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CONVERGENCE OF ROTHE'S METHOD IN HÖLDER SPACES*

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Abstract. The convergence of Rothe's method in Hölder spaces is discussed. The obtained results are based on uniform boundedness of Rothe's approximate solutions in Hölder spaces recently achieved by the first author. The convergence and its rate are derived inside a parabolic cylinder assuming an additional compatibility conditions.

Keywords: Rothe's method, method of lines, convergence of Rothe's method

MSC 2000: 65M40

Introduction

Let us consider the linear parabolic equation

$$\partial_t u + Au = f(t) \quad \text{in } I \times \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $I = (0, T)$, $T < \infty$, and

$$Au = -\sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + a_0(x)u$$

with $a_{ij}, a_0 \in L_\infty(\Omega)$. Let the boundary $\partial\Omega$ of Ω be Lipschitz continuous and let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be relatively open subsets. We assume $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\operatorname{mes}_{N-1}\Gamma_1 + \operatorname{mes}_{N-1}\Gamma_2 = \operatorname{mes}_{N-1}\partial\Omega$.

Along with the parabolic equation we consider the mixed boundary conditions

$$u(t,x) = \Psi(t,x) \text{ on } I \times \Gamma_1, \quad -\partial_{\nu_A} u = \gamma(x)u \text{ on } I \times \Gamma_2,$$

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and the initial condition

$$u(0,x)=u_0(x) \text{ on } \Omega,$$

where $u_0 \in L_2(\Omega)$ and $-\partial_{\nu_A} u = \sum_{i,j=1}^{N} a_{ij}(x)\partial_{x_j} u\nu_i$, ν being the outward normal to $\partial\Omega$.

We consider the weak solution $u(t) \in W_2^1(\Omega)$ of the introduced parabolic problem which has to satisfy the identity

$$(1)\qquad (\partial_t u(t),v)+((u(t),v))=(f(t),v) \quad \text{for a.e. } t\in I, \quad \forall v\in V,$$
$$u(0)=u_0,$$

where $V=\{v\in W_2^1(\Omega)\colon\ v|_{\Gamma_1}=0\}$ ($W_2^1(\Omega)$ being the Sobolev space), (u,v) denotes the scalar product in $L^2(\Omega)$ and $((u,v))$ is defined by

$$((u,v)) := \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}\partial_{x_j} u\, \partial_{x_i} v + a_0 uv\right) \mathrm{d}x + \int_{\Gamma_2} \gamma uv \,\mathrm{d}\sigma$$

for $u,v\in V$, where a_{ij}, a_0, γ are measurable and bounded functions. Let $u_\Psi(t)\in W_2^1(\Omega)$, $t\in I$, be such that $u_\Psi(t)=\Psi(t)$ on Γ_2 (in the sense of traces). Then, additionally to (1), the weak solution $u(t)$ has to satisfy $u(t)-u_\Psi(t)\in V$.

Let $|\cdot|$ denote the $L_2(\Omega)$ norm and $\|\cdot\|$ the norm in $W_2^1(\Omega)$. We assume that

$$(2)\qquad a_{ij}(x)\xi_i\xi_j \geqslant C_{\mathrm{e}} \sum_{i=1}^{N} \xi_i^2 \quad \forall\,\xi\in\mathbb{R}^N, \quad a_0,\gamma\geqslant 0, \quad a_{ij}=a_{ji}, \quad i,j=1,\dots,N,$$

where $C_2>0$.

R e m a r k 1. The nonhomogeneous boundary condition

$$-\partial_{\nu_A} u = \gamma(x)(u-\varphi_1)+\varphi_2 \quad \text{on } I\times\Gamma_2$$

can be reduced by linear transformation to the homogeneous one ($\varphi_1=\varphi_2=0$) provided there exists a smooth function Θ on Ω satisfying

$$-\partial_{\nu_A}\Theta = \gamma(x)(\Theta-\varphi_1)+\varphi_2 \quad \text{on } I\times\Gamma_2.$$

The very common numerical approximation of (1) is Rothe's method, the method of lines, which corresponds to Euler backward approximation of ODE. In this concept, time discretization is used and (1) is reduced to a sequence of elliptic problems in the following way.

Let $h = T/n$ be a time step for any $n \in \mathbb{N}$. On the time level $t = t_i = ih$ $(i = 1, 2, \ldots, n)$ we successively determine $u_i \in W_2^1(\Omega)$, $u_i - u_\Psi(t_i) \in V$, from elliptic problems

$$(\delta u_i, v) + ((u_i, v)) = (f_i, v) \quad \forall\, v \in V, \tag{3}$$

where $\delta u_i := (u_i - u_{i-1})/h$, $f_i = \frac{1}{\tau}\int_{t_{i-1}}^{t_i} f(s,x)\,\mathrm{d}s$ and u_0 is from (1) and u_{i-1} is known from the previous time level. By means of u_i $(i = 1, 2, \ldots, n)$ we construct approximate solutions $u^n(t)$ and $\overline{u}^n(t)$ of (1) as follows:

$$\begin{aligned} &u^n(t) := u_{i-1} + (t - t_{i-1})\delta u_i, \\ &\overline{u}^n(t) := u_i \quad \text{for } t \in (t_{i-1}, t_i],\ \ i = 1, 2, \ldots, n, \\ &u^n(0) = \overline{u}^n(0) := u_0. \end{aligned} \tag{4}$$

The function u^n and the corresponding step function $\overline{u}^n$ are called *Rothe's functions.*

For both numerical and theoretical reasons, appropriate functional space, convergence and its rate are of importance. The convergence of $\{u^n\}$ has been studied in many papers, see [2]–[5], [9], [10]. The following functional spaces have been considered: $C(I, L_2(\Omega))$, $C(I \times \Omega)$, $L_2(I, C^{0,\mu}(\Omega))$, where $C^{0,\mu}$ is the Hölder space, see [11].

The main purpose of this section is to discuss the convergence of $\{u^n\}$ in the space $C^{0,\alpha}(Q_{\mathrm{loc}})$, where $Q = I \times \Omega$ and $\overline{Q}_{\mathrm{loc}} \subset Q$.

Our convergence result is based on the following one in [6]:

Theorem K. *Let $\{u^n\}$ be defined by* (3), (4). *If* (2) *is satisfied and if $\{u^n\}$ is uniformly bounded in $L_\infty(Q_{\mathrm{loc}})$, then there exist $C > 0$ and $\mu > 0$ such that*

$$|u^n(t,x) - u^n(t',y)| \leqslant C(|t - t'|^\mu + |x - y|^{2\mu}) \quad \forall\,(t,x),\ (t',y) \in Q_{\mathrm{loc}}$$

holds uniformly for n.

We can formulate our result as follows:

If the conditions of Theorem K are fulfilled and (5), (6) (see below) are satisfied, then (cf. Theorem 1)

$$\begin{aligned} \|u^n - u\|_{C^{0,\delta\mu}(Q_{\mathrm{loc}})} \leqslant C\bigl(1/n &+ \sup_{|\tau| \leqslant h} \|\partial_t u(\cdot + \tau) - \partial_t u(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau| \leqslant 2h} \|\partial_t u_\Psi(\cdot + \tau) - \partial_t u_\Psi(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau| \leqslant 2h} \|\partial_t f(\cdot + \tau) - \partial_t f(\cdot)\|_{\infty,Q}\bigr)^{1-\delta}, \end{aligned}$$

where $\delta \in (0,1)$ and μ is from Theorem K.

Moreover, under some additional regularity of f and u_Ψ and smoothing effect of parabolic problems we can localize the term $\partial_t u$ in the time variable, i.e., to the interval (τ, T), $\tau > 0$. Then our final convergence result reads (cf. Theorem 3)

$$\|u^n - u\|_{C^{0,\delta\mu}(Q_{\rm loc})} \leqslant C(\Omega, \tau) n^{-(1-\delta)},$$

where $Q_{\rm loc} \subset (\tau, T) \times \Omega$. If, moreover, $a_{ij}, a_0, \partial_t f \in C^{0,1}(Q)$ and $\partial_t^2 u_\Psi \in L_2(I, W_2^2(\Omega)) \cap L_\infty(Q)$ then we can take arbitrary $0 < \mu < 1$ and replace $Q_{\rm loc}$ by $(\tau, T) \times \Omega$ (cf. Theorem 5).

In Section 1 we prove the $L_\infty(Q)$ convergence. In Section 2 we prove the $L_\infty((\tau, T) \times \Omega)$ convergence and in Section 3 the $C^{0,\mu}(Q_{\rm loc})$ convergence.

Section 1

In addition to (2) we assume

$$f \in L_\infty(Q), \quad u_0 \in V \cap L_\infty(\Omega), \tag{5}$$

and the following regularity concerning u_0, f:

there exists $z_0 \in L_\infty(\Omega)$ such that

$$(z_0, v) + ((u_0, v)) = (f(0), v) \quad \forall\, v \in V. \tag{6}$$

R e m a r k 2. The assumption (6) requires stronger regularity for u_0 and a compatibility condition on the boundary $\partial\Omega$ for $t = 0$. Condition (6) is clearly satisfied if u_0 satisfies the stationary problem corresponding to (1) at $t = 0$.

R e m a r k 3. The existence and uniqueness of $u_i \in V$ $(i = 1, 2, \ldots, n)$ satisfying (3) follows from the Lax-Milgram lemma.

We prove the uniform boundedness of $\{u_i\}$ in $L_\infty(\Omega)$. We set $\|g\|_{\infty,D} := \|g\|_{L_\infty(D)}$.

Lemma 1. *If* (2), (5) *and* $\partial_t u_\Psi \in L_\infty(Q)$ *are satisfied, then*

$$\|u_i\|_{\infty,\Omega} \leqslant C \quad (i = 1, 2, \ldots, n)$$

holds uniformly for n.

P r o o f. The unique weak solution u_i of (3) can be represented in the form $u_i = z_i + u_{\Psi,i}, z_i \in V$ (we denote $u_{\Psi,i} \equiv u_\Psi(t_i)$), where z_i is the unique point of minimum of the functional

$$\Phi_i(v) = \frac{1}{2h} \int_\Omega (v - u_{i-1})^2 \,\mathrm{d}x + \frac{1}{2}((v, v)) - (f_i, v)$$

over the set $u_{\Psi,i}+V$. For $K \geqslant 0$ let u_i^K be the truncation of u_i defined by $u_i^K = \min\{1, K/|u_i|\}u_i$ ($u_i^K(x)=0$ if $u_i(x)=0$). We will prove that

$$\Phi_i(u_i) - \Phi_i\big(u_i^K\big) \geqslant 0$$

for

$$K = \max\{\|u_{\Psi,i}\|_\infty, \|u_0\|_\infty, \|u_{i-1}\|_\infty + \tau\|f_i\|_\infty\}.$$

Then from the uniqueness argument it follows that $\|u_i\|_\infty \leqslant K$ and, consequently, from this recurrent estimate (K is dependent of $\|u_{i-1}\|_\infty$) we obtain

$$\|u_i\|_{\infty,\Omega} \leqslant \|u_0\|_{\infty,\Omega} + \|u_\Psi\|_{\infty,Q} + \int_I \|f(t)\|_{\infty,\Omega}\,\mathrm{d}t.$$

Here we can assume that $\|u_\Psi\|_{\infty,Q} = \|\Psi\|_{\infty,I\times\Gamma_1}$, since in the place of u_Ψ we could take u_Ψ^L with $L = \|\Psi\|_{\infty,I\times\Gamma_1}$.

We write

$$\begin{aligned}\Phi_i(u_i) - \Phi_i(u_i^K) = \frac{1}{2h}&\int_\Omega (u_i - u_i^K)(u_i + u_i^K - 2u_{i-1} - 2\tau f_i)\,\mathrm{d}x \\ &+ ((u_i - u_{i-1}^K, u_i - u_{i-1}^K)) \\ &+ \int_{\Gamma_2} (u_i - u_i^K)(u_i + u_i^K - 2u_{i-1})\,\mathrm{d}\sigma.\end{aligned}$$

We find out easily that all three terms on R.H.S. are nonnegative due to our assumptions. □

Now, we shall prove the uniform boundedness of δu_i in $L_\infty(\Omega)$.

Lemma 2. *If (2), (5), (6) hold and $\partial_t f, \partial_t u_\Psi \in L_\infty(Q)$, then*

$$\|\delta u_i\|_{\infty,\Omega} \leqslant C \quad (i = 1,2,\dots,n), \quad \|\partial_t u\|_{\infty,Q} \leqslant C$$

uniformly for n.

P r o o f. We subtract (3) with indices i, $i-1$ and obtain

$$(\delta u_i - \delta u_{i-1}, v) + h((\delta u_i, v)) = h(\delta f_i, v) \quad \forall v \in V.$$

If $i=1$ then we use (6), where δu_0 is replaced by z_0. Then we proceed in the same way as in the proof of Lemma 1 and conclude that

$$(7) \qquad \|\delta u_i\|_{\infty,\Omega} \leqslant C(\|z_0\|_{\infty,\Omega} + \|\partial_t f\|_{\infty,Q} + \|\partial_t u_\Psi\|_{\infty,Q}) \leqslant C \quad (i = 1,2,\dots,n)$$

hold uniformly for n. Hence, we get $u^n \to u$ in $C(I, L_2)$ (see, e.g. [5]) and $\partial_t u^n \rightharpoonup w$ (weakly) in $L_2(Q)$ for $n \to \infty$. Therefore, by letting $n \to \infty$ in the identity

$$(u^n(t), v) - (u_0, v) = \int_0^t (\partial_t u^n, v)\,\mathrm{d}s \quad \forall\, v \in C_0^\infty(\Omega),$$

we find that

$$(u(t) - u_0, v) = \int_0^t (w(s), v)\,\mathrm{d}s \quad \forall\, v \in C_0^\infty(\Omega),$$

which implies

$$u(t) - u_0 = \int_0^t w(s)\,\mathrm{d}s \quad \text{in } \; L_2(\Omega),$$

and hence, $\partial_t u = w$. Moreover, from (7) we deduce

$$\begin{aligned}\int_I \int_\Omega \partial_t u v\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty} \int_I \int_\Omega \partial_t u^n v\,\mathrm{d}x\,\mathrm{d}t \\ &\leqslant C(\|z_0\|_{\infty,\Omega} + \|\partial_t f\|_{\infty,Q} + \|\partial_t u_\Psi\|_{\infty,Q}) \\ &\quad \times \int_I \int_\Omega |v|\,\mathrm{d}x\,\mathrm{d}t \quad \forall\, v \in C_0^\infty(Q)\end{aligned}$$

to obtain

$$\|\partial_t u\|_{\infty,Q} \leqslant C(\|z_0\|_{\infty,\Omega} + \|\partial_t f\|_{\infty,Q} + \|\partial_t u_\Psi\|_{\infty,Q})$$

and the proof is complete. □

Now, we set $\widetilde{u}_i = h^{-1} \int_{I_i} u(t)\,\mathrm{d}t$, $\overline{u}_i = u(t_i)$, and $e_i = \widetilde{u}_i - u_i$ for $i = 1, 2, \ldots, n$, and $e_0 = 0$.

Theorem 1. *If the assumptions of Lemma* 2 *are fulfilled, then*

$$\begin{aligned}\|u^n - u\|_{\infty,Q} \leqslant C\Big(\frac{1}{n} &+ \sup_{|\tau|\leqslant 2h} \|\partial_t u(\cdot+\tau) - \partial_t u(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau|\leqslant 2h} \|\partial_t u_\Psi(\cdot+\tau) - \partial_t u_\Psi(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau|\leqslant 2h} \|f(\cdot+\tau) - f(\cdot)\|_{\infty,Q}\Big),\end{aligned}$$

where u is the unique solution of (1) *and* $\{u^n\}$ *is from* (3) *and* (4).

P r o o f. We integrate (1) over $I_i = (t_{i-1}, t_i)$, $1 \leqslant i \leqslant n$, and obtain

$$(\delta \overline{u}_i, v) + ((\widetilde{u}_i, v)) = (\widetilde{f}_i, v) \quad \forall\, v \in V.$$

Subtracting it from (3), we deduce

$$(e_i - e_{i-1}, v) + h((e_i, v)) = h(z_i, v) + h(\widetilde{f}_i - f_i, v) \quad (i = 1, 2, \ldots, n), \tag{8}$$

where

$$\begin{aligned} z_i := \delta\widetilde{u}_i - \delta\overline{u}_i &= h^{-2}\int_{I_i} (u(s) - u(s-h))\,\mathrm{d}s - h^{-1}\int_{I_i} \partial_t u(s)\,\mathrm{d}s \\ &= h^{-1}\int_{I_i} \left(h^{-1}\int_{s-h}^{s} \partial_t u(r)\,\mathrm{d}r - \partial_t u(s) \right) \mathrm{d}s. \end{aligned}$$

We recall that $u \in L_\infty(Q)$ as a consequence of Lemma 1 and $u^n(t) \to u(t)$ in $L_2(\Omega)$ by the same argument as we have proved $\partial_t u \in L_\infty(Q)$. Now we estimate

$$\begin{aligned} \|z_i\|_{\infty,\Omega} &\leqslant h^{-2}\int_{I_i}\int_{s-h}^{s} \|\partial_t u(s) - \partial_t u(r)\|_{\infty,\Omega}\,\mathrm{d}r\,\mathrm{d}s \\ &\leqslant \sup_{|\tau|\leqslant h} h^{-1}\int_{I_i} \|\partial_t u(s+\tau) - \partial_t u(s)\|_{\infty,\Omega}\,\mathrm{d}s \\ &\leqslant \sup_{|\tau|\leqslant 2h} \|\partial_t u(t+\tau) - \partial_t u(t)\|_{\infty,\Omega} \end{aligned}$$

and similarly

$$\begin{aligned} \|\widetilde{f}_i - f_i\|_{\infty,\Omega} &\leqslant \sup_{|\tau|\leqslant h} h^{-1}\int_{I_i} \|f(s+\tau) - f(s)\|_{\infty,\Omega}\,\mathrm{d}s \\ &\leqslant \sup_{|\tau|\leqslant 2h} \|f(t+\tau) - f(t)\|_{\infty,\Omega} \end{aligned}$$

for any $t \in I_i$. The error e_i in (8) can be considered as the unique point of minimum of the functional

$$\Phi_i(v) = \frac{1}{2h}\int_\Omega (v - e_{i-1})^2\,\mathrm{d}x + \frac{1}{2}((v, v)) - (f_i, v)$$

over the set $w_{\Psi,i} + V$, where $w_{\Psi,i} = \delta\widetilde{u}_{\Psi,i} - \delta\overline{u}_{\Psi,i-1}$. Proceeding as in the proof of Lemma 1 and using the above estimates, we obtain

$$\begin{aligned} \|e_i\|_{\infty,\Omega} \leqslant C(\Omega)\big(&\sup_{|\tau|\leqslant 2h} \|\partial_t u(t+\tau) - \partial_t u(t)\|_{\infty,\Omega} \\ &+ \sup_{|\tau|\leqslant 2h} \|f(t+\tau) - f(t)\|_{\infty,\Omega} \\ &+ \sup_{|\tau|\leqslant 2h} \|\partial_t u_\Psi(t+\tau) - \partial_t u_\Psi(t)\|_{\infty,\Omega}\big), \end{aligned}$$

where the term $w_{\Psi,i}$ has been estimated in the same way as the term z_i. Now, for $t \in I_i$ we estimate

$$\begin{aligned}\|u^n - u\|_{\infty,\Omega} &\leqslant \|u - \overline{u}_i\|_{\infty,\Omega} + \|\widetilde{u}_i - \overline{u}_i\|_{\infty,\Omega} + \|\widetilde{u}_i - u_i\|_{\infty,\Omega} + 2h\|\delta u_i\|_{\infty,\Omega} \\ &\leqslant C(h\|\partial_t u\|_{\infty,\Omega} + h\|\partial_t u^n\|_{\infty,\Omega} + \sup_{|\tau|\leqslant 2h} \|\partial_t u(t+\tau) - \partial_t u(t)\|_{\infty,\Omega} \\ &\qquad + \sup_{|\tau|\leqslant 2h} \|f(t+\tau) - f(t)\|_{\infty,\Omega} + \sup_{|\tau|\leqslant 2h} \|\partial_t u_\Psi(t+\tau) - \partial_t u_\Psi(t)\|_{\infty,\Omega}).\end{aligned}$$

Hence, our result follows from Lemma 2. □

Section 2

In this section we estimate $\partial_t u$, $\partial_t^2 u$ in $Q_\tau = (\tau, T) \times \Omega$ using the smoothing effect of linear parabolic problems. We follow the idea from [5], [14]. Denote by

$$\delta^p u_i := \frac{\delta^{p-1} u_i - \delta^{p-1} u_{i-1}}{h} \quad \text{for } p \geqslant 2.$$

From (3) we obtain, by subsequent subtracting,

$$(9) \qquad (\delta^2 u_i, v) + ((\delta u_i, v)) = (\delta f_i, v) \quad (i \geqslant 3), \quad \forall\, v \in V.$$

Lemma 3. *Assume* (2), (5), (6) *and that* $\partial_t f \in L_2(Q), \partial_t u_\Psi \in L_\infty(I, L_2(\Omega))$, $\partial_t u_\Psi \in L_2(I, W_2^1(\Omega))$. *Then the estimates*

$$h\sum_{i=1}^{j} |\delta u_i|^2 + \|u_j\|^2 \leqslant C \quad \text{and} \quad |\delta u_j|^2 + h\sum_{i=1}^{j} \|\delta u_i\|^2 \leqslant C$$

hold uniformly for j *and* n.

P r o o f. We test (3) with $v = u_i - u_{\Psi,i}$ and sum it up for $i = 1, \ldots, j$. Then using symmetry of $((w, w))$ and Gronwall's argument, we get the first estimate. To prove the second inequality we subtract (3) with indices i, $i-1$ and put $v = \delta u_i - \delta u_{\Psi,i}$. Summing it up for $i = 1, 2, \ldots, j$ and using (6) and Gronwall's argument, we obtain the required second inequality. □

Now we prove the smoothing effect—see [5], [14].

Lemma 4. *Assume* (2), (5), (6), $\partial_t^3 f \in L_2(I, L_2(\Omega))$, $\partial_t^3 u_\Psi \in L_\infty(I, W_2^1(\Omega))$, *and* $\partial_t^4 u_\Psi \in L_2(I, W_2^1(\Omega))$. *Then the estimates*

$$\|\delta u_i\|^2 = \frac{C}{i_0 h}, \quad |\delta^2 u_i|^2 \leqslant \frac{C}{(i_0 h)^2}, \quad \text{and} \quad |\delta^3 u_i|^2 \leqslant \frac{C}{(i_0 h)^4}$$

hold uniformly for n, *and* $i \geqslant 5i_0$, $i_0 \geqslant 4$ *is fixed.*

P r o o f. We test (9) with $v = \delta u_i - \delta u_{i-1} - (\delta u_{\Psi,i} - \delta u_{\Psi,i-1})$ and sum it up for $i = r+1, \ldots, s$. Using the symmetry of $((w,w))$, we obtain

$$h \sum_{i=r+1}^{s} |\delta^2 u_i|^2 + \frac{1}{2} C_l \|\delta u_s\|^2 \leqslant C_1 \|\delta u_r\|^2 + C_2 h \sum_{i=r+1}^{s} \|\delta^2 u_i\|^2 + C_3 h \sum_{i=r+1}^{s} \|\delta^2 f_i\|^2 + C_4 h \sum_{i=r+1}^{s} \|\delta^2 u_{\Psi,i}\|^2.$$

Then, using the estimates from Lemma 3 and Gronwall's argument, we obtain

$$\sum_{i=r+1}^{s} |\delta^2 u_i|^2 h + \|\delta u_s\|^2 \leqslant C(1 + \|\delta u_r\|^2), \tag{10}$$

where C is independent of r and s. Here we consider $s = 2i_0$ and $i_0 \leqslant r \leqslant 2i_0$. We multiply (10) by h and sum it up for $r = i_0+1, \ldots, 2i_0$. We find that (omitting the first term)

$$i_0 h \|\delta u_{2i_0}\|^2 \leqslant C\left(T + h \sum_{r=i_0+1}^{2i_0} \|\delta u_r\|^2\right) \leqslant \overline{C},$$

because of Lemma 3. Using $s \geqslant 2i_0$, from (10) we deduce $(r = 2i_0)$

$$\|\delta u_s\|^2 \leqslant \frac{C}{i_0 h} \quad \text{and} \quad h \sum_{i=r+1}^{s} |\delta^2 u_i|^2 \leqslant \frac{C}{i_0 h}. \tag{11}$$

Now, we successively subtract (9) with i, $i-1$ and put $v = \delta^2 u_i - \delta^2 u_{\Psi,i}$. We sum it up for $i = r+1, \ldots, s$ and come to

$$\frac{1}{2}|\delta^2 u_s|^2 - \frac{1}{2}|\delta^2 u_r|^2 + h \sum_{i=r+1}^{s} \|\delta^2 u_i\|^2 \leqslant C\left(h \sum_{i=r+1}^{s} |\delta^2 u_i|^2 + h \sum_{i=r+1}^{s} |\delta^2 f_i|^2\right) + \varepsilon h \sum_{i=r+1}^{s} \|\delta^2 u_i\|^2 + C_\varepsilon h \sum_{i=r+1}^{s} \|\delta^2 u_{\Psi,i}\|^2,$$

where $\varepsilon > 0$. We take into account the regularity of f and u_Ψ. From Gronwall's argument and for ε small we obtain

$$|\delta^2 u_s|^2 + h \sum_{i=r+1}^{s} \|\delta^2_{u_i}\|^2 \leqslant C(1 + |\delta^2 u_r|^2). \tag{12}$$

Here, we proceed analogously as in (10). We consider $s \geqslant 3i_0$ and $r = 2i_0+1, \ldots, 3i_0$. Using (11) we conclude $|\delta^2 u_{3i_0}|^2 \leqslant C/(i_0 h)^2$ and applying it to (12), we get

$$|\delta^2 u_s|^2 + h \sum_{i=r+1}^{s} \|\delta^2 u_i\|^2 \leqslant \frac{C}{(i_0 h)^2} \quad \text{for } s \geqslant 3i_0. \tag{13}$$

Now we proceed analogously in estimating $\delta^3 u_i$. We use

$$(\delta^3 u_i, v) + ((\delta^2 u_i, v)) = (\delta^3 f_i, v) \quad \forall v \in V$$

and apply (13), (11). Then we deduce the required estimate. □

Section 3

Our main result will be obtained from Theorem 1 and Theorem K, interpolating the spaces $C^0(Q)$ and $C^{0,\mu}(Q)$. First we verify the following equality for any $0 < \beta < \alpha < 1$:

$$[w]_\beta := \frac{|w(x) - w(y)|}{|x-y|^\beta} = |w(y) - w(x)|^{\beta/\alpha} \Big(\frac{|w(x) - w(y)|}{|x-y|^\gamma} \Big)^\chi \tag{14}$$

with $\gamma = \alpha\beta/(\alpha - \beta)$ and $\chi = (\alpha - \beta)/\alpha$.

The first convergence result can be formulated as follows.

Theorem 2. *Let the assumptions of Theorem* 1 *be satisfied. Then*

$$\begin{aligned} \|u^n - u\|_{C^{0,\delta\mu}(Q_{\mathrm{loc}})} \leqslant C\Big(\frac{1}{n} &+ \sup_{|\tau|\leqslant 2h} \|\partial_t u(\cdot + \tau) - \partial_t u(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau|\leqslant 2h} \|\partial_t u_\Psi(\cdot + \tau) - \partial_t u_\Psi(\cdot)\|_{\infty,Q} \\ &+ \sup_{|\tau|\leqslant 2h} \|f(\cdot + \tau) - f(\cdot)\|_{\infty,Q}\Big)^{1-\delta} \end{aligned}$$

holds, where μ is from Theorem K.

For the proof we apply (14), where we put $w = u^n - u$, $\beta := \delta\mu$, and choose α so that $\gamma = \mu$. Then $\beta/\alpha = 1 - \delta$. We estimate the right-hand side in (14) by means of Theorem 1 and Theorem K.

Since Theorem K is restricted only to Q_{loc} we cannot obtain the convergence result on Q. To estimate $\partial_t u(t+\tau) - \partial_t u(t)$ in the space $L_\infty(Q)$ we need a higher order regularity of $\partial_t u$ which requires very restrictive higher order compatibility conditions at $t = 0$. To avoid it we will focus only to the time interval (τ, T), $0 < \tau < T$, apply Theorem 1 only to the domain $Q_\tau = (\tau, T) \times \Omega$ and use our results from Section 2. We shall also restrict only to the Dirichlet boundary value problem, i.e., we shall assume $\Gamma_2 = \emptyset$. However, the results can be extended also to our original mixed boundary conditions. We prove the following lemma.

Lemma 5. *Let the assumptions of Lemma* 4 *be satisfied and let* $\Gamma_2 = \emptyset$, $\partial_t^2 u_\Psi \in L_\infty(Q)$. *Then there exists* $C = C(\tau, \Omega)$ *such that*

$$\|\partial_t u\|_{\infty, Q_\tau} + \|\partial_t^2 u\|_{\infty, Q_\tau} \leqslant C(\tau, \Omega)$$

holds in the case of $N = 2, 3$.

P r o o f. The function $\delta^2 u_i$ is the solution of the elliptic problem

$$((\delta^2 u_i, v)) = -(\delta^3 u_i, v) + (\delta^2 f_i, v) =: (g_i, v) \quad \forall\, v \in V, \quad i \geqslant 3,$$

where g_i is from $L_2(\Omega)$, because of Lemma 4. Thus, $\delta^2 u_i \in W_2^1(\Omega)$ and Sobolev imbedding theorem gives $\delta^2 u_i \in L_p(\Omega)$ for $p = 2N/(N-2)$ (for $N = 2$, p is arbitrarily large). This implies $g_i \in L_q(\Omega)$ for $q > N/2$ for $N = 2, 3$. Then from regularity results, see Theorem 13.1 in [12], and Lemma 4

$$\|\delta^2 u_i\|_{\infty, (t_0, T) \times \Omega} \leqslant C(t_0, \Omega)$$

uniformly for n, $i \geqslant i_0(n)$, $hi_0(n) \leqslant t_0$. Similarly, we obtain $\|\delta u_i\|_{\infty, (t_0, T) \times \Omega} \leqslant C(t_0, \Omega)$ for $i \geqslant i_0(n)$ with $hi_0(n) \geqslant t_0$. Then we construct $u^n_{(1)}(t), \overline{u}^n_{(1)}(t)$; $u^n_{(2)}(t)$, $\overline{u}^n_{(2)}(t)$ by means of δu_i; $\delta^2 u_i$ as in (4). From the above estimates we have $u^n_{(1)} \to \partial_t u$ in $L_2((t_0, T) \times \Omega)$ and since $\partial_t u^n_{(1)} = \overline{u}^n_{(2)}$, we similarly obtain $\partial_t u^n_{(1)} = \overline{u}^n_{(2)} \to \partial_t^2 u$ in $L_2((t_0, T) \times \Omega)$. By virtue of the L_∞ estimates for $\delta u_i, \delta^2 u_i$ we deduce $\partial_t u, \partial_t^2 u \in L_\infty((t_0, T) \times \Omega)$ and the proof is complete. $\square$

Our main result reads:

Theorem 3. *Let the assumptions of Lemma* 5 *be satisfied and let* $\partial_t f \in L_\infty(Q)$. *Then there exists* $C = C(t_0, \Omega)$ *such that*

$$\|u^n - u\|_{C^{0, \delta\mu}((t_0, T) \times \Omega')} \leqslant C(t_0, \Omega) n^{-(1-\delta)}$$

holds for any $\delta \in (0, 1)$, *where* μ *is from Theorem* K, $\overline{\Omega}' \subset \Omega$, u *is the variational solution of* (1) *and* u^n *is defined by* (3) *and* (4).

For the proof we apply Lemma 5 in Theorem 2, where $Q_{\mathrm{loc}} = (t_0, T) \times \Omega'$.

R e m a r k 4. Our convergence result is valid only for $a_{ij}, a_0 \in L_\infty(\Omega)$.

As a generalization of Lemmas 4 and 5 we have

Theorem 4. *Let the assumptions of Theorem 2 be fulfilled. If* $\partial_t^p f \in L_2(I\times\Omega_{\mathrm{loc}})$, $\partial_t^{p+2} u_\Psi \in L_2(I, W_2^1(\Omega))$ *and* $\partial_t^{p+1} u_\Psi \in L_\infty(I, W_2^1(\Omega))$ *for* $p = 0, 1, \ldots, L$, *then*

$$h \sum_{i=r+1}^{s} |\delta^{p+1} u_i|^2 + \|\delta^p u_s\|^2 \leqslant C(i_0 h)^{-2p+1},$$

$$h \sum_{i=r+1}^{s} \|\delta^{p+1} u_i\|^2 + |\delta^{p+1} u_s|^2 \leqslant C(i_0 h)^{-2p}$$

holds uniformly for n, $s \geqslant 3^p i_0$, *and* $p = 1, 2, \ldots, L$.

Remark 5. The results of Lemma 5 can be extended also for $N = 4$ using the $W^1_{2+\varepsilon}$ regularity for $\delta^2 u_i$, see Theorems 1 and 2 in [7].

If we assume $C^{0,1}(\Omega)$ regularity of a_{ij}, then we can prove the convergence of $\{u^n\}$ in $C^{0,1}(t_0, \Omega)$. Then, in Theorem 3, we can take an arbitrary $\mu < 1$.

Theorem 5. *Let the assumptions of Theorem 2 be satisfied. If, moreover,* $\partial\Omega \in C^2$, $f, \partial_t f \in C^{0,1}((t_0, T)\times\Omega)$ *and* $a_0, a_{ij} \in C^{0,1}((t_0, T)\times\Omega)$ $(i, j = 1, 2, \ldots, N)$, *then*

$$\|u^n - u\|_{C^{0,\delta'}((t_0,T)\times\Omega)} \leqslant C(t_0, \Omega) n^{-(1-\delta)}$$

holds for any $\delta \in (0, 1)$ *and* $\delta' < \delta$.

Proof. Because of the regularity assumptions we have—see [12], [1, Theorem 8.12], that $u_i \in W_2^2(\Omega)$ for $i = 1, 2, \ldots, n$. We can verify that $D^\beta(u_i)$ with $|\beta| = 1$ satisfies elliptic problem

$$((D^\beta(u_i), v)) = -(D^\beta(\delta u_i), v) + (D^\beta f_i, v) + (aD^1 u_i, D^1 v) \quad \forall v \in V, \tag{15}$$

where D^1 are the first order partial derivatives and a represents a matrix from the functions of the form $D^1 a_{rs}$. Then we insert $v = D^\beta(u_i)$ into (15) and using Young's inequality and (2), we obtain the estimate

$$\|D^\beta u_i\|_{W_2^1(\Omega)} \leqslant C(\Omega)(\|u_i\| + \|\delta u_i\| + \|f_i\|) \leqslant C(\Omega)/(i_0 h)^2 \quad \forall i \geqslant i_0 h \geqslant t_0$$

by virtue of Theorem 4 (with $L = 1$). This information we use again in (15) and we also apply Theorem 13.1 from [12]. Similarly as in the proof of Lemma 5 we deduce

$$D^\beta u_i \in L_\infty(\Omega) \quad \text{and} \quad \|D^\beta u_i\|_{L_\infty(\Omega)} \leqslant C(t_0, \Omega)$$

for all $i \geqslant i_0$ such that $i_0 h \geqslant t_0$. Since β is arbitrary with $|\beta| = 1$, we conclude that

$$u_i \in W^1_\infty(\Omega) \quad \text{and} \quad \|u_i\|_{W^1_\infty(\Omega)} \leqslant C(t_0, \Omega). \tag{16}$$

From Lemma 2 we obtain

$$\|\delta u_i\|_{\infty,\Omega} \leqslant C, \quad \forall\, n, \;\; i = 1, 2, \ldots, n, \tag{17}$$

which implies

$$\|u_i - u_j\|_{\infty,\Omega} \leqslant C(t_0, \Omega)(i-j)h \quad \forall\, n, \;\; i = 1, 2, \ldots, n. \tag{18}$$

As a consequence of (16) and (18) we obtain

$$|u^n(x,t) - u^n(y,t')| \leqslant C(|t-t'| + |y-x|) \tag{19}$$

uniformly in $(t_0, T) \times \Omega$, where $C = C(t_0, \Omega)$. Then from Theorem 1, Lemma 5, (14), and (19) we obtain the assertion of Theorem 5 similarly as we have obtained Theorem 3. □

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