_2)v^0 = c\varrho_3 u^0 + d\varrho_4 v^0 + g & \text{in } \Omega, \\ u^0 = v^0 = 0 \text{ on } \partial\Omega; \quad u^0, v^0 \to 0 \text{ for } |x| \to +\infty. \end{cases}$$

We conclude that (u^0, v^0) is a solution of (S). $\square$

The uniqueness of the solution follows from Lemma 1. Indeed, let $\left(u^1, v^1\right)$ be another solution of the system (S). We put $w = u^0 - u^1$ and $z = v^0 - v^1$, then (w, z) is a solution of the system (15). Lemma 1 gives $(w, z) = (0, 0)$.

Acknowledgment. We wish to thank the referee for helpful comments.

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