Jaromír Baštinec; Josef Diblík

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ONE CASE OF APPEARANCE OF POSITIVE SOLUTIONS OF DELAYED DISCRETE EQUATIONS*

Jaromír Baštinec, Josef Diblík, Brno

Abstract. When mathematical models describing various processes are analysed, the fact of existence of a positive solution is often among the basic features. In this paper, a general delayed discrete equation

$$\Delta u(k+n) = f(k, u(k), u(k+1), \dots, u(k+n))$$

is considered. Sufficient conditions concerning f are formulated in order to guarantee the existence of a positive solution for $k \to \infty$. An upper estimate for it is given as well. The appearance of the positive solution takes its origin in the nature of the equation considered since the results hold only for delayed equations (i.e. for $n > 0$) and not for the case of an ordinary equation (with $n = 0$).

Keywords: positive solution; nonlinear discrete delayed equation

MSC 2000: 39A10, 39A11

1. Introduction

The phenomenon of existence of a positive solution of differential or difference equations often arises when we analyse mathematical models describing various processes. It is an opposite case to the phenomenon of oscillation of all solutions. The existence of positive solutions is very often substantial for a concrete model considered. In biology e.g. when a model of population dynamics is described by an equation, the positivity of a solution may mean that a concrete biological species can exist in the supposed environment. This is a motivation for intensive study of conditions of existence of positive solutions of differential and difference equations, as well as of their properties. Let us note that investigations in this field can be found e.g. in [1], [3]–[12].

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In this paper conditions of existence of a positive solution are given for a general class of nonlinear delayed discrete equations. Results obtained indicate sufficient conditions for the existence of a positive solution and also give an upper estimate for it. As for the results in the existing literature we remark that only concrete classes of linear and nonlinear discrete equations were considered. Sufficient conditions in Theorem 2 are sharp in a sense. This is illustrated by considering a simple linear equation with a suitable right-hand side and with a coefficient satisfying an indicated inequality. If in this equation the coefficient is constant and satisfies the inverse inequality then in accordance with the known results all solutions oscillate. This comparison underlines the high effectiveness of the sufficient conditions. Moreover, the importance of our results consists also in the fact that the appearance of positive solutions is caused by the delay (which is quite usual e.g. in biological models) involved in the equation considered. If the delay is missing, the results lose any sense.

2. Preliminary

Let us consider the scalar delayed discrete equation

$$\Delta u(k+n) = f(k, u(k), u(k+1), \ldots, u(k+n)), \tag{1}$$

where $f(k, u_0, u_1, \ldots, u_n)$ is defined on $N(a) \times \mathbb{R}^{n+1}$, $N(a) := \{a, a+1, \ldots\}$, $a \in \mathbb{N}$ with values in $\mathbb{R}$, $a \in \mathbb{N}$ and $n \in \mathbb{N}$ are fixed, $\mathbb{N} := \{1, 2, \ldots\}$. In this paper we are interested in the existence of a positive solution of equation (1) for $k \to \infty$.

Together with the discrete equation (1) we consider an initial problem. It is posed as follows: for a given $s \in \mathbb{N}$ find the solution of (1) satisfying $n+1$ initial conditions

$$u(a+s+m) = u^{s+m} \in \mathbb{R}, \quad m = 0, 1, \ldots, n \tag{2}$$

with prescribed constants u^{s+m}.

Let us recall that the solution of the initial problem (1), (2) is defined as an infinite sequence of numbers

$$\{u(a+s) = u^s, u(a+s+1) = u^{s+1}, \ldots, u(a+s+n) = u^{s+n},$$
$$u(a+s+n+1), u(a+s+n+2), \ldots\}$$

such that for any $k \in N(a+s)$ the equality (1) holds.

The existence and uniqueness of the solution of the initial problem (1), (2) is obvious for every $k \in N(a+s)$.

We will suppose that for all

$$(k, u_0, u_1, \ldots, u_n), (k, v_0, v_1, \ldots, v_n) \in N(a) \times \mathbb{R}^{n+1},$$

the Lipschitz type condition

$$|f(k,u_0,u_1,\dots,u_n)-f(k,v_0,v_1,\dots,v_n)|\leqslant\lambda(k)\sum_{i=0}^{n}|u_i-v_i|$$

holds with a nonnegative function $\lambda(k)$ defined on $N(a)$. Then the initial problem (1), (2) depends continuously on the initial data (see e.g. [1]).

For every $k\in N(a)$ let us define a set $\omega(k)$ as

$$\omega(k):=\{u\in\mathbb{R}:b(k)<u<c(k)\},\tag{3}$$

where $b(k)$, $c(k)$, $b(k)<c(k)$ are real functions defined on $N(a)$.

The following theorem is proved in [2] and will be used below.

Theorem 1. *Let us suppose that $f(k,u_0,u_1,\dots,u_n)$ is defined on $N(a)\times\mathbb{R}^{n+1}$ with values in $\mathbb{R}$ and that for all*

$$(k,u_0,u_1,\dots,u_n),(k,v_0,v_1,\dots,v_n)\in N(a)\times\mathbb{R}^{n+1}$$

we have

$$|f(k,u_0,u_1,\dots,u_n)-f(k,v_0,v_1,\dots,v_n)|\leqslant\lambda(k)\sum_{i=0}^{n}|u_i-v_i|,$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. If, moreover, the inequalities

$$f(k,u_0,u_1,\dots,u_{n-1},b(k+n))-b(k+n+1)+b(k+n)<0\tag{4}$$

and

$$f(k,u_0,u_1,\dots,u_{n-1},c(k+n))-c(k+n+1)+c(k+n)>0\tag{5}$$

hold for every $k\in N(a)$ and every $u_0\in\omega(k),u_1\in\omega(k+1),\dots,u_{n-1}\in\omega(k+n-1)$, then there exists an initial condition

$$u^*(a+m)=u^*_m\in\mathbb{R},\quad m=0,1,\dots,n$$

with

$$u^*_0\in\omega(a),\ u^*_1\in\omega(a+1),\ \dots,\ u^*_n\in\omega(a+n)$$

such that the corresponding solution $u=u^*(k)$ of equation (1) satisfies the inequalities

$$b(k)<u^*(k)<c(k)$$

for every $k\in N(a)$.

3. Existence of a positive solution

In this part we indicate conditions under which a positive solution of Equation (1) exists. In the proof of the corresponding theorem, the following elementary lemma concerning asymptotic expansion of a certain function is necessary. The proof is omitted since it can be done easily with the aid of the binomial formula. Let us recall that the symbol O used below means the Landau order symbol.

Lemma 1. *For $k \to \infty$ and $\sigma, d \in \mathbb{R}$ fixed, the following asymptotic representation holds:*

$$\left(1+\frac{d}{k}\right)^{\sigma} = 1 + \frac{\sigma d}{k} + \frac{\sigma(\sigma-1)d^2}{2k^2} + O\left(\frac{1}{k^3}\right). \tag{6}$$

Theorem 2 (Existence of a positive solution)**.** *Let $a \in \mathbb{N}$ and $n \in \mathbb{N}$ be fixed. Let us suppose that $f(k, u_0, u_1, \ldots, u_n)$ is defined on $N(a) \times \mathbb{R}^{n+1}$ with values in $\mathbb{R}$ and for all*

$$(k, u_0, u_1, \ldots, u_n), (k, v_0, v_1, \ldots, v_n) \in N(a) \times \mathbb{R}^{n+1}$$

we have

$$|f(k, u_0, u_1, \ldots, u_n) - f(k, v_0, v_1, \ldots, v_n)| \leqslant \lambda(k) \sum_{i=0}^{n} |u_i - v_i|,$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. If, moreover, there exists a constant $\theta \in [0, 1)$ such that

$$\begin{aligned} -\left(\frac{n}{n+1}\right)^{k+n} \cdot \sqrt{k} \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right) \\ < f\left(k, u_0, u_1, \ldots, u_{n-1}, \sqrt{k+n} \cdot \left(\frac{n}{n+1}\right)^{k+n}\right) \end{aligned} \tag{7}$$

and

$$f(k, u_0, u_1, \ldots, u_{n-1}, 0) < 0 \tag{8}$$

for every $k \in N(a)$ and every $u_0 \in \omega(k)$, $u_1 \in \omega(k+1)$, …, $u_{n-1} \in \omega(k+n-1)$ with

$$b(k) := 0, \quad c(k) := \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^{k}$$

then there exists a positive integer $a_1 \geqslant a$ *and a solution* $u = u(k)$, $k \in N(a_1)$ *of equation* (1) *such that*

$$u(k) > 0 \tag{9}$$

holds for every $k \in N(a_1)$.

Proof. In the proof, Theorem 1 is used. We have (see (3))

$$\omega(k) := \{u \in \mathbb{R}\colon\ b(k) < u < c(k)\} \equiv \left\{u \in \mathbb{R}\colon\ 0 < u < \sqrt{k} \cdot \Big(\frac{n}{n+1}\Big)^k\right\}$$

for every $k \in N(a)$. Let us verify that inequality (4) holds. It is easy to see that (due to (8))

$$\begin{aligned} &f(k, u_0, u_1, \dots, u_{n-1}, b(k+n)) - b(k+n+1) + b(k+n) \\ &\qquad = f(k, u_0, u_1, \dots, u_{n-1}, 0) < 0 \end{aligned}$$

for every $k \in N(a)$ and every $u_0 \in \omega(k)$, $u_1 \in \omega(k+1)$, …, $u_{n-1} \in \omega(k+n-1)$.

Let us start the verification of inequality (5). For sufficiently large $k \in N(a)$ and for every $u_0 \in \omega(k)$, $u_1 \in \omega(k+1)$, …, $u_{n-1} \in \omega(k+n-1)$ we get

$$\begin{aligned} &f(k, u_0, u_1, \dots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ &\qquad = f\left(k, u_0, u_1, \dots, u_{n-1}, \sqrt{k+n} \cdot \Big(\frac{n}{n+1}\Big)^{k+n}\right) \\ &\qquad\qquad - \sqrt{k+n+1} \cdot \Big(\frac{n}{n+1}\Big)^{k+n+1} + \sqrt{k+n} \cdot \Big(\frac{n}{n+1}\Big)^{k+n}. \end{aligned}$$

Further, the inequality (7) yields

$$\begin{aligned} &f(k, u_0, u_1, \dots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ &\quad > -\Big(\frac{n}{n+1}\Big)^{k+n} \cdot \sqrt{k} \cdot \Big(\frac{1}{n+1} + \frac{\theta n}{8k^2}\Big) \\ &\quad\quad - \sqrt{k+n+1} \cdot \Big(\frac{n}{n+1}\Big)^{k+n+1} + \sqrt{k+n} \cdot \Big(\frac{n}{n+1}\Big)^{k+n} \\ &\quad = \Big(\frac{n}{n+1}\Big)^{k+n} \cdot \sqrt{k} \cdot \left[-\Big(\frac{1}{n+1} + \frac{\theta n}{8k^2}\Big) - \sqrt{1 + \frac{n+1}{k}} \cdot \frac{n}{n+1} + \sqrt{1 + \frac{n}{k}}\right]. \end{aligned}$$

Using formula (6) with $\sigma = 1/2$, $d = n+1$ and with $\sigma = 1/2$, $d = n$ we get

$$\sqrt{1 + \frac{n+1}{k}} = 1 + \frac{n+1}{2k} - \frac{(n+1)^2}{8k^2} + O\Big(\frac{1}{k^3}\Big)$$

and

$$\sqrt{1 + \frac{n}{k}} = 1 + \frac{n}{2k} - \frac{n^2}{8k^2} + O\Big(\frac{1}{k^3}\Big).$$

Therefore

$$\begin{aligned}
&f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\
&\quad > \Big(\frac{n}{n+1}\Big)^{k+n} \cdot \sqrt{k} \cdot \Big[-\Big(\frac{1}{n+1} + \frac{\theta n}{8k^2}\Big) \\
&\quad - \Big(1 + \frac{n+1}{2k} - \frac{(n+1)^2}{8k^2} + O\Big(\frac{1}{k^3}\Big)\Big) \cdot \frac{n}{n+1} \\
&\quad + 1 + \frac{n}{2k} - \frac{n^2}{8k^2} + O\Big(\frac{1}{k^3}\Big)\Big].
\end{aligned}$$

It is easy to verify that the expression in the square brackets equals

$$\begin{aligned}
-\Big(\frac{1}{n+1} + \frac{\theta n}{8k^2}\Big) - \Big(1 + \frac{n+1}{2k} - \frac{(n+1)^2}{8k^2} + O\Big(\frac{1}{k^3}\Big)\Big) \cdot \frac{n}{n+1} \\
+ 1 + \frac{n}{2k} - \frac{n^2}{8k^2} + O\Big(\frac{1}{k^3}\Big) = \frac{n(1-\theta)}{8k^2} + O\Big(\frac{1}{k^3}\Big)
\end{aligned}$$

and, consequently,

$$\begin{aligned}
&f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\
&\quad > \Big(\frac{n}{n+1}\Big)^{k+n} \cdot \sqrt{k} \cdot \Big[\frac{n(1-\theta)}{8k^2} + O\Big(\frac{1}{k^3}\Big)\Big].
\end{aligned}$$

Now it is obvious that there exists an integer $a_1 \geqslant a$ such that the inequality

$$\frac{n(1-\theta)}{8k^2} + O\Big(\frac{1}{k^3}\Big) > 0$$

holds for every $k \in N(a_1)$. Consequently,

$$f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) > 0,$$

i.e. the inequality (5) holds for every $k \in N(a_1)$. So, all the suppositions of Theorem 1 are valid with $a := a_1$. Hence there exists an initial problem

$$u^*(a_1 + m) = u^*_m \in \mathbb{R}, \quad m = 0, 1, \ldots, n$$

with

$$u^*_0 \in \omega(a_1),\ u^*_1 \in \omega(a_1+1),\ \ldots,\ u^*_n \in \omega(a_1+n)$$

such that the corresponding solution $u = u^*(k)$ of equation (11) satisfies $u^*(k) > 0$ for every $k \in N(a_1)$, i.e. (9) holds. The theorem is proved. □

Taking into account the form of the set $\omega(k)$ for every $k \in N(a)$, given by relation (3), it is easy to improve the assertion of Theorem 2:

Theorem 3 (Estimation of a positive solution). *Let all assumptions of Theorem 2 be valid. Then there exists a positive integer $a_1 \geqslant a$ and a solution $u = u(k)$, $k \in N(a_1)$ of equation (1) such that*

$$0 < u(k) < \sqrt{k} \cdot \Big(\frac{n}{n+1}\Big)^k \tag{10}$$

holds for every $k \in N(a_1)$.

Remark 1. Let us note that the assumption $n \in \mathbb{N}$ in Theorems 2, 3 cannot be weakened to $n \in \mathbb{N} \cup \{0\}$. Indeed, as can be seen easily, if $n = 0$, the proof of Theorem 2 as well as formula (10) lose any sense. It means that the results presented hold only in the case of (substantially) delayed discrete equations.

4. An application

Let us consider the delayed scalar linear discrete equation

$$\Delta u(k+n) = -p(k)u(k) \tag{11}$$

with fixed $n \in \mathbb{N}$ and variable $k \in N(a)$, $N(a) := \{a, a+1, \ldots\}$, $a \in \mathbb{N}$. The function $p\colon\ N(a) \to \mathbb{R}$ is supposed to be positive. Let us apply Theorems 2, 3 to the case of equation (11). The following result is a consequence of these theorems in the case when

$$f(k, u(k), u(k+1), \ldots, u(k+n)) := -p(k)u(k), \quad \lambda(k) := p(k).$$

Theorem 4. *Let $a \in \mathbb{N}$ and $n \in \mathbb{N}$ be fixed. Suppose that there exists a constant $\theta \in [0, 1)$ such that the function $p\colon\ N(a) \to \mathbb{R}$ satisfies the inequalities*

$$0 < p(k) \leqslant \Big(\frac{n}{n+1}\Big)^n \cdot \Big(\frac{1}{n+1} + \frac{\theta n}{8k^2}\Big) \tag{12}$$

for every $k \in N(a)$. Then there exists a positive integer $a_1 \geqslant a$ and a solution $u = u(k)$, $k \in N(a_1)$ of equation (11) such that the inequalities

$$0 < u(k) < \sqrt{k} \cdot \Big(\frac{n}{n+1}\Big)^k$$

hold for every $k \in N(a_1)$.

5. Concluding remarks

Problems concerning the asymptotic behaviour of solutions of discrete equations has been widely investigated. Let us remark that the result given by Theorem 4 improves a previous one, given in [10, p. 192]. Our results are sharp in a sense. Indeed, it is known (see e.g. [10, p. 179]) that if $p(k) = p = \text{const}$ and an inequality opposite to (12) holds, namely, the inequality

$$p > \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1},$$

then all solutions of (11) oscillate. This inequality is a necessary and sufficient condition for the oscillation of all solutions of the discrete equation (11) with a constant coefficient.

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Authors' addresses: *Jaromír Baštinec*, Department of Mathematics, Faculty of Electrical Engineering and Communication, Brno University of Technology, Technická 8, 616 00 Brno, Czech Republic, e-mail: `bastinec@feec.vutbr.cz`; *Josef Diblík*, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Brno University of Technology, Žižkova 17, 662 37 Brno, Czech Republic, e-mail: `diblik.j@fce.vutbr.cz`.