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AN EPIDEMIC MODEL WITH A TIME DELAY IN TRANSMISSION

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Abstract. We study a mathematical model which was originally suggested by Greenhalgh and Das and takes into account the delay in the recruitment of infected persons. The stability of the equilibria are also discussed. In addition, we show that the introduction of a time delay in the transmission term can destabilize the system and periodic solutions can arise by Hopf bifurcation.

Keywords: epidemic model, time delay, Hopf bifurcation, equilibrium analysis, differential equations

MSC 2000: 92B05

1. Introduction

We study a simple mathematical model for the effect of an endemic disease on a population. Infected people suffer a disease-related mortality. The population is divided into compartments containing susceptible, infectious and immune individuals and differential equations are used to model the spread of the disease between the classes. This model would be suitable for a disease when a typical individual starts as susceptible, at some stage catches the disease and after a short infection period becomes permanently immune. In this model it is assumed that additional mortality due to the diseases and the death rate depend on population density.

The course of an epidemic depends on the contact rate between susceptible and infected individuals and on the assumption that the net rate at which infections are acquired is proportional to the number of encounters between susceptible and infected individuals. The constant of proportionality, β, is sometimes called the transmission coefficient (Anderson and May [1]). This transmission coefficient may well depend on the population size. We assume a transmission term of the form $\beta XY/N$, where

X is the number of susceptibles and Y the number of infected individuals. This form for the disease transmission term was used in [4], [11], [22], [3], [16] and [20].

In this paper we examine a model similar to that of Greenhalgh and Das [9] and introducing time delay in the recruitment of infected persons we show that the introduction of a time delay into the transmission term can destabilize the system and periodic solutions can arise by Hopf bifurcation.

Many authors in the past studied epidemic models with time delays. We now summarise previous results obtained for an epidemic model with time delays. Hethcote and Tudor [12] consider SIR and SEIR models. It is concluded in these models that distributed delays do not change the thresholds or the asymptotic behaviour of the model. Hethcote et al. [13] survey previous work on epidemic models with time delays. Periodic solutions do not occur for an SIS model with or without delays. The SIRS model without delays does not have periodic solutions. The main results of Hethcote et al. paper [13] are that in an SIRS model with a time delay corresponding to temporary immunity for certain parameter values there is a locally asymptotically stable small-amplitude periodic solution, but an SIRS model with delayed transfer through the infective class does not have periodic solutions. Hethcote et al. [14] considered a cyclic constant-parameter SEIS model in a closed population. Distributed delays are introduced and the model is formulated as two coupled Volterra integral equations. Limit cycles are never possible. Thieme ([23], [24]) considered a SEIRS model with a constant time delay as the incubation-period distribution. A threshold condition was derived. If this threshold is exceeded and there is initially some disease present then the solutions tend to this unique endemic solution. Periodic solutions are never possible. Hethcote et al. [15] consider an SIRS epidemiological model with both a time delay in the removed class and a nonlinear incidence rate. There are multiple endemic equilibria for some parameter values, and for some of them periodic solutions arise by Hopf bifurcation from the large nontrivial endemic equilibrium state.

The study of functional differential equations is of considerable interest. There are several books devoted to the fundamental theory of delay differential equations: [2], [6] and [10]. It is recognized that time delays are natural components of the dynamic process of biology, ecology, physiology, economics, epidemiology and mechanics. For an adequate treatment of dynamic economic phenomena the mixed differential equations are much more suitable than differential equations alone, Gandolfo [7]. Systems whose dynamics are modelled by functional differential equations were introduced to economic theory by Kalecki [17] and investigated by Leontief [18]. The book of MacDonald [19] is solely concerned with time lags in biological models. Many problems related to differential equations with time delay and widely used in population dynamics are discussed in the book of Gopalsamy [8]. The main purpose of our model is to investigate the effect of time delay upon the stability of equilibria.

This paper is organized as follows: The model is formulated in Section 2 and equilibrium and stability is discussed in Section 3. Section 4 includes Hopf bifurcation of a non-zero equilibrium by taking time delay as a bifurcation parameter.

2. Description of the model

Basic models of viral diseases contain three variables, which are functions of time t: the populations of susceptible, $X(t)$, infected, $Y(t)$, and immune individuals, $Z(t)$. The total number of individuals at time t is denoted by $N(t)$. The transmission term corresponding to the incidence of new infected cases is $\beta XY/N$ where β corresponds to the constant rate at which individuals come into contact and is independent of the population size. The per capita birth rate of the population is γ and the per capita death rate in the absence of disease is $f(N)$ which depends on the population size N. Those individuals who are infected experience an additional death rate α because they have the disease. The average infectious period is S^{-1} so that infectious individuals enter the immune at rate S. A certain fraction p of newborns are vaccinated immediately at birth. Then the differential equations which describe the progress of the disease are

$$
\begin{aligned}
&\frac{\mathrm{d}X}{\mathrm{d}t} = (1-p)\gamma N - \frac{\beta XY}{N} - f(N)X;\\
&\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\beta XY}{N} - (S+\alpha+f(N))Y;\\
&\frac{\mathrm{d}Z}{\mathrm{d}t} = SY + p\gamma N - f(N)Z;\\
&\frac{\mathrm{d}N}{\mathrm{d}t} = (\gamma - f(N))N - \alpha Y.
\end{aligned}
\tag{2.1}
$$

These equations are strictly valid provided only that $N > 0$. This system of differential equations has previously been used by Greenhalgh and Das [9], to describe the spread of infectious diseases. The model does not contain a time delay in the recruitment of infected persons. In this model we assume that infection production lags by a delay τ. This implies that recruitment into infected class at time t is not given by the density $\beta XY/N$ but rather by the density of newly infected at time

$t-\tau$ who are still alive at time t. Thus the refined model can be written as

$$
\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= \gamma N - p_1 p_2 \gamma N(t-\tau) - \frac{\beta X(t-\tau)Y(t-\tau)}{N} - f(N)X; \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{\beta X(t-\tau)Y(t-\tau)}{N} - (S+\alpha+f(N))Y; \\
\frac{\mathrm{d}Z}{\mathrm{d}t} &= SY + p_1 p_2 \gamma N(t-\tau) - f(N)Z; \\
\frac{\mathrm{d}N}{\mathrm{d}t} &= [\gamma - f(N)]N - \alpha Y
\end{aligned}
\tag{2.2}
$$

with suitable initial conditions. Here one of the equations is redundant since $X(t)+Y(t)+Z(t)=N(t)$.

In equations (2.2), the term $p_1p_2\gamma N(t-\tau)$ is the total rate at which the newly born susceptible individuals vaccinated at time $t-\tau$ enter the immune class at time t after a time delay of length τ. p_1 is the fraction of newborn individuals who are vaccinated immediately at birth and p_2 is the percentage of those vaccinated who become immune (the vaccine efficacy). The rest of the variables have the same meaning and properties as in the system (2.1).

3. Equilibrium and stability analysis

There are only two types of physically relevant equilibrium points, namely,

(i) $\overline{E}_1 = (\overline{N}(1-p), 0, f^{-1}(\gamma))$ where $\gamma = f(\overline{N})$

and

(ii)

$$
\overline{E}_2 = \left\{ \frac{\overline{N}M}{\beta}, \frac{(\gamma\beta(1-p) - f(\overline{N})M)\overline{N}}{M\beta}, f^{-1}\left(\frac{\beta\gamma(M-\alpha(1-p))}{M(\beta-\alpha)} \right) \right\}
\tag{3.1}
$$

where

$$
M = S + \alpha + f(\overline{N}) \text{ and } p = p_1 p_2.
\tag{3.2}
$$

3.1. Stability analysis of equilibrium (i).

We note that as $X+Y+Z=N$, the equations (2.2) are linearly dependent. Eliminating the third equation (in Z) we next consider a small perturbation about the equilibrium point $\overline{E}_1$; i.e. $X=\overline{X}+u$, $Y=0+v$, $N=\overline{N}+w$. Substituting these into the differential equations (2.2) and neglecting the products of small quantities, we obtain the stability matrix

$$
\begin{pmatrix}
-f(\overline{N}) & \dfrac{-\beta\overline{X}\mathrm{e}^{-\tau\lambda}}{\overline{N}} & \gamma - f'(\overline{N})\overline{X} - p\gamma\mathrm{e}^{-\tau\lambda} \\
0 & \dfrac{\beta\overline{X}\mathrm{e}^{-\tau\lambda}}{\overline{N}} - M & 0 \\
0 & -\alpha & \gamma - f(\overline{N}) - \overline{N}f'(\overline{N})
\end{pmatrix}. \tag{3.3}
$$

The characteristic equation resulting from (3.3) is

$$
(f(\overline{N}) + \lambda)(\gamma - f(\overline{N}) - \overline{N}f'(\overline{N}) - \lambda)\Big(\frac{\beta\overline{X}\mathrm{e}^{-\tau\lambda}}{\overline{N}} - M - \lambda\Big) = 0; \tag{3.4}
$$

$\lambda = -f(\overline{N})$ is always a negative eigenvalue and $\lambda = \gamma - f(\overline{N}) - \overline{N}f'(\overline{N})$ is also a negative eigenvalue because $\gamma = f(\overline{N})$ and $f(N)$ is an increasing function of N.

All other eigenvalues are given by solutions of

$$
\frac{\beta\overline{X}\mathrm{e}^{-\tau\lambda}}{\overline{N}} - M = \lambda. \tag{3.5}
$$

Suppose that $\operatorname{Re}\lambda \geqslant 0$. Then from (3.5) we compute the real parts of λ and get

$$
\operatorname{Re}\lambda = \frac{\beta\overline{X}}{\overline{N}}\mathrm{e}^{-\tau\operatorname{Re}\lambda}\cos(\tau\operatorname{Im}\lambda) - M \leqslant \frac{\beta\overline{X}}{\overline{N}} - M = \beta(1-p) - M,
$$

i.e. if $\beta < (S+\alpha+\gamma)/(1-p)$ then $\operatorname{Re}\lambda < 0$, a contradiction.

We summarize the above results in

Theorem 1. *Let $\beta < \frac{S+\alpha+\gamma}{(1-p)}$. Then $\overline{E}_1$ is locally asymptotically stable.*

4. Hopf bifurcation analysis

In this section we determine criteria for Hopf bifurcation using the time delay as the bifurcation parameter. Let $\overline{E}_2 = (\overline{X}, \overline{Y}, \overline{Z}, \overline{N})$ denote the unique interior equilibrium point where $\overline{X}, \overline{Y}, \overline{Z} \geqslant 0$. Arguing as for equilibrium $\overline{E}_1$ and eliminating the equation in Z from (2.2), we obtain the stability matrix

$$
\begin{pmatrix}
-f(\overline{N}) - L\overline{Y} & -L\overline{X} & \gamma - \overline{X}f'(\overline{N}) - p\mathrm{e}^{-\lambda\tau} + C \\
L\overline{Y} & -A + L\overline{X} & -f'(\overline{N})\overline{Y} - C \\
0 & -\alpha & \gamma - f(\overline{N}) - \overline{N}f'(\overline{N})
\end{pmatrix}. \tag{4.1}
$$

Here

$$
\begin{aligned}
L &= \frac{\beta}{\overline{N}} \mathrm{e}^{-\lambda\tau}, \\
A &= S + \alpha + f(\overline{N}), \\
C &= \frac{\beta \overline{XY}}{\overline{N}^2}.
\end{aligned}
$$

The stability matrix (4.1) leads to the characteristic equation

$$
\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = \mathrm{e}^{-\tau\lambda}(T_1\lambda^2 + T_2\lambda + T_3) + \mathrm{e}^{-2\tau\lambda}\left(-\frac{\beta}{\overline{N}}\alpha\overline{Y}p\gamma\right) \tag{4.2}
$$

where

$$
\begin{aligned}
A_1 &= 3f(\overline{N}) + S + \alpha - \gamma + \overline{N}f'(\overline{N}), \\
A_2 &= (S + \alpha + f(\overline{N}))f(\overline{N}) - \gamma f(\overline{N}) + f^2(\overline{N}) + \overline{N}f(\overline{N})f'(\overline{N}) \\
&\quad - \gamma(S + \alpha + f(\overline{N})) + f(\overline{N})(S + \alpha + f(\overline{N})) \\
&\quad + \overline{N}f'(\overline{N})(S + \alpha + f(\overline{N})) - \alpha f'(\overline{N})\overline{Y} - \frac{\alpha\beta\overline{X}\,\overline{Y}}{\overline{N}^2}, \\
A_3 &= -(S + \alpha + f'(\overline{N}))\gamma f(\overline{N}) + f^2(\overline{N})(S + \alpha + f(\overline{N})) \\
&\quad + (S + \alpha + f(\overline{N})) + \overline{N}f(\overline{N})f'(\overline{N}) - \alpha f(\overline{N})f'(\overline{N})\overline{Y} \\
&\quad - \frac{\alpha\beta\overline{X}\,\overline{Y}f(\overline{N})}{\overline{N}^2}, \\
T_1 &= \frac{\beta}{\overline{N}}\overline{Y} - \frac{\beta}{\overline{N}}\overline{X}, \\
T_2 &= -\frac{\beta}{\overline{N}}\overline{X}f(\overline{N}) + (S + \alpha + f(\overline{N}))\frac{\beta}{\overline{N}}\overline{Y} - \frac{\gamma\beta}{\overline{N}}\overline{Y} \\
&\quad + \frac{\beta}{\overline{N}}\overline{Y}f(\overline{N}) + \beta\overline{Y}f'(\overline{N}) + \frac{\beta}{\overline{N}}\gamma\overline{X} \\
&\quad - \frac{\beta}{\overline{N}}f(\overline{N})\overline{X} - \beta\overline{X}f'(\overline{N}), \\
T_3 &= \frac{\beta}{\overline{N}}f(\overline{N})\gamma\overline{X} - \frac{\beta}{\overline{N}^2}f^2(\overline{N})\overline{X} - \beta\overline{X}f'(\overline{N})f(\overline{N}) \\
&\quad - \frac{\beta}{\overline{N}}\overline{Y}\gamma(S + \alpha + f(\overline{N})) + \frac{\beta}{\overline{N}}\overline{Y}(S + \alpha + f(\overline{N}))f(\overline{N}) \\
&\quad + \beta\overline{Y}f'(\overline{N})(S + \alpha + f(\overline{N})) - \frac{\beta}{\overline{N}}\overline{Y}^2\alpha f'(\overline{N}) - \frac{\beta^2\alpha\overline{X}\,\overline{Y}^2}{\overline{N}^3} \\
&\quad + \frac{\alpha\beta\gamma\overline{Y}}{\overline{N}} - \frac{\alpha\overline{X}\,\overline{Y}\beta f'(\overline{N})}{\overline{N}} + \frac{\alpha\beta^2\overline{X}\,\overline{Y}^2}{\overline{N}^3}.
\end{aligned} \tag{4.3}
$$

If we don't consider vaccination programme then the characteristic equation (4.2) reduces to

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = \mathrm{e}^{-\tau\lambda}(T_1\lambda^2 + T_2\lambda + T_3). \tag{4.4}$$

It is the sign of the real parts of the solutions λ of equation (4.4) that determines the stability of $\overline{E}_2 = (\overline{X}, \overline{Y}, \overline{Z})$. When $\tau = 0$, $\overline{E}_2$ is stable as in Greenhalgh and Das [9].

We set $\lambda = \mu + \mathrm{i}\nu$ and substituting into (4.4) we obtain the following equations:

$$\begin{aligned}(\mu^3 - 3\mu\nu^2) &+ A_1(\mu^2 - \nu^2) + A_2\mu + A_3 \\ &= \mathrm{e}^{-\tau\mu}[\{T_1(\mu^2 - \nu^2) + T_2\mu + T_3\}\cos(\tau\nu) \\ &\quad + \{2T_1\mu\nu + T_2\nu\}\sin(\tau\nu)], \\ 3\mu^2\nu^2 &- \nu^3 + 2A_1\mu\nu + A_2\nu \\ &= \mathrm{e}^{-\tau\mu}[-\{T_1(\mu^2 - \nu^2) + T_2\mu + T_3\}\sin(\tau\nu) \\ &\quad + \{2T_1\mu\nu + T_2\nu\}\cos(\tau\nu)].\end{aligned} \tag{4.5}$$

Let τ_1^* be such that $\mu(\tau_1^*) = 0$. Then equations (4.5) reduce to

$$\begin{aligned}-A_1{\nu_1^*}^2 + A_3 &= \{T_1(-{\nu_1^*}^2) + T_3\}\cos(\tau_1^*\nu_1^*) + T_2\nu_1^*\sin(\tau_1^*\nu_1^*), \\ -{\nu_1^*}^3 + A_2\nu_1^* &= -\{T_1(-{\nu_1^*}^2) + T_3\}\sin(\tau_1^*\nu_1^*) + T_2\nu_1^*\cos(\tau_1^*\nu_1^*)\end{aligned} \tag{4.6}$$

where $\nu_1^* = \nu(\tau_1^*)$.

Squaring and adding the equations (4.6) and simplifying we arrive at an equation for ν_1^* of the form

$${\nu_1^*}^6 + {\nu_1^*}^4(A_1^2 - 2A_2 - T_1^2) + {\nu_1^*}^2(A_2^2 - 2A_1A_3 + 2T_1T_3 - T_2^2) + A_3^2 - T_3^2 = 0. \tag{4.7}$$

This is a cubic equation in ${\nu_1^*}^2$ that has one or more real roots, ${\nu_0^*}^2$, since when $\nu_1^* = 0$, the left-hand side of (4.7) is negative if $A_3^2 < T_3^2$, while for sufficiently large values of ν_1^* it is positive. The left-hand side of the equation (4.7) is positive for sufficiently large values of ν_1^* and also $\nu_1^* = 0$ if $A_3^2 > T_3^2$. Hence equation (4.7) has either no or two positive real roots for ${\nu_1^*}^2$.

Lemma. *Define*

$$U = \frac{4}{27}a_2^3 - \frac{1}{27}a_1^2a_2^2 + \frac{4}{27}a_1^3a_3 - \frac{2}{3}a_1a_2a_3 + a_3^2.$$

Suppose $a_3 > 0$. Then necessary and sufficient conditions for the cubic equation

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0 \tag{4.8}$$

to have at least one simple positive root for z are:

(i) *Either* (a) $a_1 < 0$, $a_2 \geqslant 0$, *and* $a_1^2 > 3a_2$ *or* (b) $a_2 < 0$,

and

(ii) $U < 0$.

Proof. Lemma is obvious, we omit the proof. □

We now show that with this value of ν_1^* there is a τ_1^* such that $\mu(\tau_1^*) = 0$ and $\nu(\tau_1^*) = \nu_1^*$. Given ν_1^* the equations (4.6) can be written as

$$\begin{aligned} P\cos(\tau_1^*\nu_1^*) + Q\sin(\tau_1^*\nu_1^*) &= G, \\ Q\cos(\tau_1^*\nu_1^*) - P\sin(\tau_1^*\nu_1^*) &= H \end{aligned} \tag{4.9}$$

where $G^2 + H^2 = P^2 + Q^2 = C_1^2$ with $C_1 > 0$. The equations

$$\begin{aligned} P &= C_1\cos\alpha, \\ Q &= C_1\sin\alpha \end{aligned} \tag{4.10}$$

determine a unique $\alpha \in [0, 2\pi)$, and with this α we have

$$\begin{aligned} C_1\cos(\tau_1^*\nu_1^* - \alpha) &= G \\ C_1\sin(\tau_1^*\nu_1^* - \alpha) &= H. \end{aligned} \tag{4.11}$$

These equations determine $\tau_1^*\nu_1^* - \alpha$ uniquely in $[0, 2\pi)$ and hence τ_1^* uniquely in $[\alpha/\nu_1^*, (2\pi + \alpha)/\nu_1^*)$.

To apply the Hopf bifurcation theorem as was stated in Marsden and McCracken [21] we state and prove the following theorem:

Theorem 2. *Suppose that ν_1^* is the largest positive simple root of equation* (4.7). *Then $\mathrm{i}\nu(\tau_1^*) = \mathrm{i}\nu_1^*$ is a simple root of equation* (4.4) *and $\mu(\tau) + \mathrm{i}\nu(\tau)$ is differentiable with respect to τ in a neighbourhood of $\tau = \tau_1^*$.*

After computation we get that $\mathrm{i}\nu_1^*$ is a simple root of equation (4.4), which is an analytic equation, and so, using the analytic version of the Implicit Function Theorem (Chow and Hale [5]), $\mu(\tau)+\mathrm{i}\nu(\tau)$ is defined and analytic in a neighbourhood of $\tau = \tau_1^*$.

To establish the Hopf bifurcation at $\tau = \tau_1^*$, we need to show that $\mathrm{d}\mu(\tau_1^*)/\mathrm{d}\tau \neq 0$. From (4.5), differentiating with respect to τ, setting $\tau = \tau_1^*$, $\nu = \nu_1^*$, $\mu = 0$ and solving for $\mathrm{d}\mu/\mathrm{d}\tau$ and $\mathrm{d}\nu/\mathrm{d}\tau$, we get $\mathrm{d}\mu/\mathrm{d}\tau$,

$$(4.12)\quad \left.\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\right|_{\tau=\tau_1^*} = \frac{3{\nu_1^*}^6 + {\nu_1^*}^4(2A_1^2 - 4A_2 - 2T_1^2) + {\nu_1^*}^2(A_2^2 - 2A_1A_3 + 2T_1T_3 - T_2^2)}{k_1^2 + k_2^2}$$

where

$$(4.13)\quad \begin{aligned} k_1 = {} & A_2 - 3{\nu_1^*}^2 + \tau_1^*\{(-T_1{\nu_1^*}^2 + T_3)\cos(\tau_1^*\nu_1^*)\} \\ & - 2T_1\nu_1^*\sin(\tau_1^*\nu_1^*) + T_2\nu_1^*\tau_1^*\sin(\tau_1^*\nu_1^*) - T_2\cos(\tau_1^*\nu_1^*) \end{aligned}$$

and

$$\begin{aligned} k_2 = {} & -2A_1\nu_1^* + 2T_1\nu_1^*\cos(\tau_1^*\nu_1^*) + \tau_1^*(-T_1{\nu_1^*}^2 + T_3)\sin(\tau_1^*\nu_1^*) \\ & - T_2\sin(\tau_1^*\nu_1^*) - T_2\nu_1^*\tau_1^*\cos(\tau_1^*\nu_1^*). \end{aligned}$$

Let $\Psi = {\nu_1^*}^2$, then equation (4.7) reduces to

$$\varphi(\Psi) = \Psi^3 + \Psi^2(A_1^2 - 2A_2 - T_1^2) + \Psi(A_2^2 - 2A_1A_3 + 2T_1T_3 - T_2^2) + A_3^2 - T_3^2.$$

So

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\Psi} = 3\Psi^2 + 2\Psi(A_1^2 - 2A_2 - T_1^2) + (A_2^2 - 2A_1A_3 + 2T_1T_3 - T_2^2).$$

As ν_1^* is the largest positive single root of equation (4.7), we have

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}\Psi}\right|_{\nu=\nu_1^*} > 0.$$

Hence

$$\left.\frac{\mathrm{d}\mu}{\mathrm{d}\tau}\right|_{\tau=\tau_1^*} = \frac{{\nu_1^*}^2\dfrac{\mathrm{d}\varphi}{\mathrm{d}\Psi}({\nu_1^*}^2)}{k_1^2 + k_2^2} > 0.$$

Theorem 3. *Let the conditions of Lemma be satisfied and let ν_1^* be the largest simple positive root of equation* (4.7). *Then a Hopf bifurcation occurs as τ passes through τ_1^*.*

In this way, using time delay as a bifurcation parameter, we have obtained conditions for a Hopf bifurcation. Hopf bifurcation has helped us in finding the existence of a region of instability in the neighbourhood of a nonzero endemic equilibrium where the population will survive undergoing regular fluctuations.

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