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BOUNDS AND NUMERICAL RESULTS FOR HOMOGENIZED DEGENERATED p-POISSON EQUATIONS

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Abstract. In this paper we derive upper and lower bounds on the homogenized energy density functional corresponding to degenerated p-Poisson equations. Moreover, we give some non-trivial examples where the bounds are tight and thus can be used as good approximations of the homogenized properties. We even present some cases where the bounds coincide and also compare them with some numerical results.

Keywords: homogenization, bounds, degenerated, p-Poisson equation

MSC 2000: 35B27, 35J60, 74Q20

1. Introduction

In many types of materials, e.g. composites, the physical properties can be modelled by a Y-periodic function λ. For small values of ε the function $\lambda(x/\varepsilon)$ will oscillate rapidly which means that the material is strongly heterogeneous on a local scale. Nevertheless, the material will globally act as a homogeneous medium. It is extremely difficult to find the effective properties λ_{hom} which describe this homogeneous medium. The field of mathematics that rigorously defines the notion of effective properties is known as homogenization.

Consider a class of physical problems described by a minimum energy principle of the form

$$E_\varepsilon = \min_u \left\{ \int_\Omega \frac{1}{p} \lambda\Big(\frac{x}{\varepsilon}\Big) |Du|^p \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \right\}, \tag{1}$$

where u belongs to some subset of $W^{1,1}(\Omega)$ and represents the state of the material. It is known that the energy $E_\varepsilon \to E_{\mathrm{hom}}$ as $\varepsilon \to 0$, where E_{hom} is of the type

$$E_{\mathrm{hom}} = \min_u \left\{ \int_\Omega \lambda_{\mathrm{hom}}(Du) \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \right\},$$

and λ_{hom} is defined as

$$\lambda_{\text{hom}}(\xi) = \min_v \int_Y \frac{1}{p}\lambda(y)|\xi + Dv|^p \, dy. \tag{2}$$

For a proof of these homogenization results when λ is bounded between two positive constants see e.g. [13]. The degenerated case, i.e. when λ is allowed to approach zero or infinity, was studied in [6] (see also [2]). By using numerical methods it is possible to compute λ_{hom} by formula (2), see e.g. [3]. Another approach is to find bounds on λ_{hom}.

Bounds for the case when λ is bounded between two positive constants were presented in [10] (see also [9]). These bounds, combined with reiterated homogenization (i.e. introducing λ of the form $\lambda(x/\varepsilon, \dots, x/\varepsilon^m)$), have played a central role in the development of new optimal structures, see e.g. [1], [7], [8], [11], [12], [14] and [16]. Moreover, these bounds were used in an extension of the Ponte Castaneda variational principle ([4] and [5]) to obtain bounds for a class of more general nonlinear problems than those described above, see [15].

The main results of this paper are that we prove lower and upper bounds on λ_{hom} for the degenerated case (see Theorem 1 and Theorem 2), i.e. we find functions λ_{lower} and λ_{upper} such that

$$\lambda_{\text{lower}}(\xi) \leqslant \lambda_{\text{hom}}(\xi) \leqslant \lambda_{\text{upper}}(\xi).$$

Moreover, we present some illustrative examples where the bounds are tight and thus can be used as a good approximation of λ_{hom}.

2. Notation and preliminary results

Let Ω be an open bounded subset of $\mathbb{R}^n$, Y the unit cube in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ the Euclidean inner product. Let $1 < p < \infty$, $1/p + 1/q = 1$, and let λ be a Y-periodic (weight) function such that

$$\lambda > 0 \text{ a.e., and } \lambda,\ \lambda^{-1/(1-p)} \text{ are in } L^1_{\text{loc}}(\mathbb{R}^n).$$

The set of all real-valued functions u in $L^1_{\text{loc}}(\Omega)$ such that $u\lambda^{1/p}$ is in $L^p(\Omega)$ is denoted by $L^p(\Omega, \lambda)$. The set of functions u in $W^{1,1}_{\text{loc}}(\Omega)$ such that u and $|Du|$ are in $L^p(\Omega, \lambda)$ is denoted by $W^{1,p}(\Omega, \lambda)$. Moreover, by $W^{1,p}_0(\Omega, \lambda)$ we mean the closure of $C^1_0(\Omega)$ in $W^{1,p}(\Omega, \lambda)$ and $W^{1,p}_{\text{per}}(Y, \lambda)$ is the set of real functions u in $W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ such that u is Y-periodic and $u \in W^{1,p}(Y, \lambda)$.

Define the family (u_ε) as the set of solutions of the variational problems

$$E_\varepsilon = \min_{u \in W^{1,p}_0(\Omega, \lambda(x/\varepsilon))} \left\{ \int_\Omega \frac{1}{p}\lambda\Big(\frac{x}{\varepsilon}\Big)|Du|^p \, dx - \int_\Omega fu \, dx \right\}.$$

Under the additional assumption that λ belongs to the Muckenhoupt class A_p it was proved in [6] that (u_ε) converges weakly in $W_0^{1,1}(\Omega)$ to the unique solution u_{hom} of the homogenized problem

$$E_{\text{hom}} = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega \lambda_{\text{hom}}(Du)\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x \right\}$$

where λ_{hom} is defined as

$$\lambda_{\text{hom}}(\xi) = \min_{v \in W^{1,p}_{\text{per}}(Y,\lambda)} \int_Y \frac{1}{p}\lambda(y)|\xi + Dv|^p\,\mathrm{d}y. \tag{3}$$

It was also proved that

$$E_\varepsilon \to E_{\text{hom}}.$$

We remark that the family of solutions (u_ε) of the minimization problems described above are also solutions to the weighted weak formulations of the p-Poisson equations, namely

$$\begin{cases} \displaystyle\int_\Omega \Big\langle \lambda\Big(\frac{x}{\varepsilon}\Big)|Du_\varepsilon|^{p-2}Du_\varepsilon, D\varphi\Big\rangle\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \\ u_\varepsilon \in W_0^{1,p}(\Omega,\lambda(x/\varepsilon)) \end{cases}$$

for every φ in $W_0^{1,p}(\Omega,\lambda(x/\varepsilon))$. The homogenized solution u_{hom} satisfies the homogenized problem

$$\begin{cases} \displaystyle\int_\Omega \langle b(Du_{\text{hom}}), D\varphi\rangle\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \\ u_{\text{hom}} \in W_0^{1,p}(\Omega), \end{cases}$$

for every φ in $W_0^{1,p}(\Omega)$, where b is given by

$$b(\xi) = \int_Y \lambda(y)|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi)\,\mathrm{d}y$$

and w_ξ is the solution of the local problem

$$\begin{cases} \displaystyle\int_Y \langle \lambda(y)|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi), D\varphi\rangle\,\mathrm{d}y = 0, \\ w_\xi \in W^{1,p}_{\text{per}}(Y,\lambda), \end{cases} \tag{4}$$

for every φ in $W^{1,p}_{\text{per}}(Y,\lambda)$.

Remark 1. The solution w_ξ of the local problem (4) is also the minimizer in the local minimization problem (3) and

$$\lambda_{\text{hom}}(\xi) = \Big\langle \frac{1}{p}b(\xi), \xi \Big\rangle.$$

3. Bounds

In this section we present upper and lower bounds on the homogenized energy density functional λ_{hom} defined in (3). The bounds are given in the following two theorems:

Theorem 1. *Let λ_{hom} be defined as in (3). Then we have the upper bound*

$$\lambda_{\mathrm{hom}}(ke_i) \leqslant \lambda_{\mathrm{upper}}(ke_i) \stackrel{\mathrm{def}}{=} |k|^p \frac{1}{p} \left(\int_0^1 \langle \lambda \rangle_i^{1/(1-p)} \,\mathrm{d}y_i \right)^{1-p},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$ and

$$\langle \lambda \rangle_i = \int_0^1 \ldots \int_0^1 \lambda \,\mathrm{d}y_1 \ldots \mathrm{d}y_{i-1} \,\mathrm{d}y_{i+1} \ldots \mathrm{d}y_n.$$

Theorem 2. *Let λ_{hom} be defined as in (3). Then we have the lower bound*

$$\begin{aligned} \lambda_{\mathrm{hom}}(ke_i) &\geqslant \lambda_{\mathrm{lower}}(ke_i) \\ &\stackrel{\mathrm{def}}{=} |k|^p \frac{1}{p} \int_0^1 \ldots \int_0^1 \langle \lambda^{1/(1-p)} \rangle_i^{1-p} \,\mathrm{d}y_1 \ldots \mathrm{d}y_{i-1} \,\mathrm{d}y_{i+1} \ldots dy_n, \end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$ and

$$\langle \lambda^{1/(1-p)} \rangle_i = \int_0^1 \lambda^{1/(1-p)} \,\mathrm{d}y_i.$$

Remark 2. By linearity, we get lower and upper bounds on λ_{hom} for all $\xi \in \mathbb{R}^n$ when $p = 2$.

For the proofs of Theorem 1 and Theorem 2 we need the following two lemmata, which themselves are of independent interest:

Lemma 3. *Let D be a measurable set in $\mathbb{R}^d$ such that $|D| = 1$. Moreover, let $a \geqslant 0$ be a weight function such that $a \in L^1(D)$ and $a^{1/(1-r)} \in L^1(D)$, where $1 < r < \infty$. Then*

$$\min_{u \in U} \int_D a(x) |1 + u(x)|^r \,\mathrm{d}x = \left(\int_D a(x)^{1/(1-r)} \,\mathrm{d}x \right)^{1-r},$$

where

$$U = \left\{ u \in L^r(D, a) \colon \int_D u \,\mathrm{d}x = 0 \right\}. \tag{5}$$

Moreover, the minimum is attained for

$$\widetilde{u} = \left(\int_D a^{1/(1-r)}\,\mathrm{d}x\right)^{-1} a^{1/(1-r)} - 1.$$

Proof. The reversed Hölder inequality and (5) imply that

$$\begin{aligned}\int_D a(x)|1+u(x)|^r\,\mathrm{d}x &\geqslant \left(\int_D a(x)^{1/(1-r)}\,\mathrm{d}x\right)^{1-r}\left|\int_D 1+u(x)\,\mathrm{d}x\right|^r\\ &= \left(\int_D a(x)^{1/(1-r)}\,\mathrm{d}x\right)^{1-r}.\end{aligned}$$

Equality holds in Hölder's inequality when

$$ca^{1/(1-r)} = |1+u| = 1+u.$$

The zero average constraint implies that

$$c = \left(\int_D a^{1/(1-r)}\,\mathrm{d}x\right)^{-1}.$$

Since $a \in L^1(D)$ and $a^{1/(1-r)} \in L^1(D)$, it follows that $0 < c < \infty$. Moreover, $\widetilde{u} \in L^r(D,a)$ since

$$\begin{aligned}\int_D |\widetilde{u}|^r a\,\mathrm{d}x &= \int_D |ca^{1/(1-r)} - 1|^r a\,\mathrm{d}x\\ &\leqslant C\left(\int_D a^{1/(1-r)}\,\mathrm{d}x + \int_D a\,\mathrm{d}x\right) < \infty,\end{aligned}$$

where C is a constant. □

Lemma 4. *Let λ_{hom} be defined as in* (3). *Then*

$$\lambda^*_{\mathrm{hom}}(\xi) = \inf_{\substack{\sigma\in V\\ \int_Y \sigma\,\mathrm{d}y=0}} \int_Y \frac{1}{q}\lambda^{1-q}|\xi+\sigma|^q\,\mathrm{d}y,$$

where λ^*_{hom} is the Legendre transform of λ_{hom} and V is defined as

$$V = \left\{\sigma \in L^q(Y,\lambda^{1-q})\colon \int_Y \langle\sigma, Dv\rangle\,\mathrm{d}y = 0 \ \textit{for all}\ v \in W^{1,p}_{\mathrm{per}}(Y,\lambda)\right\}.$$

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(\xi) = \frac{1}{p}\lambda|\xi + s|^p,$$

where s is a fixed vector in $\mathbb{R}^n$. The Legendre transform f^* of f is

$$f^*(\sigma) \stackrel{\text{def}}{=} \sup_{\xi\in\mathbb{R}^n} \{\langle\sigma,\xi\rangle - f(\xi)\} = \frac{1}{q}\lambda^{1-q}|\sigma|^q - \langle\sigma, s\rangle.$$

This implies Young's inequality

$$\langle\sigma,\xi\rangle \leqslant f(\xi) + f^*(\sigma) \quad \text{for all } \sigma,\xi \in \mathbb{R}^n, \tag{6}$$

with equality for

$$\sigma = \lambda|\xi + s|^{p-2}(\xi + s). \tag{7}$$

Inequality (6) implies that for any measurable function σ we have

$$\begin{aligned}\lambda_{\text{hom}}(\xi) &= \min_{v\in W^{1,p}_{\text{per}}(Y,\lambda)} \int_Y \frac{1}{p}\lambda(y)|\xi + Dv|^p \,\mathrm{d}y \\ &\geqslant \min_{v\in W^{1,p}_{\text{per}}(Y,\lambda)} \int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q + \langle\sigma, Dv\rangle \,\mathrm{d}y.\end{aligned}$$

This implies

$$\lambda_{\text{hom}}(\xi) \geqslant \sup_{\sigma\in V} \int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y. \tag{8}$$

Actually we have equality in (8). This fact will be clear if we prove that

$$\lambda_{\text{hom}}(\xi) \leqslant \sup_{\sigma\in V} \int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y. \tag{9}$$

Let w_ξ be the minimizer in (3) and let σ_1 be defined as

$$\sigma_1 = \lambda|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi). \tag{10}$$

Then it follows by (6) and (7) that

$$\lambda_{\text{hom}}(\xi) = \int_Y \langle\sigma_1,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma_1|^q + \langle\sigma_1, Dw_\xi\rangle \,\mathrm{d}y.$$

Next we note that $\sigma_1 \in V$ and thus (9) holds. Indeed, by (10) and Remark 1 we have

$$\int_Y \langle \sigma_1, D\varphi \rangle \,\mathrm{d}y = \int_Y \langle \lambda |\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi), D\varphi \rangle \,\mathrm{d}y = 0$$

for every $\varphi \in W^{1,p}_{\mathrm{per}}(Y,\lambda)$ and (10) implies that

$$\int_Y |\sigma_1|^q \lambda^{1-q} \,\mathrm{d}y = \int_Y |\xi + Dw_\xi|^p \lambda \,\mathrm{d}y < \infty.$$

We now proceed as follows:

$$\begin{aligned} \lambda_{\mathrm{hom}}(\xi) &= \sup_{\sigma \in V} \int_Y \langle \sigma, \xi \rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y \\ &= \sup_\eta \Bigg[\langle \eta, \xi \rangle - \inf_{\substack{\sigma \in V \\ \int_Y \sigma \,\mathrm{d}y = \eta}} \int_Y \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y \Bigg]. \end{aligned} \tag{11}$$

Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be defined as

$$F(\eta) = \inf_{\substack{\sigma \in V \\ \int_Y \sigma \,\mathrm{d}y = \eta}} \int_Y \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y = \inf_{\substack{\sigma \in V \\ \int_Y \sigma \,\mathrm{d}y = 0}} \int_Y \frac{1}{q}\lambda^{1-q}|\eta + \sigma|^q \,\mathrm{d}y. \tag{12}$$

In view of (11) and (12) it follows that

$$\lambda_{\mathrm{hom}}(\xi) = \sup_\eta [\langle \eta, \xi \rangle - F(\eta)] = F^*(\xi).$$

Since F is convex and lower semicontinuous we have

$$\lambda^*_{\mathrm{hom}}(\xi) = F^{**}(\xi) = F(\xi)$$

and the proof is complete. $\square$

Proof of Theorem 1. Without loss of generality we prove the result for $k = 1$. Let $M_i = \{v \in W^{1,p}_{\mathrm{per}}(Y,\lambda)\colon\ v = v(y_i)\}$. Lemma 3 then gives

$$\begin{aligned} \lambda_{\mathrm{hom}}(e_i) &= \min_{v \in W^{1,p}_{\mathrm{per}}(Y,\lambda)} \int_Y \frac{1}{p}\lambda(y)|e_i + Dv(y)|^p \,\mathrm{d}y \\ &\leqslant \min_{v \in M_i} \int_Y \frac{1}{p}\lambda(y)|1 + D_i v(y_i)|^p \,\mathrm{d}y \\ &= \min_{v \in M_i} \int_0^1 \frac{1}{p}\langle \lambda \rangle_i |1 + D_i v(y_i)|^p \,\mathrm{d}y_i \\ &= \frac{1}{p}\left(\int_0^1 \langle \lambda \rangle_i^{1/(1-p)} \,\mathrm{d}y_i \right)^{1-p}. \end{aligned}$$

$\square$

Proof of Theorem 2. Without loss of generality we prove the result for $k = 1$. Let

$$S_i = \left\{\sigma \in V\colon\ \sigma = (0, \dots, \sigma_i(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n), \dots, 0) \text{ and } \int_Y \sigma \,\mathrm{d}y = 0\right\}.$$

By using Lemma 4 and Lemma 3 we obtain

$$\begin{aligned}
\lambda^*_{\mathrm{hom}}(e_i) = {} & \inf_{\substack{\sigma \in V \\ \int_Y \sigma\,\mathrm{d}y = 0}} \int_Y \frac{1}{q} \lambda^{1-q} |e_i + \sigma|^q \,\mathrm{d}y \\
& \leqslant \inf_{\sigma \in S_i} \int_Y \frac{1}{q} \lambda^{1-q} |1 + \sigma_i(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)|^q \,\mathrm{d}y \\
& = \inf_{\sigma \in S_i} \int_0^1 \dots \int_0^1 \frac{1}{q} \langle \lambda^{1-q} \rangle_i |1 + \sigma_i|^q \,\mathrm{d}y_1 \dots \mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1} \dots \mathrm{d}y_n \\
& = \frac{1}{q} \left[\int_0^1 \dots \int_0^1 \langle \lambda^{1-q} \rangle_i^{1-p} \,\mathrm{d}y_1 \dots \mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1} \dots \mathrm{d}y_n\right]^{1-q}.
\end{aligned}$$

This implies the following lower bound on $\lambda_{\mathrm{hom}}(e_i)$:

$$\begin{aligned}
\lambda_{\mathrm{hom}}(e_i) = {} & \sup_{\xi \in \mathbb{R}^n} \{\langle e_i, \xi \rangle - \lambda^*_{\mathrm{hom}}(\xi)\} \\
& \geqslant \sup_{te_i \in \mathbb{R}^n} \{t - \lambda^*_{\mathrm{hom}}(te_i)\} \\
& = \sup_{t \in \mathbb{R}} \{t - |t|^q \lambda^*_{\mathrm{hom}}(e_i)\} \\
& \geqslant \sup_{t \in \mathbb{R}} \left\{t - \frac{|t|^q}{q} \left[\int_0^1 \dots \int_0^1 \langle \lambda^{1-q} \rangle_i^{1-p} \,\mathrm{d}y_1 \dots \mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1} \dots \mathrm{d}y_n\right]^{1-q}\right\} \\
& = \frac{1}{p} \int_0^1 \dots \int_0^1 \langle \lambda^{1/(1-p)} \rangle_i^{1-p} \,\mathrm{d}y_1 \dots \mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1} \dots \mathrm{d}y_n.
\end{aligned}$$

$\square$

4. Some examples

In this section we apply the bounds from Theorem 1 and Theorem 2 in two illustrative examples. The examples are presented in $\mathbb{R}^2$ for simplicity. Let us first remark that when the upper and lower bounds are equal we know the effective energy density functional exactly. For instance, this is the case when λ is of the type

$$\lambda(y) = f(y_1)g(y_2), \quad \lambda(y) \text{ is } Y\text{-periodic}.$$

Then it follows from Theorem 1 and Theorem 2 that

$$\lambda_{\mathrm{hom}}(e_1) = \frac{1}{p}\left(\int_0^1 f(y_1)^{1/(1-p)}\,\mathrm{d}y_1\right)^{1-p} \int_0^1 g(y_2)\,\mathrm{d}y_2,$$
$$\lambda_{\mathrm{hom}}(e_2) = \frac{1}{p}\left(\int_0^1 g(y_2)^{1/(1-p)}\,\mathrm{d}y_2\right)^{1-p} \int_0^1 f(y_1)\,\mathrm{d}y_1.$$

We now give one example of this situation where the conductivity degenerates on the unit cell boundary. We note that it is easy to make the mistake of believing that we have zero conductivity in one direction and infinitely high conductivity in the other direction. However, as we show below, the homogenized energy density functional is nonzero and finite in both directions.

Example 5. Consider the special case when $p = 2$ and let $\lambda\colon \mathbb{R}^2 \to \mathbb{R}$ be Y-periodic and defined as

$$\lambda(y) = |y_1(1-y_1)|^{-1/2}\left|y_1 - \frac{1}{2}\right|^{1/2} \quad \text{on } Y.$$

Then, by Theorem 1 and Theorem 2, we have

$$\lambda_{\mathrm{hom}}(e_1) = \frac{1}{2}\left(\int_0^1 \lambda^{-1}(y_1)\,\mathrm{d}y_1\right)^{-1} = \frac{1}{\sqrt{2}}\left(\frac{2\sqrt{2}}{3}K\left(\frac{1}{\sqrt{2}}\right)\right)^{-1},$$
$$\lambda_{\mathrm{hom}}(e_2) = \frac{1}{2}\int_0^1 \lambda(y_1)\,\mathrm{d}y_1 = \frac{\pi}{2K\left(\frac{1}{\sqrt{2}}\right)},$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. This means that

$$\lambda_{\mathrm{hom}}(e_1) \approx 0.40451, \quad \lambda_{\mathrm{hom}}(e_2) \approx 0.84721,$$

that is, the effective conductivity in the y_2-direction is only about twice as high as the effective conductivity in the y_1-direction.

We now consider an example where the upper and lower bounds are very tight. This means that we have a good explicit estimate of the effective energy density functional. This fact can be used to obtain error estimates for numerical computations. We demonstrate this by comparing the bounds with numerical computations done in MATLAB using the FEMLAB toolbox.

Example 6. Let

$$D_r = \left\{y \in Y\colon \left|y - \left(\frac{1}{2}, \frac{1}{2}\right)\right| \leqslant r,\ 0 \leqslant r \leqslant \frac{1}{2}\right\}$$

and let $\lambda\colon\ \mathbb{R}^2 \to \mathbb{R}$ be Y-periodic and defined as

$$\lambda(y) = \begin{cases} \left(\dfrac{1}{r}\left|y - \left(\dfrac{1}{2}, \dfrac{1}{2}\right)\right|\right)^{\alpha}, & y \in D_r, \\ 1, & y \in Y \setminus D_r, \end{cases}$$

where $-2 < \alpha < 2p - 2$. By symmetry, the homogenized energy density functional $\lambda_{\mathrm{hom}}(e_1) = \lambda_{\mathrm{hom}}(e_2)$. Let $r = 0.4$ and let λ^-, λ^+ denote the lower and upper bounds, respectively.

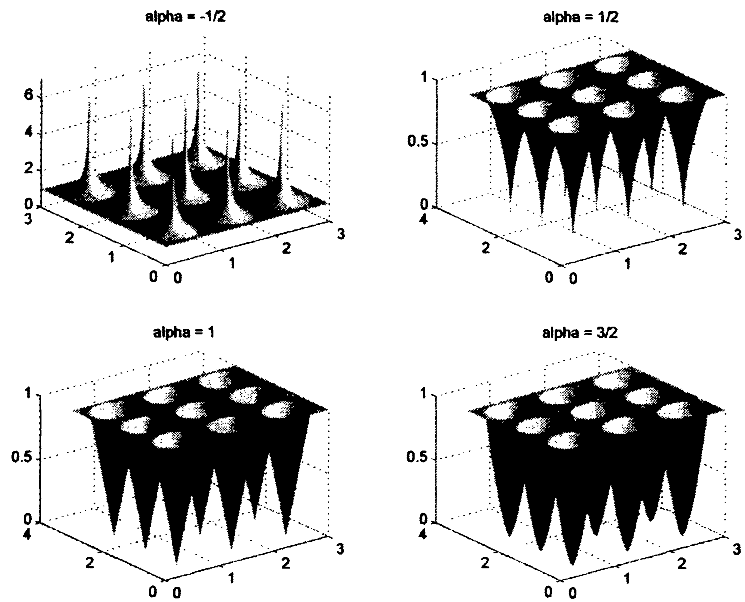

Figure 1. This picture shows 9 unit cells for each of the values $\alpha = -1/2$, $\alpha = 1/2$, $\alpha = 1$ and $\alpha = 3/2$. The radius is $r = 0.4$.

(a) The linear case, $p = 2$:

We present the results rounded to five digits.

α	$\lambda_{\mathrm{hom}}(e_i)$	λ^-	λ^+
$-3/2$	0.70147	0.67425	0.74023
-1	0.63470	0.62059	0.65433
$-1/2$	0.56597	0.56203	0.57082
$1/2$	0.44172	0.43797	0.44482
1	0.39389	0.38207	0.40284
$3/2$	0.35639	0.33773	0.37078

(b) The nonlinear case, $p = 3$:

We present the results rounded to four digits.

α	$\lambda_{\text{hom}}(e_i)$	λ^-	λ^+
-1	0.4325	0.4260	0.4482
1	0.2649	0.2608	0.2708
2	0.2219	0.2121	0.2353
3	0.1952	0.1821	0.2132

As we see from these tables, the lower and upper bounds are very tight, which means that they can be used as a good approximation of the homogenized energy density functional.

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BOUNDS AND NUMERICAL RESULTS FOR HOMOGENIZED DEGENERATED p-POISSON EQUATIONS

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Abstract. In this paper we derive upper and lower bounds on the homogenized energy density functional corresponding to degenerated p-Poisson equations. Moreover, we give some non-trivial examples where the bounds are tight and thus can be used as good approximations of the homogenized properties. We even present some cases where the bounds coincide and also compare them with some numerical results.

Keywords: homogenization, bounds, degenerated, p-Poisson equation

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1. Introduction

In many types of materials, e.g. composites, the physical properties can be modelled by a Y-periodic function λ. For small values of ε the function $\lambda(x/\varepsilon)$ will oscillate rapidly which means that the material is strongly heterogeneous on a local scale. Nevertheless, the material will globally act as a homogeneous medium. It is extremely difficult to find the effective properties $\lambda_{\rm hom}$ which describe this homogeneous medium. The field of mathematics that rigorously defines the notion of effective properties is known as homogenization.

Consider a class of physical problems described by a minimum energy principle of the form

$$E_\varepsilon = \min_u \left\{ \int_\Omega \frac{1}{p} \lambda\Big(\frac{x}{\varepsilon}\Big) |Du|^p \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \right\}, \tag{1}$$

where u belongs to some subset of $W^{1,1}(\Omega)$ and represents the state of the material. It is known that the energy $E_\varepsilon \to E_{\rm hom}$ as $\varepsilon \to 0$, where $E_{\rm hom}$ is of the type

$$E_{\rm hom} = \min_u \left\{ \int_\Omega \lambda_{\rm hom}(Du) \,\mathrm{d}x - \int_\Omega f u \,\mathrm{d}x \right\},$$

and λ_{hom} is defined as

$$\lambda_{\text{hom}}(\xi) = \min_{v} \int_{Y} \frac{1}{p} \lambda(y) |\xi + Dv|^p \, \mathrm{d}y. \tag{2}$$

For a proof of these homogenization results when λ is bounded between two positive constants see e.g. [13]. The degenerated case, i.e. when λ is allowed to approach zero or infinity, was studied in [6] (see also [2]). By using numerical methods it is possible to compute λ_{hom} by formula (2), see e.g. [3]. Another approach is to find bounds on λ_{hom}.

Bounds for the case when λ is bounded between two positive constants were presented in [10] (see also [9]). These bounds, combined with reiterated homogenization (i.e. introducing λ of the form $\lambda(x/\varepsilon, \ldots, x/\varepsilon^m)$), have played a central role in the development of new optimal structures, see e.g. [1], [7], [8], [11], [12], [14] and [16]. Moreover, these bounds were used in an extension of the Ponte Castaneda variational principle ([4] and [5]) to obtain bounds for a class of more general nonlinear problems than those described above, see [15].

The main results of this paper are that we prove lower and upper bounds on λ_{hom} for the degenerated case (see Theorem 1 and Theorem 2), i.e. we find functions λ_{lower} and λ_{upper} such that

$$\lambda_{\text{lower}}(\xi) \leqslant \lambda_{\text{hom}}(\xi) \leqslant \lambda_{\text{upper}}(\xi).$$

Moreover, we present some illustrative examples where the bounds are tight and thus can be used as a good approximation of λ_{hom}.

2. Notation and preliminary results

Let Ω be an open bounded subset of $\mathbb{R}^n$, Y the unit cube in $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ the Euclidean inner product. Let $1 < p < \infty$, $1/p + 1/q = 1$, and let λ be a Y-periodic (weight) function such that

$$\lambda > 0 \text{ a.e., and } \lambda,\ \lambda^{-1/(1-p)} \text{ are in } L^1_{\text{loc}}(\mathbb{R}^n).$$

The set of all real-valued functions u in $L^1_{\text{loc}}(\Omega)$ such that $u\lambda^{1/p}$ is in $L^p(\Omega)$ is denoted by $L^p(\Omega, \lambda)$. The set of functions u in $W^{1,1}_{\text{loc}}(\Omega)$ such that u and $|Du|$ are in $L^p(\Omega, \lambda)$ is denoted by $W^{1,p}(\Omega, \lambda)$. Moreover, by $W^{1,p}_0(\Omega, \lambda)$ we mean the closure of $C^1_0(\Omega)$ in $W^{1,p}(\Omega, \lambda)$ and $W^{1,p}_{\text{per}}(Y, \lambda)$ is the set of real functions u in $W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ such that u is Y-periodic and $u \in W^{1,p}(Y, \lambda)$.

Define the family (u_ε) as the set of solutions of the variational problems

$$E_\varepsilon = \min_{u \in W^{1,p}_0(\Omega, \lambda(x/\varepsilon))} \left\{ \int_\Omega \frac{1}{p} \lambda\Big(\frac{x}{\varepsilon}\Big) |Du|^p \, \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x \right\}.$$

Under the additional assumption that λ belongs to the Muckenhoupt class A_p it was proved in [6] that (u_ε) converges weakly in $W_0^{1,1}(\Omega)$ to the unique solution u_{hom} of the homogenized problem

$$E_{\text{hom}} = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega \lambda_{\text{hom}}(Du)\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x \right\}$$

where λ_{hom} is defined as

$$\lambda_{\text{hom}}(\xi) = \min_{v \in W_{\text{per}}^{1,p}(Y,\lambda)} \int_Y \frac{1}{p}\lambda(y)|\xi + Dv|^p\,\mathrm{d}y. \tag{3}$$

It was also proved that

$$E_\varepsilon \to E_{\text{hom}}.$$

We remark that the family of solutions (u_ε) of the minimization problems described above are also solutions to the weighted weak formulations of the p-Poisson equations, namely

$$\begin{cases} \displaystyle\int_\Omega \Big\langle \lambda\Big(\frac{x}{\varepsilon}\Big)|Du_\varepsilon|^{p-2}Du_\varepsilon, D\varphi\Big\rangle\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \\ u_\varepsilon \in W_0^{1,p}(\Omega, \lambda(x/\varepsilon)) \end{cases}$$

for every φ in $W_0^{1,p}(\Omega, \lambda(x/\varepsilon))$. The homogenized solution u_{hom} satisfies the homogenized problem

$$\begin{cases} \displaystyle\int_\Omega \langle b(Du_{\text{hom}}), D\varphi\rangle\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x, \\ u_{\text{hom}} \in W_0^{1,p}(\Omega), \end{cases}$$

for every φ in $W_0^{1,p}(\Omega)$, where b is given by

$$b(\xi) = \int_Y \lambda(y)|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi)\,\mathrm{d}y$$

and w_ξ is the solution of the local problem

$$\begin{cases} \displaystyle\int_Y \langle \lambda(y)|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi), D\varphi\rangle\,\mathrm{d}y = 0, \\ w_\xi \in W_{\text{per}}^{1,p}(Y,\lambda), \end{cases} \tag{4}$$

for every φ in $W_{\text{per}}^{1,p}(Y,\lambda)$.

Remark 1. The solution w_ξ of the local problem (4) is also the minimizer in the local minimization problem (3) and

$$\lambda_{\text{hom}}(\xi) = \Big\langle \frac{1}{p} b(\xi), \xi \Big\rangle.$$

3. Bounds

In this section we present upper and lower bounds on the homogenized energy density functional λ_{hom} defined in (3). The bounds are given in the following two theorems:

Theorem 1. *Let λ_{hom} be defined as in* (3). *Then we have the upper bound*

$$\lambda_{\text{hom}}(ke_i) \leqslant \lambda_{\text{upper}}(ke_i) \overset{\text{def}}{=} |k|^p \frac{1}{p} \biggl(\int_0^1 \langle \lambda \rangle_i^{1/(1-p)} \,\mathrm{d}y_i \biggr)^{1-p},$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$ and

$$\langle \lambda \rangle_i = \int_0^1 \ldots \int_0^1 \lambda \,\mathrm{d}y_1 \ldots \mathrm{d}y_{i-1} \,\mathrm{d}y_{i+1} \ldots \mathrm{d}y_n.$$

Theorem 2. *Let λ_{hom} be defined as in* (3). *Then we have the lower bound*

$$\begin{aligned} &\lambda_{\text{hom}}(ke_i) \geqslant \lambda_{\text{lower}}(ke_i) \\ &\qquad \overset{\text{def}}{=} |k|^p \frac{1}{p} \int_0^1 \ldots \int_0^1 \langle \lambda^{1/(1-p)} \rangle_i^{1-p} \,\mathrm{d}y_1 \ldots \mathrm{d}y_{i-1} \,\mathrm{d}y_{i+1} \ldots dy_n, \end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis in $\mathbb{R}^n$ and

$$\langle \lambda^{1/(1-p)} \rangle_i = \int_0^1 \lambda^{1/(1-p)} \,\mathrm{d}y_i.$$

Remark 2. By linearity, we get lower and upper bounds on λ_{hom} for all $\xi \in \mathbb{R}^n$ when $p = 2$.

For the proofs of Theorem 1 and Theorem 2 we need the following two lemmata, which themselves are of independent interest:

Lemma 3. *Let D be a measurable set in $\mathbb{R}^d$ such that $|D| = 1$. Moreover, let $a \geqslant 0$ be a weight function such that $a \in L^1(D)$ and $a^{1/(1-r)} \in L^1(D)$, where $1 < r < \infty$. Then*

$$\min_{u \in U} \int_D a(x) |1 + u(x)|^r \,\mathrm{d}x = \biggl(\int_D a(x)^{1/(1-r)} \,\mathrm{d}x \biggr)^{1-r},$$

where

$$U = \Bigl\{ u \in L^r(D, a) \colon \int_D u \,\mathrm{d}x = 0 \Bigr\}. \tag{5}$$

Moreover, the minimum is attained for

$$\widetilde{u} = \Big(\int_D a^{1/(1-r)}\,\mathrm{d}x\Big)^{-1} a^{1/(1-r)} - 1.$$

P r o o f. The reversed Hölder inequality and (5) imply that

$$\begin{aligned}\int_D a(x)|1+u(x)|^r\,\mathrm{d}x &\geqslant \Big(\int_D a(x)^{1/(1-r)}\,\mathrm{d}x\Big)^{1-r}\Big|\int_D 1+u(x)\,\mathrm{d}x\Big|^r \\ &= \Big(\int_D a(x)^{1/(1-r)}\,\mathrm{d}x\Big)^{1-r}.\end{aligned}$$

Equality holds in Hölder's inequality when

$$ca^{1/(1-r)} = |1+u| = 1+u.$$

The zero average constraint implies that

$$c = \Big(\int_D a^{1/(1-r)}\,\mathrm{d}x\Big)^{-1}.$$

Since $a \in L^1(D)$ and $a^{1/(1-r)} \in L^1(D)$, it follows that $0 < c < \infty$. Moreover, $\widetilde{u} \in L^r(D,a)$ since

$$\begin{aligned}\int_D |\widetilde{u}|^r a\,\mathrm{d}x &= \int_D |ca^{1/(1-r)} - 1|^r a\,\mathrm{d}x \\ &\leqslant C\Big(\int_D a^{1/(1-r)}\,\mathrm{d}x + \int_D a\,\mathrm{d}x\Big) < \infty,\end{aligned}$$

where C is a constant. □

Lemma 4. *Let λ_{hom} be defined as in* (3). *Then*

$$\lambda^*_{\mathrm{hom}}(\xi) = \inf_{\substack{\sigma\in V \\ \int_Y \sigma\,\mathrm{d}y=0}} \int_Y \frac{1}{q}\lambda^{1-q}|\xi+\sigma|^q\,\mathrm{d}y,$$

where λ^*_{hom} is the Legendre transform of λ_{hom} and V is defined as

$$V = \Big\{\sigma \in L^q(Y,\lambda^{1-q})\colon \int_Y \langle\sigma, Dv\rangle\,\mathrm{d}y = 0 \ \textit{for all}\ v \in W^{1,p}_{\mathrm{per}}(Y,\lambda)\Big\}.$$

P r o o f. Let $f\colon\ \mathbb{R}^n \to \mathbb{R}$ be defined as

$$f(\xi) = \frac{1}{p}\lambda|\xi + s|^p,$$

where s is a fixed vector in $\mathbb{R}^n$. The Legendre transform f^* of f is

$$f^*(\sigma) \overset{\text{def}}{=} \sup_{\xi\in\mathbb{R}^n}\{\langle\sigma,\xi\rangle - f(\xi)\} = \frac{1}{q}\lambda^{1-q}|\sigma|^q - \langle\sigma, s\rangle.$$

This implies Young's inequality

$$\langle\sigma,\xi\rangle \leqslant f(\xi) + f^*(\sigma) \quad \text{for all } \sigma,\xi\in\mathbb{R}^n, \tag{6}$$

with equality for

$$\sigma = \lambda|\xi + s|^{p-2}(\xi + s). \tag{7}$$

Inequality (6) implies that for any measurable function σ we have

$$\begin{aligned}\lambda_{\text{hom}}(\xi) &= \min_{v\in W^{1,p}_{\text{per}}(Y,\lambda)}\int_Y \frac{1}{p}\lambda(y)|\xi + Dv|^p\,\mathrm{d}y\\ &\geqslant \min_{v\in W^{1,p}_{\text{per}}(Y,\lambda)}\int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q + \langle\sigma, Dv\rangle\,\mathrm{d}y.\end{aligned}$$

This implies

$$\lambda_{\text{hom}}(\xi) \geqslant \sup_{\sigma\in V}\int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q\,\mathrm{d}y. \tag{8}$$

Actually we have equality in (8). This fact will be clear if we prove that

$$\lambda_{\text{hom}}(\xi) \leqslant \sup_{\sigma\in V}\int_Y \langle\sigma,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q\,\mathrm{d}y. \tag{9}$$

Let w_ξ be the minimizer in (3) and let σ_1 be defined as

$$\sigma_1 = \lambda|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi). \tag{10}$$

Then it follows by (6) and (7) that

$$\lambda_{\text{hom}}(\xi) = \int_Y \langle\sigma_1,\xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma_1|^q + \langle\sigma_1, Dw_\xi\rangle\,\mathrm{d}y.$$

Next we note that $\sigma_1 \in V$ and thus (9) holds. Indeed, by (10) and Remark 1 we have

$$\int_Y \langle \sigma_1, D\varphi\rangle \,\mathrm{d}y = \int_Y \langle \lambda|\xi + Dw_\xi|^{p-2}(\xi + Dw_\xi), D\varphi\rangle \,\mathrm{d}y = 0$$

for every $\varphi \in W^{1,p}_{\mathrm{per}}(Y,\lambda)$ and (10) implies that

$$\int_Y |\sigma_1|^q \lambda^{1-q} \,\mathrm{d}y = \int_Y |\xi + Dw_\xi|^p \lambda \,\mathrm{d}y < \infty.$$

We now proceed as follows:

$$\begin{aligned}\lambda_{\mathrm{hom}}(\xi) &= \sup_{\sigma\in V} \int_Y \langle \sigma, \xi\rangle - \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y \\ &= \sup_\eta \Bigg[\langle \eta, \xi\rangle - \inf_{\substack{\sigma\in V\\ \int_Y \sigma \,\mathrm{d}y = \eta}} \int_Y \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y\Bigg].\end{aligned} \tag{11}$$

Let $F\colon \mathbb{R}^n \to \mathbb{R}$ be defined as

$$F(\eta) = \inf_{\substack{\sigma\in V\\ \int_Y \sigma \,\mathrm{d}y = \eta}} \int_Y \frac{1}{q}\lambda^{1-q}|\sigma|^q \,\mathrm{d}y = \inf_{\substack{\sigma\in V\\ \int_Y \sigma \,\mathrm{d}y = 0}} \int_Y \frac{1}{q}\lambda^{1-q}|\eta + \sigma|^q \,\mathrm{d}y. \tag{12}$$

In view of (11) and (12) it follows that

$$\lambda_{\mathrm{hom}}(\xi) = \sup_\eta [\langle \eta, \xi\rangle - F(\eta)] = F^*(\xi).$$

Since F is convex and lower semicontinuous we have

$$\lambda^*_{\mathrm{hom}}(\xi) = F^{**}(\xi) = F(\xi)$$

and the proof is complete. □

P r o o f of Theorem 1. Without loss of generality we prove the result for $k = 1$. Let $M_i = \{v \in W^{1,p}_{\mathrm{per}}(Y,\lambda)\colon\ v = v(y_i)\}$. Lemma 3 then gives

$$\begin{aligned}\lambda_{\mathrm{hom}}(e_i) &= \min_{v\in W^{1,p}_{\mathrm{per}}(Y,\lambda)} \int_Y \frac{1}{p}\lambda(y)|e_i + Dv(y)|^p \,\mathrm{d}y \\ &\leqslant \min_{v\in M_i} \int_Y \frac{1}{p}\lambda(y)|1 + D_i v(y_i)|^p \,\mathrm{d}y \\ &= \min_{v\in M_i} \int_0^1 \frac{1}{p}\langle\lambda\rangle_i |1 + D_i v(y_i)|^p \,\mathrm{d}y_i \\ &= \frac{1}{p}\Big(\int_0^1 \langle\lambda\rangle_i^{1/(1-p)} \,\mathrm{d}y_i\Big)^{1-p}.\end{aligned}$$

□

P r o o f of Theorem 2. Without loss of generality we prove the result for $k=1$. Let

$$S_i=\Big\{\sigma\in V\colon\ \sigma=(0,\dots,\sigma_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n),\dots,0)\ \text{ and }\ \int_Y\sigma\,\mathrm{d}y=0\Big\}.$$

By using Lemma 4 and Lemma 3 we obtain

$$\begin{aligned}
\lambda^*_{\mathrm{hom}}(e_i) &= \inf_{\substack{\sigma\in V\\ \int_Y\sigma\,\mathrm{d}y=0}}\int_Y\frac{1}{q}\lambda^{1-q}|e_i+\sigma|^q\,\mathrm{d}y\\
&\leqslant \inf_{\sigma\in S_i}\int_Y\frac{1}{q}\lambda^{1-q}|1+\sigma_i(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)|^q\,\mathrm{d}y\\
&= \inf_{\sigma\in S_i}\int_0^1\dots\int_0^1\frac{1}{q}\langle\lambda^{1-q}\rangle_i|1+\sigma_i|^q\,\mathrm{d}y_1\dots\mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1}\dots\mathrm{d}y_n\\
&= \frac{1}{q}\Big[\int_0^1\dots\int_0^1\langle\lambda^{1-q}\rangle_i^{1-p}\,\mathrm{d}y_1\dots\mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1}\dots\mathrm{d}y_n\Big]^{1-q}.
\end{aligned}$$

This implies the following lower bound on $\lambda_{\mathrm{hom}}(e_i)$:

$$\begin{aligned}
\lambda_{\mathrm{hom}}(e_i) &= \sup_{\xi\in\mathbb{R}^n}\{\langle e_i,\xi\rangle-\lambda^*_{\mathrm{hom}}(\xi)\}\\
&\geqslant \sup_{te_i\in\mathbb{R}^n}\{t-\lambda^*_{\mathrm{hom}}(te_i)\}\\
&= \sup_{t\in\mathbb{R}}\{t-|t|^q\lambda^*_{\mathrm{hom}}(e_i)\}\\
&\geqslant \sup_{t\in\mathbb{R}}\Big\{t-\frac{|t|^q}{q}\Big[\int_0^1\dots\int_0^1\langle\lambda^{1-q}\rangle_i^{1-p}\,\mathrm{d}y_1\dots\mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1}\dots\mathrm{d}y_n\Big]^{1-q}\Big\}\\
&= \frac{1}{p}\int_0^1\dots\int_0^1\langle\lambda^{1/(1-p)}\rangle_i^{1-p}\,\mathrm{d}y_1\dots\mathrm{d}y_{i-1}\,\mathrm{d}y_{i+1}\dots\mathrm{d}y_n.
\end{aligned}$$

□

4. Some examples

In this section we apply the bounds from Theorem 1 and Theorem 2 in two illustrative examples. The examples are presented in $\mathbb{R}^2$ for simplicity. Let us first remark that when the upper and lower bounds are equal we know the effective energy density functional exactly. For instance, this is the case when λ is of the type

$$\lambda(y)=f(y_1)g(y_2),\quad \lambda(y)\ \text{ is } Y\text{-periodic}.$$

Then it follows from Theorem 1 and Theorem 2 that

$$\begin{aligned}\lambda_{\text{hom}}(e_1) &= \frac{1}{p}\biggl(\int_0^1 f(y_1)^{1/(1-p)}\,\mathrm{d}y_1\biggr)^{1-p}\int_0^1 g(y_2)\,\mathrm{d}y_2,\\ \lambda_{\text{hom}}(e_2) &= \frac{1}{p}\biggl(\int_0^1 g(y_2)^{1/(1-p)}\,\mathrm{d}y_2\biggr)^{1-p}\int_0^1 f(y_1)\,\mathrm{d}y_1.\end{aligned}$$

We now give one example of this situation where the conductivity degenerates on the unit cell boundary. We note that it is easy to make the mistake of believing that we have zero conductivity in one direction and infinitely high conductivity in the other direction. However, as we show below, the homogenized energy density functional is nonzero and finite in both directions.

Example 5. Consider the special case when $p = 2$ and let $\lambda\colon \mathbb{R}^2 \to \mathbb{R}$ be Y-periodic and defined as

$$\lambda(y) = |y_1(1-y_1)|^{-1/2}\biggl|y_1 - \frac{1}{2}\biggr|^{1/2} \quad \text{on } Y.$$

Then, by Theorem 1 and Theorem 2, we have

$$\begin{aligned}\lambda_{\text{hom}}(e_1) &= \frac{1}{2}\biggl(\int_0^1 \lambda^{-1}(y_1)\,\mathrm{d}y_1\biggr)^{-1} = \frac{1}{\sqrt{2}}\biggl(\frac{2\sqrt{2}}{3}K\Bigl(\frac{1}{\sqrt{2}}\Bigr)\biggr)^{-1},\\ \lambda_{\text{hom}}(e_2) &= \frac{1}{2}\int_0^1 \lambda(y_1)\,\mathrm{d}y_1 = \frac{\pi}{2K\bigl(\frac{1}{\sqrt{2}}\bigr)},\end{aligned}$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. This means that

$$\lambda_{\text{hom}}(e_1) \approx 0.40451, \quad \lambda_{\text{hom}}(e_2) \approx 0.84721,$$

that is, the effective conductivity in the y_2-direction is only about twice as high as the effective conductivity in the y_1-direction.

We now consider an example where the upper and lower bounds are very tight. This means that we have a good explicit estimate of the effective energy density functional. This fact can be used to obtain error estimates for numerical computations. We demonstrate this by comparing the bounds with numerical computations done in MATLAB using the FEMLAB toolbox.

Example 6. Let

$$D_r = \Bigl\{y \in Y\colon \Bigl|y - \Bigl(\frac{1}{2}, \frac{1}{2}\Bigr)\Bigr| \leqslant r,\ 0 \leqslant r \leqslant \frac{1}{2}\Bigr\}$$

and let $\lambda\colon\ \mathbb{R}^2 \to \mathbb{R}$ be Y-periodic and defined as

$$\lambda(y) = \begin{cases} \left(\dfrac{1}{r}\left|y - \left(\dfrac{1}{2}, \dfrac{1}{2}\right)\right|\right)^{\alpha}, & y \in D_r, \\ 1, & y \in Y \setminus D_r, \end{cases}$$

where $-2 < \alpha < 2p - 2$. By symmetry, the homogenized energy density functional $\lambda_{\text{hom}}(e_1) = \lambda_{\text{hom}}(e_2)$. Let $r = 0.4$ and let λ^-, λ^+ denote the lower and upper bounds, respectively.

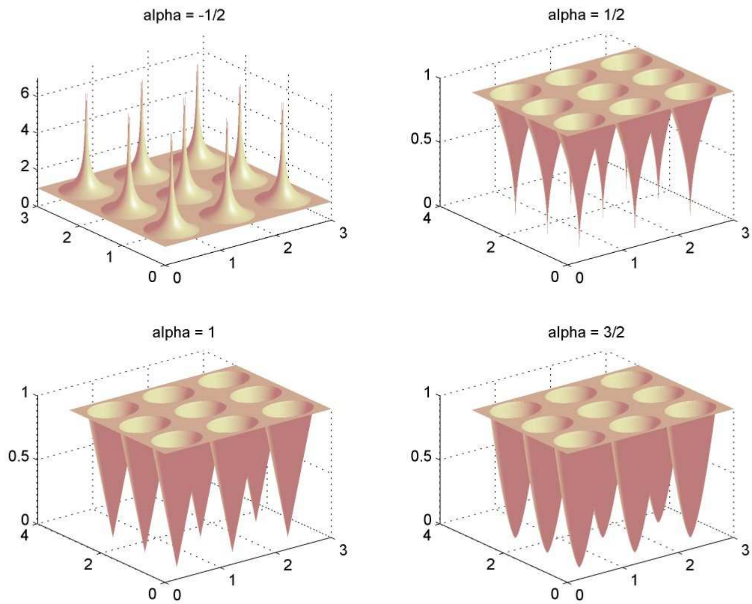

Figure 1. This picture shows 9 unit cells for each of the values $\alpha = -1/2$, $\alpha = 1/2$, $\alpha = 1$ and $\alpha = 3/2$. The radius is $r = 0.4$.

(a) The linear case, $p = 2$:

We present the results rounded to five digits.

α	$\lambda_{\text{hom}}(e_i)$	λ^-	λ^+
$-3/2$	0.70147	0.67425	0.74023
-1	0.63470	0.62059	0.65433
$-1/2$	0.56597	0.56203	0.57082
$1/2$	0.44172	0.43797	0.44482
1	0.39389	0.38207	0.40284
$3/2$	0.35639	0.33773	0.37078

(b) The nonlinear case, $p = 3$:

We present the results rounded to four digits.

α	$\lambda_{\text{hom}}(e_i)$	λ^-	λ^+
-1	0.4325	0.4260	0.4482
1	0.2649	0.2608	0.2708
2	0.2219	0.2121	0.2353
3	0.1952	0.1821	0.2132

As we see from these tables, the lower and upper bounds are very tight, which means that they can be used as a good approximation of the homogenized energy density functional.

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