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ON IMPLICIT CONSTITUTIVE THEORIES

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Abstract. In classical constitutive models such as the Navier-Stokes fluid model, and the Hookean or neo-Hookean solid models, the stress is given explicitly in terms of kinematical quantities. Models for viscoelastic and inelastic responses on the other hand are usually implicit relationships between the stress and the kinematical quantities. Another class of problems wherein it would be natural to develop implicit constitutive theories, though seldom resorted to, are models for bodies that are constrained. In general, for such materials the material moduli that characterize the extra stress could depend on the constraint reaction. (E.g., in an incompressible fluid, the viscosity could depend on the constraint reaction associated with the constraint of incompressibility. In the linear case, this would be the pressure.) Here we discuss such implicit constitutive theories. We also discuss a class of bodies described by an implicit constitutive relation for the specific Helmholtz potential that depends on both the stress and strain, and which does not dissipate in any admissible process. The stress in such a material is not derivable from a potential, i.e., the body is not hyperelastic (Green elastic).

Keywords: constitutive relations, constraint, Lagrange multiplier, Helmholtz potential, rate of dissipation, elasticity, inelasticity, viscoelasticity

MSC 2000: 76A02, 76A05, 76A10, 74D10, 74A20, 74C99

1. Introduction

I discuss here a general framework for describing the thermomechanics of continuous media. Most of the theories that are in place are specializations of the framework which is being proposed. The new framework however allows for a much more general structure to study the response of bodies when subject to external as well as internal stimuli and allows one to bring under one unifying theme a much richer and wider class of material response. In fact, the framework provides an embarrassment of riches that can and has to be kept in check by appealing to sound physics.

It is customary to start by defining what is meant by a body and then delineating the set of states that are accessible to the bodies that belong to the Material Universe of interest (of course some of the bodies might be defined by a subset of the state parameters that have been identified, i.e., the state space associated with them is a subspace or submanifold of the state space of choice). In fact, identifying functions which will be associated with (defined on) our body of interest (i.e., the state functions) and their co-domains defines the class of bodies of interest (i.e., we are interested in those abstract sets B on which these state functions are defined).

However, the notion of a fixed abstract set B with which a certain set of properties can be associated (i.e., the assumption that certain functions can be defined on $B \times \mathbb{R}$) is quite untenable as a general framework for defining a body. For instance, if one is interested in growth or adaptation of biological matter due to external and internal stimuli, then the only set that qualifies to be designated as the body is the set that is present at current time, the body everchanging with time. Thus, the only tenable descriptor of a body is a (differentiable) manifold from an eulerian point of view: the set of particles that exist at a certain instant of time having a certain measure and topological structure. This set, frozen in time, compared with other sets similarly frozen at other instants of time, allow us to make measurements concerning members of subsets which have a commonality that can be mathematically identified. Moreover, no body or for that matter system is ever a closed system which receives no stimulus from that which is external to it. All systems are open systems. However, in some of them, we can neglect the external stimuli and treat them as closed systems. The only system, if any, that can be viewed as a closed system, is the total universe of objects that have nothing external to them, but one can immediately see the dangerous path down which the existence of such a set leads towards as one is then postulating a set of all bodies that qualifies to be a body. Trying to put such a general mathematical structure into place is not only beyond my ability, but were I able to do so, would make what follows so complicated that it would defeat the purpose of the article, namely a discussion of the physics of deformable continua.

Thus, I will dispense with the attempt to infuse such rigor and discuss issues from an intuitive standpoint.

The classical definition of a body that endows a set with a measure theoretic structure is itself a partial specification of what we mean by the state space of a body for it endows an abstract set (of "particles") with the property of mass. Associating other properties with the abstract set of particles is not any the different. Thus, the state space is nothing more than the space defined by the properties associated with an abstract set (of particles). With the same set of particles we can associate different properties with the passage of time. At different times, with the same abstract set of particles we thus can associate different response capabilities. The paths traversed in the state space are called processes. Not all points in the state space can be connected by processes, for a given set of particles, i.e., a particular body cannot be subject to every possible process. Thus, with the same set of particles that resembles a "solid" body at one instant of time, we might associate a body that is a "fluid" or a gas at a later instant of time. At this stage we of course cannot make precise what we mean by a solid, fluid or a gas. These comments are merely meant to motivate the main ideas.

The same set of particles, let us say a piece of steel, would respond like what is referred to as a linearized elastic solid when subject to sufficiently small deformations; behave in an inelastic manner when subject to sufficiently large deformations; if the conditions are right it could twin or undergo solid to solid phase transition; melt or for that matter exist in a gaseous state. It could thus happen that we need to associate different parts of the state space with the same set of particles, the body. It is important to recognize that for a specific set of particles, say in its "solid-like" form, not all the "states" that go to define the state space are necessary to describe its response in the sense that the body's response is very well approximated by a certain subset of states. This does not mean that the other functions that go to define the state-space are unimportant, it is just that they do not play a significant role in describing the body's behavior, given the state that it is in, for a certain class of processes of interest (one could think of the case of space, time and velocity as being state variables, and at sufficiently high velocities our constitutive theories would have to be different). Thus, for a body that is formed by the liquid that oozes from the bark of a rubber tree, *under certain conditions*, we need to know only the extent of its deformation to describe its state of stress reasonably well. (Both the stress and the deformation gradient are state functions. Constitutive relations are merely relations between the values of these state functions. Constitutive relations define subsets in the state-space.) This does not mean that how fast the body deforms or other such information is unnecessary. It merely means that for a class of processes that one is interested in, the response seems to be independent of how fast the body deforms. Thus, it might be more accurate to state that, as long as the body is deformed "sufficiently" slowly, the stress in the material is captured quite accurately by the deformation (which also has to be within certain limits). Thus, we do not need to know exactly how slowly we deform, provided it is below a certain value. Of course similar caveats would apply for the rate of heating, etc. We cannot exhaust the specification of all the quantities that need to meet certain restrictions in order that the stress response depend only on the state of the deformation. Nonetheless, we do find it most useful to discuss this ideal case without detailing all the caveats, and this idealization explains the response of materials such as rubber reasonably well for a large class of processes.

How a body responds may depend on how the body got to the state (i.e., its past history), but on the other hand it need not. In fact, theories which are currently available that describe the response of a body in terms of the history of certain states, can be recast within a framework where such a historical description is unnecessary; an explicit relationship between the stress and the history of the strain, say, can be described by an implicit theory in which the history of the strain is replaced by an equivalent differential equation and an initial condition.

Constitutive theories and constraints can be thought of as defining subsets in the state space. In fact, one could argue for a more general framework where even "balance laws" define appropriate subsets of the state space; for instance the balance of mass might define a subdomain in the state space with the possibility of radioactive decay outside this subdomain. That is, there could be processes in the state space wherein the balance of mass is violated. Such a process cannot lie within the subset of the state space wherein the balance of mass is always met. I shall not try to create such a grand framework. Here, I shall limit myself and rest content with a state-space wherein processes are required to meet the usual balance laws of continuum mechanics.

While discussing the state-space and process classes, it might be appropriate to discuss invariance requirements on constitutive quantities and balance laws. In different domains of the state-space and the process classes we might require different invariance requirements. For instance, we might require Galilean invariance for a certain class of processes. In classical Newtonian Mechanics, we require such an invariance for our balance laws. The requirement of frame-indifference for the constitutive relations which is routinely enforced for continua is somewhat questionable as they are ultimately substituted into the balance laws wherein we merely require Galilean invariance. For other classes of processes, we might require more general balance laws, say Maxwell's equations and invariance under Lorentz transformations. The requirement of Galilean invariance could be viewed as the invariance that this class of processes satisfies under certain conditions, i.e., Galilean invariance could be viewed as being an appropriate reduction of Lorentz invariance under the conditions under consideration. While both Galilean and Lorentz invariance seem to be appropriate restrictions in that they are consistent with one another, frame-indifference that allows for rotations that are time dependent between frames is not consistent with Lorentz invariance. It has been shown in [37] that frame-indifference cannot be required amongst all observers and that the rotation of the earth is sufficient to create problems as time goes on. In fact, it is easy to see that if $\boldsymbol{Q}(t)$ is not zero velocities can exceed the speed of light for sufficiently large $|\boldsymbol{x}|$.

It is also worth noting that to obtain the constitutive models for the stress in elasticity, one does not need to require frame-indifference, Galilean invariance would suffice. The standard argument that is used in developing the constitutive model for a viscous fluid (i.e., Newtonian fluid or Navier-Stokes fluid) makes use of the requirement of frame-indifference. However, it is possible to arrive at this model as a limit of a viscoelastic fluid with instantaneous elasticity, under the requirement of Galilean invariance. Thus, requirement of Galilean invariance is sufficient for arriving at the classical Navier-Stokes model in the sense of a limit. This point cannot be overemphasized as all models are merely limits and even fluids such as air and water can exhibit elastic response if the conditions are correct (or to be more precise, response which is predominantly elastic).

This fact is obscured by the manner in which the models for viscoelastic bodies are represented. In the traditional approach, a model such as the Maxwell model is described by a rate equation in which the time rate of the stress and the symmetric part of the velocity gradient appear explicitly. However, the manner in which such models are expressed within the context of bodies with multiple natural configurations is such that the stress is expressed in terms of the elastic response from an evolving set of natural configurations. Galilean invariance suffices for obtaining sensible models both with regard to the elastic response and the evolution equation that is given in terms of the deformation gradient from the evolving set of natural configurations (Galilean invariance would imply that the evolution equation be given in terms of the Cauchy-Green stretch tensor obtained from the evolving natural configuration). The evolution equation and the representation for the stress guarantee that one can obtain the classical Navier-Stokes fluid response as an appropriate limit for this viscoelastic model. Thus, the classical Navier-Stokes model can be obtained without ever requiring frame-indifference. Galilean invariance will suffice. This becomes clear from the discussion that follows.

Since we discuss rate type fluid models later, we refer the reader to equations (69)–(72) which are a generalization of the Maxwell model in the sense that this model corresponds to a body that stores energy like a neo-Hookean elastic solid while the Maxwell model stores energy like a linearized elastic solid. Linearization of (69)–(72) leads to the Maxwell model and we discuss it later. What is important to note is that the traditional way of expressing the Maxwell model is the following:

$$\begin{aligned} \boldsymbol{T} &= -p\mathbf{1} + \boldsymbol{S}, \\ \boldsymbol{S} + \tau \overset{\triangledown}{\boldsymbol{S}} &= 2\mu \boldsymbol{D}. \end{aligned}$$

Unfortunately, in the above representation $\boldsymbol{D}$ appears explicitly in the model while in the representation (69)–(72), which is equivalent to the above classical representation, only the elastic response from the evolving natural configurations occurs, and the classical Navier-Stokes model is obtained in the limit of absence of elastic response, i.e., with $\boldsymbol{B}_{\kappa_{p(t)}}$ in (69)–(72) tending to $\mathbf{1}$. This immediately ensures that $\boldsymbol{T} \to -p\mathbf{1} + 2\mu\boldsymbol{D}$. Thus, we have only to contend with restrictions that need to be imposed on elastic response from the evolving natural configurations and it is for this reason that the requirement of Galilean invariance suffices. There might be some concern that the rationale offered above is questionable as use is made of the upper convected derivative of the identity tensor being the negative of twice the symmetric part of the velocity gradient, which happens to be a frame-indifferent derivative. This is not the case. It is not that one needs to require that a frame-indifferent derivative be used. To the contrary, the time derivative of the deformation gradient of the elastic response is used, the current configuration being held fixed, and it happens that this time derivative is frame-indifferent (see [36] for a discussion of time derivatives while different configurations are being held fixed). This derivative occurs naturally within the thermodynamic framework that has been used to study bodies that have more than one natural configuration. It is a matter merely of taking appropriate time derivatives that have a specific physical meaning namely which configuration is held fixed, not whether the derivative is frame-indifferent. Thus, it is not necessary to require frame-indifference and make choices for values of the orthogonal transformation that relates the two frames, and its time derivative, as is usually done in standard treatments of the derivation of the Navier-Stokes model. It is sufficient merely to require the invariance required in elasticity. It transpires that the natural way of expressing the evolution of the natural configuration is in terms of a time derivative that corresponds to the current configuration of the body being held fixed.

The most popular constitutive theories for modeling fluids and solids, the Euler and Navier-Stokes fluid models, and the linearized elastic and the Neo-Hookean solid models are explicit models for the stress in that an explicit expression is provided for the stress in terms of kinematical variables. While implicit constitutive theories arise naturally within the context of viscoelasticity and inelasticity, the pregnant implications of such implicit theories have not been recognized. Some of these models for viscoelastic or inelastic materials are not truly implicit in that the differential equation governing the extra stress can be integrated to obtain an explicit representation, given an appropriate initial condition for the stress, the Maxwell model being a case in point.

A truly implicit constitutive theory requires careful evaluation of several notions that are taken for granted in explicit theories, for example the notion of material symmetry. Symmetry restrictions are now connected with invariances associated with certain implicit relationships. Moreover, such implicit theories are much richer than explicit theories in that there is bifurcation in two senses (more appropriately two levels) associated with such theories, one on the level of constitutive representations and the other the possibility of multiple solutions for the equations of motion for each constitutive representation.

Interestingly, the history of implicit constitutive modeling is very old. In his seminal paper on the classical Navier-Stokes-Poisson fluid[1], Stokes [46] recognized that the viscosity in a fluid could depend on the normal stresses. Now, if the fluid is incompressible[2], then the pressure is merely the negative of a third of the trace of the Cauchy stress tensor. Thus, if the viscosity depends on the pressure, it depends on the stress and we obtain an implicit model, in the limit of incompressibility. Implicit models for viscoelastic materials can be traced back to the seminal work of Maxwell [18], and those for the inelastic behavior of solids to the pioneering work of Prandtl [25].

The treatment of constraints, especially within the context of the response of continua, leaves much to be desired. Tenets have been put into place on the basis of D'Alembert's work [7] in particle dynamics that imply that constraint forces do no work. The assumption that constraint forces do no work stems from the work of Bernoulli and was articulated within the context of dynamics by D'Alembert [7]. Lagrange [15] expresses D'Alembert's ideas thus: *"If a motion is impressed upon several bodies so that they are forced to move consistent with their mutual interaction, it is clear that these motions can be viewed as composed of those which the bodies would actually follow and of other motions which are negated, from which it follows that these later motions must be such that the bodies following only these motions are in equilibrium."* Thus, the motions that are negated are essentially those due to the constraints and with regard to such motions, the body according to D'Alembert is in equilibrium, hence the forces associated with the constraint would do no work (as equilibrium with regard to such motion precludes any work being done). This idea of D'Alembert is widely used in particle dynamics, though it is recognized fully well that such an idea is not universally valid as the following comments of

[1] While the fluid is usually referred to as the Navier-Stokes fluid, Stokes [46] observes that his work is much closer in philosophy to the work of Poisson [24] than that of Navier [21], hence my inclusion of Poisson's name in the model. In fact, Stokes [46] remarks: "I afterward found that Poisson had written a memoir on the same subject, and on referring to it found that he had arrived at the same equations. The method which he employed was however so different from mine that I feel justified in lying the latter before this society." … "The same equations have also been obtained by Navier in the case of an incompressible fluid (Mém. de l'Académie, t. VI., p. 389), but his principles differ from mine still more than do Poisson's."

[2] Of course, incompressibility is merely an approximation and the density of the material remains essentially constant. In such fluids there is no equation of state that connects the pressure and the density. Though no real fluid is truly incompressible, it is possible that the changes in the pressure cause a small change in the density while changing the viscosity by several orders of magnitude.

Goldstein [11] make evident: "*We now restrict ourselves to systems for which the net virtual work of forces of constraint is zero. We have seen that this condition holds true for rigid bodies and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. This is no longer true if sliding friction forces are present, and we must exclude such systems from our formulation. The restriction is not unduly hampering, since the friction is essentially a macroscopic phenomenon.*" Even within the confines of particle dynamics, the appropriateness of D'Alembert's assumption was questioned by no less a person than Gauss [10] who suggested that the constraints in place lead to constraint responses that are "minimal" in order to enforce the constraint: "*The motion of a system of material points connected together in any manner whatsoever, whose motions are modified by any external restraints whatsoever, proceeds in every instance in the greatest possible accordance with free motion, or under the least possible constraint; the measure of the constraint which the whole system suffers in every particle of time being considered equal to the sum of the products of the square of the deviation of every point from its free motion into its mass. Let* $m, m', m'', \ldots$ *be the masses of the points;* $a, a', a'', \ldots$ *their places at the time* t*;* $b, b', b'', \ldots$ *the places which they would occupy if entirely free in their motion after the infinitely small particle of time* dt*, in consequence of the forces acting upon them during this time, and of the velocities and directions acquired by them at the time* t*. Their real places* $c, c', c'', \ldots$ *will then be those for which of all places compatible with the conditions of the system the quantity* $m(bc)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$ *is a minimum. The equilibrium is evidently a particular case only of the general law, and the condition for this case is, that* $m(bc)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$ *itself is a minimum, or that the continunce of the system in a state of rest more accords with the free motion of the single points than any possible change which the system could undergo*". While Gauss does not address whether the constraint force corresponding to this minimum does, or does not, do work, his statement allows for the possibility that work can be done by the constraint force. There is no notion of equilibrium of the body in the direction in which the constraint force acts in Gauss' discussion, and thus there is no reason to conclude that there is no work done by the constraint force. Gauss is not addressing problems involving friction. However, even in the absence of friction, the requirement that the constraint force does no work is an additional assumption.

D'Alembert's ideas have to be used with great caution and applied merely to a subclass of processes that are possible, even when one restricts attention to the class of rigid bodies. D'Alembert's requirement that the constraint response does no work is, as we shall see, imprecise and misleading. This will become immediately clear if we consider the following example of a rigid block of weight W sliding on a *rough planar surface* due to an applied force. If one were to depict the forces acting on the body, one would show the weight of the body, the reaction that acts normal to the surface, and the force due to friction at the surface of contact between the plane and the body. To accord special status to the normal reaction as being the only constraint force should be subject to a critical reassessment as the constraint is a "*rough*" planar surface and thus the force due to friction (due to the roughness of the surface) is also a constraint reaction. Were the *rough* surface not present, there would be no resistance due to the friction. This is not a mere semantic quibble. But even if this challenge to what constitutes the reaction due to the constraint is to be resolved in favor of just the normal force[3] being the constraint reaction, there is yet another serious problem. The question that ought to be asked is whether the constraint reaction has any effect on the work that is done by or on the body, i.e., does the constraint response influence the work done on or by the body? The answer to this is an unequivocal yes. As the force due to friction depends on the normal force, the work done due to the force of friction is influenced by the constraint response in the normal direction. The implication of this fact has far reaching consequences. Allowing the constraint response to influence the work allows the material moduli that appear in what is usually identified as the extra stress to depend on the constraint response. Thus, in an incompressible fluid, it is possible that the viscosity depends on the pressure. The usual procedure of splitting the Cauchy stress into a constraint stress that does no work and does not depend on the state variable and a constitutively determinate part that does not depend on the constraint response (see Truesdell [48]) is unwarranted in general on two counts. The constraint stress could do work and the part that is usually thought of as being constitutively determined is not determined only in terms of the state variables, it can depend on the constraint response.

The discussions in this paper are restricted to isothermal processes but can be easily extended to a fully thermodynamical framework. In marked contrast to much of what is done in classical continuum mechanics we allow the possibility that the internal energy and the rate of dissipation are given implicitly, thereby providing implicit constitutive relations for the stress. Also, the forms assumed for the rate of dissipation allow us to incorporate the rate of dissipation corresponding to Coloumb friction and classical viscous dissipation.

Topologically, the body B is a three dimensional differentiable manifold in $\mathbb{R}^3$. We understand by a placer Φ a one to one mapping that takes elements (particles) belonging to B to a three-dimensional Euclidean space. We call $\Phi(B)$ the configuration

[3] Of course by its very definition, a force normal to the planar surface on which motion is taking place would do no work.

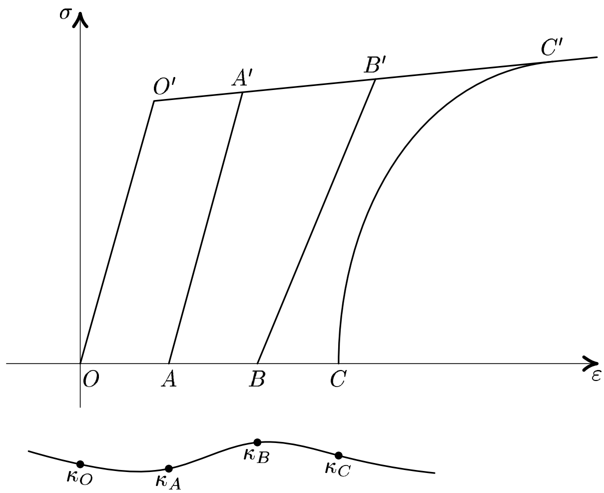

Figure 1. A schematic sketch showing the notion of natural configurations.

of the body B. We would however like to define a local notion at a point $X \in B$, namely that of a local configuration which is essentially an equivalence class of configurations $\kappa(B)$, $\hat{\kappa}(B)$ etc. Let $N(X)$ denote a neighborhood of a particle $X \in B$ and let κ be a smooth homeomorphism of $N(X)$ into the translational space V of the three dimensional Euclidean space E such that X is mapped into the null vector, i.e., $\kappa(X) = \mathbf{0}$. We say that the homeomorphism κ is equivalent to the homeomorphism $\hat{\kappa}$ if and only if $\kappa(Y) = \hat{\kappa}(Y)$ for all $Y \in N(X)$, for some neighborhood $N(X)$. The equivalence class of homeomorphisms to which κ belongs is called the local placer of X. We call the corresponding equivalence class $\{\ldots, \kappa(N(X)), \hat{\kappa}(N(X)), \ldots\}$ a local configuration at X. We refer the reader to Noll [22] for a detailed discussion of the notions of deformations and local configurations. Here we shall consider only bodies subject to homogeneous deformations. When bodies are subject to inhomogeneous deformations, it is necessary to involve the notion of local configurations of the body, wherein in a sufficiently small neighborhood $N(X)$ of X, the deformation can be "approximated" as a homogeneous deformation. We shall not get into a detailed discussion of these issues here.

Many processes involving dissipative materials can be viewed as non-dissipative responses from an evolving class of natural configurations, the evolution of the natural configuration being accompanied by dissipation (in general entropy production). For instance, classical plasticity that has been developed to describe the response of metals concerns a class of elastic response functions from an evolving set of natural configurations (see Fig. 1). Getting from any point (state) on the curve OO' to any of the curves AA', BB' or CC' would involve the production of entropy. It is possible to mathematically construct theories in which it is possible to produce entropy without the natural configuration changing (an example is provided by the response depicted in Fig. 2 wherein the reference configuration κ_0 does not change and a variety of elastic responses is possible from that natural configuration. This is a pathological possibility that will not be permitted within the scope of the framework that is being described. While the response depicted in Fig. 2 resembles the

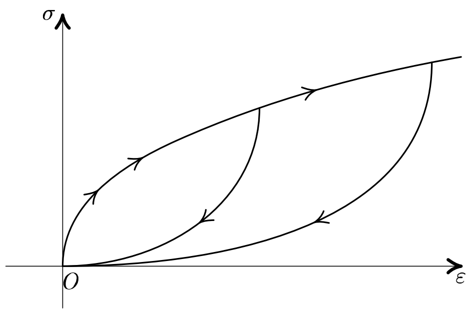

Figure 2. Dissipative process without a change in the natural configuration.

response of some geological materials, it is not an accurate depiction. The actual elastic responses of a real material will not all pass exactly through the same point O. As long as the natural configuration changes, i.e., as long as all the responses do not pass exactly through O, the framework could apply, however close the other natural configurations might be corresponding to points close to O. Natural configurations are exactly what they mean, a minimal set of configurations (possibly more than one) from which measurements need to be made to describe the response of the body. There is usually an infinite set of configurations which could serve this purpose. However, some of them are "natural" in that the expression is simplified. Each member of this minimal set belongs to an equivalence class of configurations, their equivalence being defined through the non-production of entropy when going from one configuration to another. For example, in an elastic material any configuration, deformed or otherwise, can be used to describe the stress response. We however find the choice of the undeformed configuration as the reference configuration to be most amenable to manipulations. Among the many configurations it is a "natural" choice. However, in many materials it is necessary to know kinematical measures from a variety of configurations in order to describe the stress. Such bodies usually have multiple stress free configurations (multiple natural configurations). Of course, any rigid body motion of a stress free configuration leads to another stress free configuration. Here, it is understood that configurations are to be understood modulo rigid body motion. The fact that certain materials possess multiple natural configurations also allows one to account for symmetry changes that are noticed in such materials, which cannot be accomplished within the framework of the theory of simple materials proposed by Noll in [23]. While the materials with multiple natural configurations can be cast into Noll's theory of 1972 (and to do this one has to inject a great deal of physical concepts and thermodynamical assumptions that Noll's theory is oblivious to, Noll [23]), the general framework proposed here that appeals to an implicit relationship between the relevant physical quantities cannot be cast into Noll's framework.

Deformation of traditional polycrystalline plasticity, twinning, solid to solid phase transitions, viscoelastic materials capable of instantaneous elastic response, crystallization in polymers, superplasticity, solidification and melting, response of multinetwork polymers, and flows of anisotropic liquids can all be captured within the framework of bodies with an evolving set of natural configurations. Classical elasticity is a degenerate special case which has one natural configuration and the classical linearly viscous fluid is another case in which every current configuration is a natural configuration. All the above mentioned responses of materials have been discussed within a thermodynamic framework wherein natural configurations are allowed to evolve: multinetwork polymers (Rajagopal and Wineman [27], Wineman and Rajagopal [52]), traditional plasticity (Rajagopal and Srinivasa [32], [33]), twinning (Rajagopal and Srinivasa [29], [30]), crystallization of polymers (Rao and Rajagopal [38], [40], [41]), solidification of atactic polymer melts into amorphous solids (Kannan et al. [14], Kannan and Rajagopal [13]), viscoelasticity (Rajagopal and Srinivasa [35], Murali Krishnan and Rajagopal [20]), anisotropic liquids (Rajagopal and Srinivasa [36]), shape memory alloys (Rajagopal and Srinivasa [35]), and growth and adaptation (Rao et al. [39]).

Implicit constitutive theories within a purely mechanical format have been studied earlier by Morgan [19], Bernstein [3] and Rajagopal and Wineman [26], among others. However, these early studies do not recognize the important role thermodynamics has to play in the response of materials, nor do they provide a systematic framework that allows one to incorporate issues concerning constraints and material symmetry. All these studies are adhoc special examples that have not recognized the important underpinnings of implicit constitutive theories. Rajagopal and Wineman [26] use branching theory to allow for the distinct response of materials. While the branching of response is determined by a selection criterion which could be thermodynamic in its origins, they do not develop a systematic and comprehensive thermodynamic framework. A general thermodynamic framework for implicit constitutive theories is not yet available, but before such a theory can be developed it is essential to get an understanding of the types of implicit models that are in place. In this paper we discuss a variety of issues that have to be recognized if one is to develop a coherent thermodynamic basis for such implicit models.

2. Kinematics

Let us start with our usual identification of a body with a set of particles that has a topological and measure theoretic structure. Let B denote this body and let κ be a reference placer that maps the abstract body onto its configuration $\kappa(B)$ in a three dimensional Euclidean space. Let $\kappa_t(B)$ denote the configuration at time t of the body B. Then, by the motion χ of the body, we mean a one to one mapping at each instant of time t, that associates a particle $\boldsymbol{X}_\kappa \in \kappa(B)$ with a particle $\boldsymbol{x} \in \kappa_t(B)$, i.e.,

$$\boldsymbol{x} = \chi_\kappa(\boldsymbol{X}_\kappa, t). \tag{1}$$

Properties associated with the body can be defined relying either on the reference configuration or on the current configuration (or for that matter any other possible configuration that the body can be placed in), i.e., a property φ can be defined through

$$\varphi = \varphi_\kappa(\boldsymbol{X}_\kappa, t) = \varphi_{\kappa_t}(\boldsymbol{x}, t). \tag{2}$$

We shall use the following notation:

$$\nabla\varphi = \frac{\partial \varphi_\kappa}{\partial \boldsymbol{X}_\kappa}, \quad \operatorname{grad}\varphi = \frac{\partial \varphi_{\kappa_t}}{\partial \boldsymbol{x}}, \quad \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial \varphi_\kappa}{\partial t}, \quad \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi_{\kappa_t}}{\partial t}. \tag{3}$$

The gradient of the motion (usually called the deformation gradient) is defined through

$$\mathbf{F}_\kappa = \frac{\partial \chi_\kappa}{\partial \boldsymbol{X}_\kappa}. \tag{4}$$

The velocity $\boldsymbol{v}$ and the velocity gradient $\boldsymbol{L}$ are defined through

$$\boldsymbol{v} = \frac{\partial \chi_\kappa}{\partial t}, \tag{5}$$

$$\boldsymbol{L} = \operatorname{grad}\boldsymbol{v}. \tag{6}$$

We define the Cauchy-Green stretch tensors $\boldsymbol{B}_\kappa$ and $\boldsymbol{C}_\kappa$ through

$$\boldsymbol{B}_\kappa = \boldsymbol{F}_\kappa \boldsymbol{F}_\kappa^T, \quad \boldsymbol{C}_\kappa = \boldsymbol{F}_\kappa^T \boldsymbol{F}_\kappa, \tag{7}$$

and the Green-St. Venant strain $\boldsymbol{E}_\kappa$ and Almansi-Hamel strain $\boldsymbol{e}_\kappa$ through

$$\boldsymbol{E}_\kappa = \frac{1}{2}(\boldsymbol{C}_\kappa - \boldsymbol{1}), \quad \boldsymbol{e}_\kappa = \frac{1}{2}(\boldsymbol{1} - \boldsymbol{B}_\kappa^{-1}). \tag{8}$$

It immediately follows that

$$\boldsymbol{L} = \dot{\boldsymbol{F}}_\kappa \boldsymbol{F}_\kappa^{-1}. \tag{9}$$

The symmetric part of the velocity gradient $\boldsymbol{D}$ is defined through[4]

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T). \tag{10}$$

The minimal kinematical definitions provided above suffice for our discussions. A detailed treatment of the kinematics of continua can be found in Truesdell [48] and Truesdell and Noll [50]. Let us consider a purely homogeneous deformation[5] of the body B. In order to fully describe the material, we have to describe the various methods of storing energy, various methods of producing entropy (mixing, conduction, conversion of mechanical work into energy in the thermal form (hereinafter referred to as heat), phase transition, growth, etc.). Here, as we shall be interested in purely isothermal problems we shall restrict ourselves to a specification of the stored energy and the rate of dissipation of the body (by dissipation we mean the process of converting mechanical work into energy in its thermal form).

3. Thermodynamical considerations

Many of the interesting applications for implicit theories are to be found within a fully thermodynamic framework with the temperature playing a critical role in the switching that takes place between the various distinct constitutive relations allowed by the implicit theory (see Rajagopal and Srinivasa [34], Rao and Rajagopal [40], [41], Murali Krishnan and Rajagopal [20]). Here we shall discuss the theory within a partial thermodynamic setting in the sense that we introduce the notions of the energy storing mechanism in the body and the rate of dissipation of the body, but we restrict our discussion to isothermal processes.

When energy is supplied to a body, part of it is stored in the form of kinetic energy, part as potential energy and part as stored energy, etc. Sometimes it is necessary to distinguish between the various types of energy that all go by the nomenclature stored energy: strain energy that can be recovered in a purely mechanical process, stored energy that can be only recovered in a thermomechanical process such as

[4] It would be more appropriate to refer to $\boldsymbol{L}$ and $\boldsymbol{D}$ as $\boldsymbol{L}_{\kappa_t}$ and $\boldsymbol{D}_{\kappa_t}$, but we shall drop the suffix κ_t for measurements made with reference to the current configuration κ_t.

[5] A deformation is said to be homogeneous if straight material filaments remain straight. This implies that the deformation gradient, in a Cartesian co-ordinate system, has constant entries in its matrix representation.

annealing wherein the energy is stored in the dislocation patterns, the stored energy being coaxed out due to heating, etc. Part of the energy that is supplied is dissipated. Part could be used to provide the latent energy required for a change of phase, while part could go towards the latent heat, etc.[6]

Most bodies can exist in more than one stress free configuration, the configurations not being related to one another by a rigid body motion. While the "natural configuration" of a body might not necessarily be associated with a stress free configuration, for our present purpose it suffices to make such an association. In fact, as we saw earlier the "natural configuration" of a body is not a configuration but an equivalence class of configurations. We shall not get into a discussion of these issues here (see Rajagopal [28] for a detailed discussion of the role of natural configurations in mechanics).

It is important to recognize that with a stressed state $\kappa_t(B)$ of the body B at time t, one could associate many natural configurations modulo rigid motion (i.e., the stress free configuration that the body would relax to, depends on what is done with the external stimuli). Thus, which natural configuration the body would tend to depends on the class of processes that the body is allowed to undergo. For instance, in the case of a viscoelastic body that is capable of instantaneous elastic response, the body could tend to a stress free state through an instantaneous elastic response, in an adiabatic process (the fact that the process is instantaneous does not allow any time for the transfer of energy in the thermal form). Let us suppose that this natural configuration is $\kappa^{\mathrm{ad}}_{p(t)}(B)$. On the other hand, the same viscoelastic material could be allowed to stress relax, ever so slowly, in an isothermal process and tend to a different natural configuration $\kappa^{\mathrm{is}}_{p(t)}(B)$. Which of these natural configurations comes into play in describing the response of bodies as they undergo complex processes is dictated by the class of the process under consideration.

Let $\kappa_{p(t)}$ denote the natural configuration corresponding to the configuration κ_t (see Fig. 3). Let $\boldsymbol{F}_{\kappa_{p(t)}}$ denote the deformation gradient with reference to the natural configuration, i.e.,

$$\boldsymbol{F}_{\kappa_{p(t)}} = \frac{\partial \chi_{\kappa_{p(t)}}(\boldsymbol{X}_{\kappa_{p(t)}}, t)}{\partial \boldsymbol{X}_{\kappa_{p(t)}}}. \tag{11}$$

Let us introduce a linear transformation $\boldsymbol{G}$ through

$$\boldsymbol{G} := \boldsymbol{F}_{\kappa\to\kappa_{p(t)}} = \boldsymbol{F}^{-1}_{\kappa_{p(t)}}\boldsymbol{F}_{\kappa}, \tag{12}$$

[6] Latent energy is different from latent heat and this distinction is important in a fully thermomechanical process such as phase transition (see Rajagopal and Srinivasa [35]).

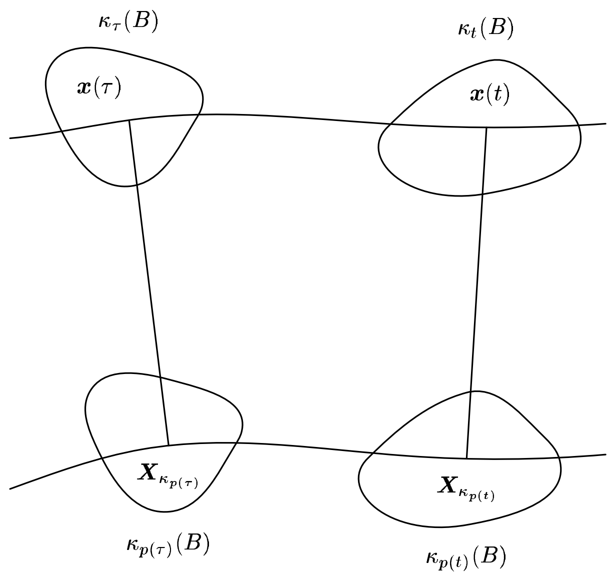

Figure 3. Natural configurations.

and

$$\boldsymbol{L}_{\kappa_{p(t)}} := \dot{\boldsymbol{G}}\boldsymbol{G}^{-1}. \tag{13}$$

In general, $\boldsymbol{F}_{\kappa_{p(t)}}$ and $\boldsymbol{G}$ will not be the gradients of a mapping. However, if the deformation is homogeneous then they will become the gradient of a mapping (see Rajagopal [28], Rajagopal and Srinivasa [32], [33]).

In general, the stored energy of the material can be a function of the stress and possibly various time derivatives and spatial gradients of stress as well as of kinematical quantities such as the deformation gradient and various time derivatives and spatial gradients of the deformation gradient. Such assumptions, while leading to models within the framework that is proposed, are quite useless as such higher gradients generally lead to theories that require boundary conditions in addition to the usual boundary conditions in classical theories. In the absence of a sensible means for generating or prescribing boundary conditions such theories are merely recondite studies. It is possible to cast such problems within a variational approach which can provide "natural" boundary conditions, but these conditions may not, and usually do not, have clear physical meaning.

We have to recognize that the mechanism for the storage of energy as well as the rate of dissipation can change with the configuration, that is, the stored energy and the rate of dissipation can depend on $\boldsymbol{G}$ and various time and spatial derivatives of $\boldsymbol{G}$ (see Rajagopal and Srinivasa [30], [32], [33], [35], Rao and Rajagopal [40], [41]).

In the case of the response of single crystals with dislocations, energy can be trapped in dislocation patterns and there is also a rate of dissipation associated with the movement of dislocations. In order to incorporate these effects it is necessary to include the dependence of the rate of dissipation on the dislocation density curl $\boldsymbol{G}$ (or grad $\boldsymbol{G}$) which can once again lead to problems concerning the specification of boundary conditions.

Here, I am merely interested in illustrating the richness of implicit theories and thus I shall restrict myself to a very special class of implicit models. It might be perfectly legitimate to raise an objection that the generality accorded by equations (14)–(16) below is quite unnecessary as simpler models seem to suffice when dealing with the mechanics of most bodies.

In general, in an implicit theory, the Helmholtz potential can depend on the stress as well as on the kinematical quantities. In fact, in implicit theories it is possible that we could have the Helmholtz potential, the stress, the kinematical quantities, etc., satisfy an equation of the form

$$f\big(\psi_{\kappa_{p(t)}}, \boldsymbol{F}_{\kappa_{p(t)}}, \boldsymbol{G}, \dot{\boldsymbol{F}}_{\kappa_{p(t)}}, \dot{\boldsymbol{G}}\big) = 0 \tag{14}$$

with more than one possible solution for the Helmholtz potential, i.e.,

$$\psi^i = \psi^i_{\kappa_{p(t)}}\big(\boldsymbol{F}_{\kappa_{p(t)}}, \boldsymbol{G}, \dot{\boldsymbol{F}}_{\kappa_{p(t)}}, \dot{\boldsymbol{G}}\big), \quad i = 1, \ldots, N. \tag{15}$$

Similarly, the rate of dissipation could be related to the stress, its rates and various kinematical quantities through

$$g\big(\xi_{\kappa_{p(t)}}, \boldsymbol{T}, \dot{\boldsymbol{T}}, \boldsymbol{F}_{\kappa_{p(t)}}, \boldsymbol{G}, \dot{\boldsymbol{F}}_{\kappa_p(t)}, \dot{\boldsymbol{G}}, \ldots\big) = 0 \tag{16}$$

leading to more than one structure for the rate of dissipation, i.e.,

$$\xi^i = \xi^i_{\kappa_{p(t)}}\big(\boldsymbol{T}, \dot{\boldsymbol{T}}, \boldsymbol{F}_{\kappa_{p(t)}}, \boldsymbol{G}, \dot{\boldsymbol{F}}_{\kappa_{p(t)}} \dot{\boldsymbol{G}}, \ldots\big). \tag{17}$$

Thus, we could have a process during which the form for the stored energy and the rate of dissipation change. Such a change is caused by structural changes in the material, and we would have to prescribe when such structural changes take place. This change in structure would be determined by the deformation, thermal and other histories that are relevant, and when a certain condition is reached, the old forms for the stored energy, rate of dissipation etc., cease to be valid and new forms come into play. This transformation could be discrete or continuous, that is, we might have just a finite number of functions for the stored energy, rate of dissipation, etc., or we could have a one parameter family of them, that is, the structure is continually modified as the process takes place.

As an example, let us consider traditional plasticity. Let us suppose a body, in its virgin stress free as well as strain free state, is deformed. Its initial response, up to some value for the strain or stress would be elastic, by which we mean there is no dissipation and all the energy stored in the body can be recovered in a purely mechanical process. When the "yield criterion" is met (this is the criterion for a change in the material's response), on continued loading the material starts to dissipate energy. Now, it could happen that on unloading to the natural state, the material behaves elastically in a manner similar to its response from the virgin state. In this case, the elastic response from the current configurations would, in the one-dimensional response shown in Fig. 1, be parallel. On the other hand, the elastic response could change as the process progresses and this would correspond to the lines with different slopes as shown in Fig. 1.

Thus, we could introduce a yield condition (activation criterion) that delineates different regimes of material response. This activation criterion could depend on a variety of variables, the stress, the (total) strain, the mapping $\boldsymbol{G}_{\kappa_{p(t)}}$, and possibly even on the energy stored and the way in which the material dissipates, that is, activation could be determined by a function (or a functional if need be) of the form

$$\mathrm{h}(\psi_{\kappa_{p(t)}}, \xi_{\kappa_{p(t)}}, \boldsymbol{T}, \boldsymbol{F}_{\kappa_{p(t)}}, \boldsymbol{G}, \theta) = 0 \tag{18}$$

Such a complicated structure for activation is hardly ever used and it usually takes a much simpler structure in, for instance, plasticity.

In isothermal processes, the manner in which the natural configurations evolve is determined by the rate of dissipation. During elastic response, there is no dissipation and the natural configuration does not evolve. (The converse is not true. It is possible that the natural configuration does not evolve though the process is dissipative (see Fig. 2). We shall not consider such possibilities.) We shall suppose that the evolution of the natural configuration is determined by maximizing the rate of dissipation (this criterion has been used with much success in modeling diverse responses of characteristics (see Rajagopal and Srinivasa [29]). There have been other studies in which the maximization of dissipation has been advocated (see Ziegler [53]) but this was not proposed for determining the evolution of the natural configuration, and in fact the analysis as carried out in Ziegler [53] is incorrect (see Rajagopal and Srinivasa [32], [33] for a discussion of the relevant issues).

It is customary in continuum thermodynamics to use the second law of thermodynamics to obtain restrictions on the allowable constitutive forms. This procedure involves allowing the body to be subject to *"arbitrary"* processes. We shall however depart from such a procedure. It is highly unlikely that a body could be subject to "arbitrary" processes and yet retain the constitutive structure assumed for it. For instance, a body that is idealized, say as an elastic body, could hardly be subject to arbitrary deformations as the model would cease to be valid for sufficiently large deformations, or if the deformation is very rapid (for instance rubber which can be thought of as an elastic body crystallizes under very rapid deformations). Of course, we can ask what happens to the idealized model when subject to arbitrarily large deformations but it has little relevance to the real body in question. It would be more reasonable to recognize that the structure of the body may change and assume forms for the constitutive relations so that the second law is automatically met. That is, we could choose a constitutive model that is "sufficient" to ensure that the second law is met. It is possible that other constitutive relations could also meet the model. However, after all, we need to use our physical insight in developing our constitutive models, and there can be no substitute to this. Another problem with the manner in which the second law (usually in the form of the Clausius-Duhem inequality) is employed concerns the absence of the entropy production due to the radiant heating in the reduced dissipation inequality. While the radiation is not subject to any restrictions, it is required to be whatever it takes to meet the balance of energy. While ignoring the radiation completely might be reasonable for certain processes, it is completely untenable for many other processes (a case in point is problems involving microwave radiation or any problem concerning electromagnetic radiation, see Rajagopal and Tao [37]). There are yet other difficulties associated with the use of the Clausius-Duhem inequality (or for that matter any form of the second law) such as, whether it ought to be enforced locally or whether it ought only to be enforced globally, etc., but we shall not get into a discussion of such issues here.

In an isothermal process within a purely mechanical context the rate of dissipation ξ is given by

$$\xi = \boldsymbol{T} \cdot \boldsymbol{D} - \varrho\dot{\psi}. \tag{19}$$

$\boldsymbol{T} \cdot \boldsymbol{D}$ is usually referred to as stress power. We shall assume constitutive relations for ξ so that

$$\xi \geqslant 0. \tag{20}$$

This automatically ensures that the second law is met.

Here, I shall merely present a few types of implicit models that can be used to describe the behavior of fluids and solids.

4.1. Incompressible fluids with pressure dependent viscosity.

We shall start with an implicit model that has relevance to many important technological applications. The classical Navier-Stokes model is unmatched in the interest it has evoked amongst mathematicians, physicists and engineers, and forms the backbone of the body of fluid mechanics. The classical incompressible Navier-Stokes model is characterized by the following relationship between the stress and the symmetric part of the velocity gradient:

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu\boldsymbol{D}, \tag{21}$$

where p denotes the indeterminate part of the stress due to the constraint of incompressibility, and μ denotes the viscosity that is usually assumed to be a constant. It is well known that the viscosity μ can depend on the temperature θ.

If one starts with the assumption that the stress in the fluid depends on the density, temperature and the velocity gradient, then it immediately follows from the assumptions of frame-indifference and isotropy that the stress $\boldsymbol{T}$ in such a fluid is given by

$$\begin{aligned}\boldsymbol{T} = {}& \alpha_0(\varrho, \theta, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\mathbf{1} + \alpha_1(\varrho, \theta, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D} \\ & + \alpha_2(\varrho, \theta, \mathrm{I}_{\boldsymbol{D}}, \mathrm{II}_{\boldsymbol{D}}, \mathrm{III}_{\boldsymbol{D}})\boldsymbol{D}^2,\end{aligned} \tag{22}$$

where $\mathrm{I}_{\boldsymbol{D}} = \operatorname{tr}\boldsymbol{D}$, $\mathrm{II}_{\boldsymbol{D}} = \frac{1}{2}[(\operatorname{tr}\boldsymbol{D})^2 - \operatorname{tr}\boldsymbol{D}^2]$ and $\mathrm{III}_{\boldsymbol{D}} = \det\boldsymbol{D}$.

If one further requires that the stress be linear in $\boldsymbol{D}$, then one obtains

$$\boldsymbol{T} = \hat{\alpha}_0(\varrho, \theta)\mathbf{1} + \hat{\alpha}_1(\varrho, \theta)(\operatorname{tr}\boldsymbol{D})\mathbf{1} + \hat{\alpha}_2(\varrho, \theta)\boldsymbol{D}. \tag{23}$$

We shall adopt the usual convention that

$$\hat{\alpha}_0(\varrho, \theta) = -p(\varrho, \theta), \quad \hat{\alpha}_1(\varrho, \theta) = \lambda(\varrho, \theta) \quad \alpha_2(\varrho, \theta) = 2\mu(\varrho, \theta), \tag{24}$$

and thus

$$\boldsymbol{T} = -p(\varrho, \theta)\mathbf{1} + [\lambda(\varrho, \theta)](\operatorname{tr}\boldsymbol{D})\mathbf{1} + 2[\mu(\varrho, \theta)]\boldsymbol{D}. \tag{25}$$

A rather interesting consequence follows when asking what happens to the representation (25) if we require that the fluid be incompressible. If we merely require that $\varrho = \text{const}$ in an incompressible material, and that motions are restricted so that $\operatorname{tr}\boldsymbol{D} = 0$, then it follows from (25) (by suppressing the dependence on the temperature θ) that the stress has the representation

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu\boldsymbol{D}. \tag{26}$$

To obtain (26) from (25), we merely follow the usual procedure that the stress can be split into a constraint stress and a constitutively determined part, the constraint stress doing no work.

On the other hand, suppose the relation for the thermodynamic pressure $p = p(\varrho, \theta)$ is invertible in the sense that $\varrho = \varrho(p, \theta)$. Then we can rewrite (25) in the form

$$\boldsymbol{T} = -p\mathbf{1} + \hat{\lambda}(p, \theta)(\operatorname{tr}\boldsymbol{D})\mathbf{1} + 2\hat{\mu}(p, \theta)\boldsymbol{D}. \tag{25$'$}$$

Now, requiring that the fluid be incompressible leads to (by suppressing the temperature θ)

$$\boldsymbol{T} = -p\mathbf{1} + 2\hat{\mu}(p)\boldsymbol{D}, \tag{26$'$}$$

where I have used the fact that the incompressibility requires that $\operatorname{tr}\boldsymbol{D} = 0$ and that the pressure be indeterminate. The model for the incompressible Navier-Stokes fluid is obtained as a limit of the compressible Navier-Stokes fluid by expanding the pressure in a Taylor series in terms of the density about the desired constant value, ignoring all but the first two terms of the expansion, and allowing the bulk modulus (the derivative of the pressure with respect to the density) to go to infinity in such a manner that the product of the bulk modulus and the difference between the value of density and its limiting value is finite (this is a standard procedure and we shall not get into the details here). Thus, in general, the standard procedure of splitting the stress into a constraint stress and a constitutively determined part that does not depend on the constraint response, and the limiting procedure do not lead to the same model.

As remarked earlier, Stokes [46] recognized that the viscosity in a fluid can depend on the pressure. In a compressible fluid wherein one has an equation of state for the pressure, the model (25)$'$ yet retains its character as an explicit relationship between the stress $\boldsymbol{T}$, the density ϱ and the kinematic tensor $\boldsymbol{D}$. However, in the case of an incompressible fluid of the class (26)$'$, the pressure denotes the mean normal stress and the relationship between $\boldsymbol{T}$ and $\boldsymbol{D}$ takes the form

$$\boldsymbol{T} = +\Big(\frac{1}{3}\operatorname{tr}\boldsymbol{T}\Big)\mathbf{1} + 2[\mu(\theta, \operatorname{tr}\boldsymbol{T})]\boldsymbol{D}. \tag{27}$$

The model (27) provides an inkling of the type of implicit models that can come into play when we deal with fluids wherein the material moduli that appear in the extra stress depend on the constraint response. That such relationships are truly implicit becomes obvious when we consider a generalization of (27), namely

$$\boldsymbol{T} = -p\mathbf{1} + [\mu(p,\theta,\boldsymbol{D})]\boldsymbol{D}, \tag{28}$$

where $p = -\frac{1}{3}\operatorname{tr}\boldsymbol{T}$.

We should expect the viscosity of a fluid to depend on the pressure, if our understanding of the resistance to the motion between two solids is to be a guide. In the case of the resistance to sliding between two solids, the frictional force that needs to be overcome depends on the normal force at the point of contact between the two sliding surfaces.[7] While this fact does not manifest itself at the pressures one encounters in everyday applications, such as flows in pipes, it becomes significant at high pressures such as those encountered in elastohydrodynamics and other applications. It would be natural to expect the frictional resistance between adjacent layers at the bottom of the Pacific Ocean to be much greater than that close to the surface.

There has been a considerable amount of work on the dependence of viscosity on the pressure and the literature prior to 1931 can be found in the magisterial treatise by Bridgman [5]. Andrade [2] had suggested the following relationship between the viscosity, the pressure, density and temperature:

$$\mu(p,\varrho,\theta) = A\varrho^{1/2}\exp\Big(p + \varrho^2 r)\frac{s}{\theta}\Big),$$

where θ denotes the temperature and r, s and A are constants. In many liquids while the changes in the density are negligible over a wide range of pressure, the changes in the viscosity could be several orders of magnitude, say even a factor of 10^8! Thus, such liquids could yet be considered incompressible with a viscosity that is pressure dependent.

It is quite straightforward, within the context of implicit constitutive theories, to have a fluid whose material moduli depend on the constraint stress. The model of a fluid with a pressure dependent viscosity can be shown to be a natural consequence of requiring that the Helmholtz potential depend only on the temperature, i.e., $\psi = \psi(\theta)$, while the rate of dissipation depends on both the stress $\boldsymbol{T}$ and the symmetric

[7] This result traces its origin to the work of Amontons [1] and was studied in detail by Coulomb [6]. In fact, Coulomb erroneously concluded that one method of distinguishing between a solid and a fluid is whether the friction between surfaces depends on the normal force. His conclusion was based on not allowing for high enough pressures in his experiments on fluids.

part of the velocity gradient $\boldsymbol{D}$, and that the fluid is incompressible. In fact, assuming the rate of dissipation to have the special form

$$\xi = 2[\mu(\theta, \operatorname{tr} \boldsymbol{T})]\boldsymbol{D} \cdot \boldsymbol{D} \tag{29}$$

and maximizing the rate of dissipation subject to (19) as a constraint as well as the constraint of incompressibility leads to the representation (26).

A further generalization of the rate of dissipation to be

$$\xi = 2[\mu(\theta, \operatorname{tr} \boldsymbol{T}, \mathrm{II}_{\boldsymbol{D}})]\boldsymbol{D} \cdot \boldsymbol{D} \tag{29$'$}$$

and the requirement that the fluid be incompressible would lead to models of the form

$$\boldsymbol{T} = -p\mathbf{1} + 2\mu(\theta, p, \mathrm{II}_{\boldsymbol{D}})\boldsymbol{D}. \tag{30}$$

Recently, Hron, Málek and Rajagopal [12] studied flows of a fluid modeled by (30), and they found that the solutions corresponding to such a model are markedly different than the classical Navier-Stokes model. It is not our aim here to discuss the model (30) in any detail; the interested reader can find a detailed treatment of such models in Hron, Málek and Rajagopal [12]. Existence of solutions to such models has been proved for the space periodic case by Málek, Nečas and Rajagopal [17] and for the Dirichlet boundary conditions by Franta, Málek and Rajagopal [9].

Next, I provide a formal computation concerning a fluid model within an implicit constitutive framework from a purely mechanical perspective. Let us consider the class of implicit models

$$\boldsymbol{f}(\boldsymbol{T}, \boldsymbol{D}) = \mathbf{0}. \tag{31}$$

The assumption that the body is isotropic leads to (see Spencer [44])

$$\begin{aligned} &\alpha_0\mathbf{1} + \alpha_1\boldsymbol{T} + \alpha_2\boldsymbol{D} + \alpha_3\boldsymbol{T}^2 + \alpha_4\boldsymbol{D}^2 + \alpha_5(\boldsymbol{TD} + \boldsymbol{DT}) + \alpha_6(\boldsymbol{T}^2\boldsymbol{D} + \boldsymbol{DT}^2) \\ &\quad + \alpha_7(\boldsymbol{TD}^2 + \boldsymbol{D}^2\boldsymbol{T}) + \alpha_8(\boldsymbol{T}^2\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{T}^2) = \mathbf{0} \end{aligned} \tag{32}$$

where the α_i, $i = 1, \dots$ depend on the density and the invariants

$$\operatorname{tr} \boldsymbol{T}, \operatorname{tr} \boldsymbol{D}, \operatorname{tr} \boldsymbol{T}^2, \operatorname{tr} \boldsymbol{D}^2, \operatorname{tr} \boldsymbol{T}^3, \operatorname{tr} \boldsymbol{D}^3, \operatorname{tr} \boldsymbol{TD}), \operatorname{tr}(\boldsymbol{T}^2\boldsymbol{D}), \operatorname{tr} \boldsymbol{D}^2\boldsymbol{T}), \operatorname{tr}(\boldsymbol{D}^2\boldsymbol{T}^2). \tag{33}$$

We notice that

$$\boldsymbol{T} + \beta(\varrho, \operatorname{tr} \boldsymbol{T})\boldsymbol{D} = \mathbf{0} \tag{34}$$

is a special subclass of fluids of the form (32), with β being an arbitrary function of the density and $\operatorname{tr} \boldsymbol{T}$. The stress in the incompressible counterpart of (34) is given by

$$\boldsymbol{T} = -p\mathbf{1} + \lfloor \hat{\beta}(\operatorname{tr} \boldsymbol{T}) \rfloor \boldsymbol{D}, \tag{35}$$

i.e., a model in which the viscosity is pressure dependent. The above development is purely formal but it can be shown that (35) can be developed in a systematic manner.

4.2. Non-hyperelastic materials that do not dissipate.

Next, I shall consider a rather intriguing model within the context of implicit theories that stems from generalizing (15) so that the Helmholtz potential also depends on the stress. This generalization makes it possible to give meaning to the notion of elastic bodies in which the stress is not derivable from a potential within a thermodynamic framework.

Truesdell [49] introduced the notion of hypo-elasticity as a possible model for the non-linear behavior of solids that reduces appropriately to the classical approximation of linearized elasticity. Bernstein [4] carefully reassessed the work of Truesdell and postulated certain additional requirements to render the theory meaningful. We shall not discuss these issues here as we are interested in providing another explanation for what constitutes an elastic response in which the stress is not derived from a potential.

According to Truesdell, the stress in a hypo-elastic material is given by

$$\dot{\boldsymbol{T}} = \hat{\boldsymbol{g}}(\boldsymbol{T}, \boldsymbol{L}) \tag{36}$$

or more precisely (see Truesdell [49]),

$$\dot{\boldsymbol{T}} = \boldsymbol{T}\boldsymbol{W}^T + \boldsymbol{W}\boldsymbol{T} + [\boldsymbol{A}(\boldsymbol{T})]\boldsymbol{D}, \tag{37}$$

where $\boldsymbol{A}$ is a fourth order tensor that depends on $\boldsymbol{T}$, i.e.,

$$\dot{T}_{ij} = T_{ik}W_{jk} + T_{jk}W_{jk} + A_{ijkl}(\boldsymbol{T})D_{kl}. \tag{38}$$

First, note that in the above representation a time derivative of $\boldsymbol{T}$ appears and such a representation stands for a class of simple materials (Noll [22]) rather than a single simple material (see Truesdell and Noll [50]). Bernstein [4] recognized this quite clearly as his following remarks make evident: “By a hypo-elastic material we shall mean the assignment of a set of equations (3.1) or (3.5) and a corresponding equivalence class of stress-configuration classes. By a representation of a hypo-elastic material we shall mean an assignment of a set of equations (3.1) or (3.5) and a corresponding stress-configuration class." In the above remarks (3.1) stands for (38).

The model (37) has been advanced by many as a possible representation for the inelastic response exhibited by some bodies. I do not share this point of view. The formulation as stated lacks a proper thermodynamic basis and the concept of dissipation that is the hallmark of inelastic response is totally obscured if in fact there is any. Also, in a model such as (37) it is not evident how the body stores energy. Here I would like to consider an implicit theory that ensures that the body does not dissipate during any process and in which the stored energy is not a function of just the deformation gradient. Moreover, for this body the stress cannot be expressed in terms of the derivative of the stored energy. Thus, the material is not hyperelastic and I shall refer to this as implicit non-hyperelastic response.

The key difference between elasticity and implicit non-hyperelasticity as identified in what follows is that the stored energy depends on both the stress and the deformation gradient. The rate of dissipation in any process that the body undergoes has to be zero and this can be ensured for an appropriate class of stored energy functions as shown below.

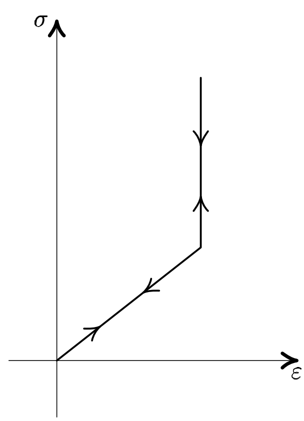

Figure 4. Non-dissipative response wherein the stress cannot be expressed explicitly as a function of the strain.

At first glance it might seem more than strange that we would have a response where the stored energy depends on both the stress $\boldsymbol{T}$ and the deformation gradient $\boldsymbol{F}$. As we shall consider a body that has a single natural configuration, i.e. the configuration κ, we shall not use the suffix κ to denote the quantities as it will be clear from the context. However, it is easy to see that such situations are possible. Consider a one dimensional response as depicted in Fig. 4. The stored energy ψ for such a material depends on both the stress σ and the strain ε:

$$\psi = \psi(\sigma, \varepsilon) = \begin{cases} \hat{\psi}(\varepsilon) & \forall\, 0 \leqslant \sigma \leqslant \sigma_{\rm cr}, \\ \psi_{\rm cr} = \text{constant} & \forall\, \sigma > \sigma_{\rm cr}. \end{cases} \tag{39}$$

The constitutive relation in this case is given by

$$\varepsilon = \begin{cases} g(\sigma) & \forall\, \sigma \leqslant \sigma_{\rm cr}, \\ \varepsilon_{\rm cr} & \forall\, \sigma > \sigma_{\rm cr}. \end{cases} \tag{40}$$

The implicit relation between σ and ε that captures the response depicted in Fig. 4 is

$$f(\sigma, \varepsilon) = \{(\varepsilon - E\sigma)[H(\sigma) - H(\sigma - \sigma_{\rm cr})]\}\{(\varepsilon - \varepsilon_{\rm cr})H(\sigma - \sigma_{\rm cr})\} = 0 \tag{41}$$

where for ease we have assumed $g(\sigma) = E\sigma$; H denotes the Heaviside function. Mechanical analogs for such a response are depicted in Fig. 5 where we have a combination of an inextensible string and a linear spring, or rigid links and a linear spring. During the deformation of these mechanical analogs there is no dissipation. However, these bodies are not hyperelastic bodies. The stress cannot be expressed as the derivative of the stored energy, or for that matter a function of the strain. We can also encounter such response in a biological material where a teleological critical strain or stress can trigger a biological modification that leads to infinite stiffness. Of course, this is an idealization that is no different from perfect inextensibility or rigidity. However, while there is no dissipation there might be yet entropy production due to the biological adaptation which we ignore within the mechanical context. The point to be borne in mind is that as idealized limits, such responses are possible.

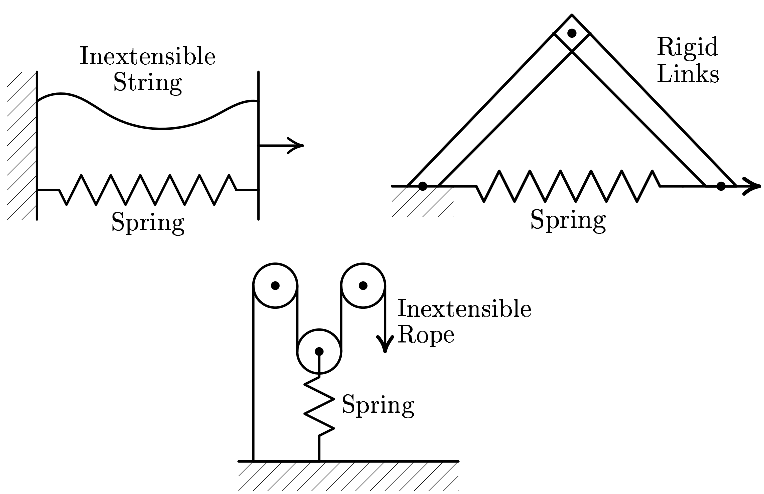

Figure 5. Mechanical analogs for the response depicted in Fig. 4.

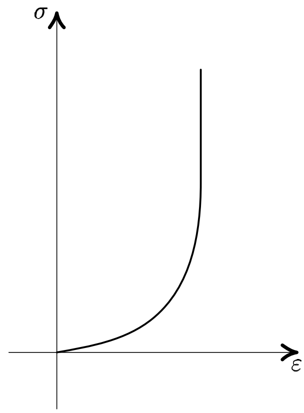

Figure 6. Smooth non-dissipative response wherein the stress cannot be expressed as a function of the strain.

While the response depicted in Fig. 4 is not differentiable at the point $(\varepsilon_{\text{cr}}, \sigma_{\text{cr}})$ it is easy to construct a smooth curve wherein the strain is a constant ε_{cr} and one does not have a one to one relationship between the stress and the strain (see Fig. 6).

In the response depicted in Fig. 4 the body does not dissipate. While the response is reversible, one cannot derive the stress from a potential (stored energy) for the whole range of response. Thus, such a response is not that of an hyperelastic body. We will discuss some other possibilities of a body's response that is non-dissipative, the body's response being characterized by implicit equations.

It is not enough to say that a body is incapable of dissipation in order for it to be called elastic. Its response has to make physical sense. For instance, we can have an implicit relation of the form

$$f(\sigma, \varepsilon) = \sigma^2 + \varepsilon^2 - k^2 = 0, \quad k\text{—constant} \tag{41$'$}$$

for which

$$\xi = \Big(\sigma - \varrho\psi_\varepsilon + \varrho\psi_\sigma \frac{\varepsilon}{\sigma}\Big)\dot{\varepsilon}$$

by virtue of (19) and (41)$'$. Thus, if we have a stored energy such that

$$\sigma - \varrho\psi_\varepsilon + \varrho\frac{\varepsilon}{\sigma}\psi_\sigma = 0,$$

we will have the rate of dissipation $\xi = 0$. However, a response of the form (41)$'$ is not physically meaningful.

On the other hand, we can have a response as that depicted in Fig. 7. The non-monotone part may not be actually realized due to a snap through instability wherein the body proceeds along the dotted part of the curve on reaching a critical strain.

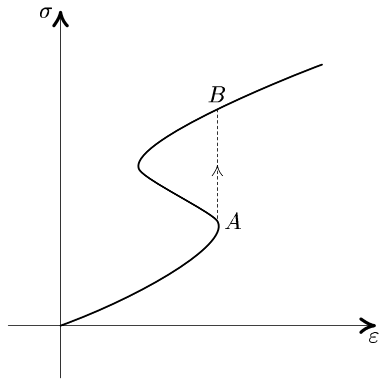

Figure 7. Possible implicit elastic response. Response proceeds vertically from A to B.

Such a response cannot be described by classical hyperelastic models, though it may be possible and physically meaningful.

Let us consider a body whose stored energy and rate of dissipation are given by

$$\psi = \hat{\psi}(\boldsymbol{T}, \boldsymbol{F}), \tag{42}$$

$$\xi = \hat{\xi}(\boldsymbol{T}, \boldsymbol{F}, \boldsymbol{L}). \tag{43}$$

We shall further suppose that[8]

$$\dot{\boldsymbol{T}} = [\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{F})]\boldsymbol{L}. \tag{44}$$

I am not making any claims that there are materials that can be described by (42)–(44). I am merely interested in investigating the consequences of such a model and particularly interested in defining a class of responses that can be described as elastic bodies in the sense that their response is "non-dissipative". Notice that materials defined by (44) are not hypo-elastic in the sense of Truesdell. It then follows from (42)–(44) that

$$\xi = (\boldsymbol{T} - \varrho\psi_{\boldsymbol{F}}\boldsymbol{F}^T - \varrho\boldsymbol{g}^T\psi_{\boldsymbol{T}}) \cdot \boldsymbol{L}. \tag{45}$$

This is an implicit non-hyperelastic body if $(\boldsymbol{T} - \varrho\psi_{\boldsymbol{F}}\boldsymbol{F}^T - \varrho\psi_{\boldsymbol{T}})$ is perpendicular to $\boldsymbol{L}$. If the body is incompressible, we then need

$$\boldsymbol{T} - \varrho\psi_{\boldsymbol{F}}\boldsymbol{F}^T - \varrho\boldsymbol{g}^T\psi_{\boldsymbol{T}} = \Phi\mathbf{1}. \tag{46}$$

[8] We have assumed here that $\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{F}) \in \mathrm{Lin}(V, V)$. I could also consider the possibility that $\boldsymbol{g}(\boldsymbol{T}, \boldsymbol{F}) \in \mathrm{Lin}(\mathrm{Lin}(V, V), \mathrm{Lin}(V, V))$ and carry out a formulation similar to what follows.

Let us consider, for the purpose of illustration, a one-dimensional analysis. Let us suppose that we are dealing with a class of bodies characterized by a Helmholtz potential, and a rate of dissipation that is given as implicit relations of the state of stress σ and strain ε. Furthermore, let us suppose that the stress and strain are implicitly related. That is

$$\psi = \hat{\psi}(\sigma, \varepsilon), \tag{47}$$

$$\xi = \hat{\xi}(\sigma, \varepsilon, \dot{\varepsilon}) \tag{48}$$

The above model includes classical elasticity as a special case

$$\psi = \psi(\varepsilon), \quad \xi = 0. \tag{49}$$

Now, the rate of dissipation needs to meet (19), i.e.,

$$\xi = \sigma\dot{\varepsilon} - \varrho\dot{\psi} \tag{50}$$

$$= \sigma\dot{\varepsilon} - rho\psi_{\varepsilon}\dot{\varepsilon} - \varrho\psi_{\sigma}\dot{\sigma}. \tag{51}$$

Now, the one-dimensional counterpart of (45) is

$$\dot{\sigma} = g(\sigma, \varepsilon)\dot{\varepsilon}. \tag{52}$$

Thus, we find that it is possible that the dissipation is zero provided

$$\sigma - \psi_{\varepsilon} - \varrho g(\sigma, \varepsilon)\psi_{\sigma} = 0. \tag{53}$$

Given a $g(\sigma, \varepsilon)$, we have thus reduced the problem to that of finding the existence of a stored energy that satisfies the above partial differential equation. In general, we cannot integrate (52) (in the three-dimensional case (44)) with appropriate initial conditions for the stress and the appropriate kinematical quantity, and obtain an explicit class of non-dissipative models. For example, if $g(\sigma, \varepsilon)$ were a constant, we would obtain a class of elastic models with different initially strained states.

Instead of the choice (52) that follows from (44) we could assume that the stress and strain are related through an implicit equation of the form

$$f(\sigma, \varepsilon) = 0. \tag{54}$$

Equation (54) implies that

$$\dot{\sigma} = -\frac{f_{\varepsilon}}{f_{\sigma}}\dot{\varepsilon}. \tag{55}$$

Here we are assuming that f_σ is not zero.

The question is whether these are actual implicit relations wherein f_σ is not zero, which leads to sensible models for the non-dissipative response of materials. There are several implicit relations that do lead to describing the non-dissipative response of bodies, wherein f_σ is not zero except at certain critical points (branch points, see Wineman and Rajagopal [52]) and an additional criterion has to be used to switch from one branch of response to another. A simple example is an implicit relationship of the form

$$f(\sigma, \varepsilon) = (\sigma - k_1\varepsilon)(\sigma - k_2\varepsilon + \alpha)(\sigma - k_3\varepsilon + \beta) = 0 \tag{56}$$

where k_1, k_2, α and β and are constants. In this case f_σ is zero at branch points (points of intersection of the straight lines that comprise the response). In the case of such an implicit relation, we need to use a criterion to switch from one type of response to the other. We now get back to a discussion when $f_\sigma \neq 0$.

It follows from (51) and (55) that

$$\xi = \big(\sigma - \varrho\psi_\varepsilon + \varrho\psi_\sigma(f_\varepsilon/f_\sigma)\big)\dot{\varepsilon} \tag{57}$$

and thus if

$$\sigma - \varrho\psi_\varepsilon + \varrho\psi_\sigma(f_\varepsilon/f_\sigma) = 0 \tag{58}$$

then in such a material, $\xi \equiv 0$. Thus, a material in which the stored energy depends on σ, ε in such a manner that (58) is met, will not dissipate.

A variety of models can be constructed that correspond to different stored energies ψ. We shall not get into a discussion of these models here.

It is not that there is no stored energy associated with the body. The material has a stored energy. However, in such a material the stress is not given explicitly in terms of the strain. It is possible that implicit constitutive theories could lead to more than one possible response from a branch point, the selection of the response being determined by some thermodynamic criterion (see Rajagopal and Wineman [26]).

4.3. Viscoelastic fluids.

We now turn our discussion to an implicit model for viscoelastic fluids. Maxwell [18] was the first to recognize that even air could store energy and exhibit dissipative behavior. This led him to develop the model for viscoelastic fluids that bears his name. This study has been followed by several others that have led to implicit models for viscoelastic materials; the popular models due to Burgers and Oldroyd belong to that type. In a recent paper, Rajagopal and Srinivasa [35] have used the framework of multiple natural configurations to provide a thermodynamic basis for viscoelastic fluids that are capable of instantaneous elastic response. Interestingly, the usual representation for such fluids has an implicit structure. However, Rajagopal and Srinivasa [35] show that they can be expressed as explicit constitutive relations from an evolving set of natural configurations. Thus, such models are not inherently implicit. It is possible to generalize models such as the Maxwell fluid and the Oldroyd fluid to implicit models, with the material moduli depending on both the stress and the velocity gradient. In fact, it is natural to obtain such models when we are dealing with constraints such as incompressibility, with the material moduli now depending on the constraint response.

Let us suppose

$$\varrho\psi = W = W(I_1, II_2), \tag{59}$$

where

$$I_1 = \operatorname{tr} \boldsymbol{B}_{\kappa_{p(t)}}, \quad I_2 = \frac{1}{2}[(\operatorname{tr} \boldsymbol{B}_{\kappa_{p(t)}})^2 - \operatorname{tr} \boldsymbol{B}^2_{\kappa_{p(t)}}] \tag{60}$$

and

$$\xi = \hat{\xi}(\boldsymbol{B}_{\kappa_{p(t)}}, \boldsymbol{D}_{\kappa_{p(t)}}). \tag{61}$$

We are thus supposing that the instantaneous elastic response is like that of an incompressible isotropic elastic material from the appropriate natural configuration $\kappa_{p(t)}$ corresponding to its current configuration κ_t, and the dissipation is that of a fluid that takes into account the stretch associated with the instantaneous elastic deformation. We shall require that

$$\hat{\xi}(\cdot, \mathbf{0}) = \mathbf{0}. \tag{62}$$

As the material can only undergo isochoric motions, we shall require that

$$\operatorname{tr} \boldsymbol{D} = \operatorname{tr} \boldsymbol{D}_{\kappa_{p(t)}} = 0. \tag{63}$$

Moreover, as we are interested in isotropic materials, we can without loss of generality set

$$\boldsymbol{F}_{\kappa_{p(t)}} = \boldsymbol{V}_{\kappa_{p(t)}}. \tag{64}$$

We now have to maximize the dissipation subject to (19) and (32). We can use the method of Lagrange multipliers to do this and it follows that

$$\boldsymbol{T} - \lambda_1 \frac{\partial \hat{\xi}}{\partial \boldsymbol{D}_{\kappa_p(t)}} - \lambda_2 \mathbf{1} = \mathbf{0} \tag{65}$$

where λ_1 and λ_2 are the Lagrange multipliers.

Next, let us make the special choice

$$W = \frac{\mu}{2}(I_1 - 3), \tag{66}$$

$$\xi = \eta \boldsymbol{D}_{\kappa_{p(t)}} \cdot \boldsymbol{B}_{\kappa_{p(t)}} \boldsymbol{D}_{\kappa_{p(t)}}. \tag{67}$$

A straightforward computation leads to (see Rajagopal and Srinivasa [32])

$$\boldsymbol{T} = -p\mathbf{1} + \boldsymbol{S}, \tag{68}$$

$$\boldsymbol{S} = \mu \boldsymbol{B}_{\kappa_{p(t)}}, \tag{69}$$

and

$$-\frac{1}{2}\overset{\triangledown}{\boldsymbol{B}}_{\kappa_{p(t)}} = \frac{\mu}{\eta}[\boldsymbol{B}_{\kappa_{p(t)}} - \lambda \mathbf{1}] \tag{70}$$

where

$$\lambda = \frac{3}{(\operatorname{tr} \boldsymbol{B}^{-1}_{\kappa_{p(t)}})}. \tag{71}$$

In equation (70), the inverted triangle superscript is the upper convected frame-indifferent Oldroyd derivative given by

$$\overset{\triangledown}{\boldsymbol{A}} = \dot{\boldsymbol{A}} - \boldsymbol{L}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{L}^T. \tag{72}$$

The model (68)–(71) is a generalization of the model due to Maxwell, though it is not apparent that it is so (see Rajagopal and Srinivasa [35] for a derivation of the equivalence).

If we suppose the elastic response to be small, that is the strain is linearized from the current natural configuration, in the sense that

$$\|\boldsymbol{B}_{\kappa_{p(t)}} - \mathbf{1}\| = O(\delta), \quad \delta \ll 1, \tag{73}$$

then the system could be linearized to yield the three dimensional version of the Maxwell model. Thus, the linearized Maxwell model essentially corresponds to a material which stores its energy like a linearized elastic solid and dissipates in a manner given by (67), that is, it dissipates like a viscous fluid as observed from the evolving natural configuration.

It is easy to show (see Rajagopal and Srinivasa [35]) that the Oldroyd-B model, within the context of the framework of bodies with multiple natural configurations, is a linearized version (with respect to (73)) of

$$(\alpha) \qquad \boldsymbol{T} = -p\mathbf{1} + \mu \boldsymbol{B}_{\kappa_{p(t)}} + \eta \boldsymbol{D},$$

$$(\beta) \qquad -\frac{1}{2}\overset{\triangledown}{\boldsymbol{B}}_{\kappa_{p(t)}} = \frac{\mu}{\eta}[\boldsymbol{B}_{\kappa_{p(t)}} - \lambda \mathbf{1}].$$

At this juncture, it is important to point to a philosophical difference between Oldroyd's approach that led to his model

$$(\gamma) \qquad \boldsymbol{T} = -p\mathbf{1} + \boldsymbol{S},$$

$$(\delta) \qquad \boldsymbol{S} + \lambda_1 \overset{\triangledown}{\boldsymbol{S}} = \eta[\boldsymbol{D} + \lambda_2 \overset{\triangledown}{\boldsymbol{D}}],$$

and the model (α), (β). In Oldroyd's approach he essentially introduces a constitutively determinate part of the stress that obeys the rate equations (γ, δ). There is no identification as to what it actually is. In the model developed by Rajagopal and Srinivasa [29] a very clear physical meaning is assigned to the extra stress $\boldsymbol{S}$. It is the elastic response from the natural configuration corresponding to the current state. Models using conformation tensors also assign a clear physical meaning to the extra stress and a correspondence can be made between $\boldsymbol{B}_{\kappa_{p(t)}}$ and the conformation tensor.

We could have implicit models which use generalizations of (67) wherein η could depend on the pressure.

4.4. Inelasticity.

We finally consider an implicit model used to describe the behavior of solids. In fact all models that have been introduced to describe the inelastic response of solids are implicit models. The early works on the inelastic response of metals by Tresca [47], St. Venant [43], Levy [16] and von Mises [51] influenced greatly the work that was to follow on the inelastic behavior of metals. However, these early studies modeled the response as "rigid-plastic", that is the material was essentially rigid upto a certain level of the stress, and then starts "flowing" when a "yield" condition is reached. This however ignores the elastic response that most metals exhibit for sufficiently small strains. Ignoring the elastic response might be reasonable if one is interested in applications where very large deformations are involved, such as metal forming, especially if one is primarily interested in some crude characterization of the response. However, even in metal forming applications, if one is interested in "spring back" as in the formation of say a door panel for an automobile, the elastic response of the material cannot be ignored. The later work of Prandtl [25] and Reuss [42] takes the elastic response of the material prior to "yielding" into account. However, their work and much of the work that has followed it assume that the total strain that the body is subject to is small, even though the material yields. Thus, the total strain comprises an "elastic" and "plastic" part, and the material is presumed to respond as an elastic material until a yield condition is reached and starts to deform "plastically" on reaching the yield condition.

Here we would like to discuss the inelastic response of materials subject to large deformations that have an elastic range. We shall discuss the modeling within the framework of bodies with multiple natural configurations. When the material is deformed form its virgin natural configuration, for sufficiently small deformations the body will respond elastically. However, if the body is subject to sufficiently large deformation, it will reach a state when the yield condition is met, and on continued loading and then unloading, the body will be unable to get back to its original natural configuration and during this process the material has dissipated energy. As the material has deformed beyond the yield condition, the body's natural configuration evolves. I shall not discuss here whether the body can indeed be unloaded to a single global natural configuration, this issue is discussed in detail in Rajagopal and Srinivasa [32], [33]. Here, I shall merely present an implicit constitutive relation that is used to describe the inelastic response of bodies subject to large deformation. Lest the readers think that all inelastic response is captured by the model described below, it is important to recognize that the departure from elastic behavior is manifest in a variety of ways; some examples of inelastic behavior are twinning, solid to solid phase transitions in metals, viscoplasticity exhibited by metals, the inelastic response exhibited by polymers, etc. The model presented below is probably applicable to the response of a small class of metals for a small class of processes.

Here we shall follow the work of Rajagopal and Srinivasa [32], [33]. Given a certain natural configuration, we first need to define the elastic range associated with this natural configuration, that is, the range of loadings for which unloading leads to the original natural configuration. Put differently, loading inside the elastic range leads to the natural configuration remaining the same. However, when the yield condition is reached, on continued loading the associated natural configuration changes. Thus, we can define the elastic domain through

$$\hat{\boldsymbol{L}}_{\kappa_{p(t)}}(\boldsymbol{E}_\kappa, \boldsymbol{G}) = \boldsymbol{0}. \tag{74}$$

From the definition of $\boldsymbol{L}_{\kappa_{p(t)}}$ we note that $\boldsymbol{L}_{\kappa_{p(t)}}$ would be zero whenever $\boldsymbol{G}$ does not change. Thus, the above definition for $\boldsymbol{L}_{\kappa_{p(t)}}$ provides the range of values of $\boldsymbol{E}_\kappa$ for a fixed value of $\boldsymbol{G}$ for which the response is elastic. Next, the yield surface that encloses the elastic domain needs to be introduced, and we do this through

$$g(\boldsymbol{E}_\kappa, \boldsymbol{G}) = 0. \tag{75}$$

Of course g coincides with the boundary of the set defined by (74). The elastic domain defined through (74) can also be defined through $g(\boldsymbol{E}_\kappa, \boldsymbol{G}) \leqslant 0$ for any fixed natural configuration.

We shall assume that the Helmholtz potential and the rate of dissipation are given through

$$\psi = \psi(\boldsymbol{E}_\kappa, \boldsymbol{G}), \tag{76}$$

$$\xi = \xi(\boldsymbol{G}, \boldsymbol{L}_{\kappa_{p(t)}}). \tag{77}$$

We once again notice that the dissipation is not affected by the elastic response, that is, ξ does not depend on $\boldsymbol{E}_\kappa$. We shall pick ξ so that

$$\xi \geqslant 0. \tag{78}$$

We shall find it convenient to work with the Piola-Kirchhoff stress $\boldsymbol{S}$, and introduce the dissipation equation (19) as

$$\boldsymbol{S} \cdot \boldsymbol{E}_\kappa - \varrho_0 \dot{\psi} = \hat{\xi}(\boldsymbol{G}, \boldsymbol{L}_{\kappa_{p(t)}}), \tag{79}$$

where ϱ_0 denotes the reference density.

While the rate of dissipation ξ determines the manner in which the natural configurations evolve, there is another quantity that plays a role in the evolution of the natural configurations, and we shall introduce this quantity:

$$\boldsymbol{A} = -\varrho_0 \frac{\partial \psi}{\partial \boldsymbol{G}} \boldsymbol{G}^T. \tag{80}$$

$\boldsymbol{A}$ is referred to usually as the "driving force" (see Rajagopal and Srinivasa [32], [33] for the reason for using such a nomenclature). It then follows that the rate of dissipation ξ can be expressed in terms of $\boldsymbol{A}$ and $\boldsymbol{G}$ instead of $\boldsymbol{E}_\kappa$ and $\boldsymbol{G}$ (this result involves certain technical assumptions and the reader is referred to Rajagopal and Srinivasa [32], [33] for details).

We now turn to a specific example. Let us consider a Helmholtz potential of the form

$$\psi = \psi(\boldsymbol{F}_{\kappa_{p(t)}}). \tag{81}$$

It immediately follows that the stress is given by

$$\boldsymbol{T} = \frac{\varrho}{[\det \boldsymbol{F}_\kappa]} \frac{\partial \psi}{\partial \boldsymbol{F}_{\kappa_{p(t)}}} \boldsymbol{F}^T_{\kappa_{p(t)}}, \tag{82}$$

and

$$\boldsymbol{A} = [\det \boldsymbol{F}_\kappa](\boldsymbol{F}_{\kappa_{p(t)}})^T \boldsymbol{T} (\boldsymbol{F}_{\kappa_{p(t)}})^{-T} \tag{83}$$

where the superscript $-T$ denotes the transpose of the inverse of the linear transformation.

Next, we pick a rate of dissipation through

$$\xi = f(\boldsymbol{A}, \boldsymbol{G}) = \|\boldsymbol{A}\| - K^2, \tag{84}$$

where K could depend on the extent of plastic deformation, i.e., $K = K(\boldsymbol{G}) > 0$. The parameter K is the strain hardening parameter. The boundary of the elastic domain is defined through $\|\boldsymbol{A}\| = K^2$.

It follows from the assumption for the rate of dissipation that (see Rajagopal and Srinivasa [32], [33])

$$\hat{\boldsymbol{L}}_{\kappa_{p(t)}}(\boldsymbol{A}, \boldsymbol{G}) = \begin{cases} \boldsymbol{0} & \text{whenever } \boldsymbol{A} \cdot \boldsymbol{A} \leqslant K^2, \\ \Big(\dfrac{\boldsymbol{A} \cdot \boldsymbol{A} - K^2}{\|\boldsymbol{A}\|}\Big) \boldsymbol{A} & \text{whenever } \boldsymbol{A} \cdot \boldsymbol{A} > K^2. \end{cases} \tag{85}$$

Picking specific choices for the Helmholtz potential leads to a variety of generalizations of the Prandtl-Reuss model for rate-dependent plastic response.

5. Boundary conditions

No discussion of constitutive theory for bodies would be complete which does not discuss the constitutive relations that hold at the boundary of two bodies. That boundary conditions are constitutive relations that depend on the constitutive structure of the two bodies that abut one another is unfortunately oftentimes lost. In reality this "boundary" is rarely sharply defined, with molecules belonging to the two bodies in question moving across an arbitrarily decided upon boundary. This is obviously the case between two fluids but not so obvious when one of the bodies under consideration is a solid. Boundaries, thus demand more intricate constitutive theories and by their very nature are fuzzy zones. Unfortunately due diligence and care are rarely exercised in the specification of boundary conditions and the applied mathematician seems content in specifying Dirichlet, Neumann or periodic boundary conditions.

The pioneers of the field recognized that boundary conditions ought to be derived. The proper condition that ought to apply between a solid and a fluid was the subject of much debate between the likes of Navier, Girard, Poisson and many others. I shall not get into a lengthy discussion of this issue here. However, I shall point out that boundary conditions, by their very nature lead to implicit relationships.

It would be instructive to note what Stokes [46] had to say with regard to boundary conditions as it is now commonly accepted that is was his imprimatur that led to the acceptance of the "no-slip" boundary condition. However, on reading Stokes carefully one can hardly conclude that he was championing the use of the "no-slip" boundary condition for general flows of liquids past an impervious solid boundary. Stokes [46] remarks: "Besides the equations which must hold good at any point in the interior of the mass, it will be necessary to form also the equations which must be satisfied at its boundaries." Stokes is very clear that such boundary conditions ought to "be formed", i.e., they need to be derived. Stokes did precisely this. He went ahead to form the equations that ought to hold when a liquid is exposed to a free surface, between two liquids and between a liquid and a solid. The equations he derived might or might not be appropriate, but that is not the point. What is more important is that he recognized that boundary conditions are constitutive relations. Stokes was not unique in this. In fact Navier, Poisson and others had derived such boundary conditions on the basis of molecular arguments.

The ambivalence of Stokes concerning the status of the "no-slip" boundary condition should become transparent from his following remarks (Stokes [46]):

"Du Buat found by experiment that when the mean velocity of water flowing through a pipe is less than about one inch in a second, the water near the inner surface of the pipe is at rest. If these experiments may be trusted, the conditions to be satisfied in the case of small velocities are those which first occured to me, and which are included in those just given by supposing $\nu = \infty$."

"I have said that when the velocity is not very small the tangential force called into action by the sliding of water over the inner surface of a pipe varies nearly as the square of the velocity ..."

"The most interesting questions connected with this subject require for their solution a knowledge of the conditions which must be satisfied at the surface of a solid in contact with the fluid, which, except in the case of very small motions, are unknown."

Such feelings were shared by most of the masters of the subject. An interesting circle seems to have been completed in that the "no-slip" boundary condition is again being questioned, for a variety of fluids, for a variety of flow conditions. I end with a brief discussion of the "slip" and "threshold-slip" conditions that might be reasonable during the flow of some fluids that are inherently implicit in their nature.

Navier [21] advocated slip at the boundary and a generalization of his derivation would be the condition

$$\boldsymbol{v} \cdot \boldsymbol{\tau} = -k(\boldsymbol{Tn} \cdot \boldsymbol{\tau}), \quad k > 0,$$

where $\boldsymbol{\tau}$ and $\boldsymbol{n}$ are the unit tangent and normal vectors at the boundary. If we allow k to be a function of the normal stress as well as the shear rate, then such a generalization would be an implicit constitutive relation.

Another boundary condition that is sometimes referred to as "threshold-slip" takes the form

$$\begin{aligned} |\boldsymbol{Tn} \cdot \boldsymbol{\tau}| \leqslant \alpha |\boldsymbol{Tn} \cdot \boldsymbol{n}| \Rightarrow \boldsymbol{v} \cdot \boldsymbol{\tau} = 0, \\ |\boldsymbol{Tn} \cdot \boldsymbol{\tau}| > \alpha |\boldsymbol{Tn} \cdot \boldsymbol{n}| \Rightarrow \boldsymbol{v} \cdot \boldsymbol{\tau} \neq 0 \end{aligned}$$

and

$$\boldsymbol{Tn} \cdot \boldsymbol{\tau} = -\gamma \frac{\boldsymbol{v} \cdot \boldsymbol{\tau}}{|\boldsymbol{v} \cdot \boldsymbol{\tau}|},$$

with

$$\gamma = \hat{\gamma}(\boldsymbol{v} \cdot \boldsymbol{\tau}, \boldsymbol{Tn} \cdot \boldsymbol{n}) > 0.$$

We could also make γ depend on the shear rate at the wall. Once again, such boundary conditions are implicit relations.

The above boundary conditions are but two of a plethora of conditions that can be applied at a boundary between a fluid and an imperious solid. If the solid is porous, other conditions could apply. In the case of two solids we can once again apply a variety of boundary conditions that are implicit in nature.

The above examples should suffice to illustrate the richness of implicit constitutive theories for the bulk material as well as the boundary, and the need to carry out a systematic study of them.

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