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Vladimír Janovský; David Just

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ON A 1-D MODEL OF STRESS RELAXATION IN AN ANNEALED GLASS*

Vladimír Janovský, Praha, David Just, Sázava

Abstract. A 1-D model of a slab of glass of a small thickness is considered. The governing equations are those of the classical 1-D linear viscoelasticity. A load due to the temperature gradients is assumed. The aim is to model the process called annealing.

It is shown that an additional load due to *structural strain* is crucial for the success of the model. Algorithms of a numerical solution of the governing equations are proposed. Numerical results are presented and commented.

Keywords: viscoelasticity, relaxation functions, method of lines

MSC 2000: 65M20, 74D05, 74F05

1. Introduction

The industrial process called *annealing* consists in regulated heating and subsequent cooling of sheet glass. The most delicate problem is to propose a suitable *cooling schedule*. The consequence of a fast cooling would be a large residual strain and the development of cracks. On the other hand, a slow cooling in not economical and it would also lead to crystallization (phase change).

Recently, a small size *annealing furnace* have appeared on the market. In this particular device, the cooling is symmetric, i.e. both faces of the sheet have the same temperature. The cooling schedule is given by a nonincreasing function $t \mapsto \vartheta_e(t)$, $0 \leqslant t \leqslant T$. The typical ranges are $\vartheta_e(0) \sim 600\,^\circ\mathrm{C}$, $\vartheta_e(T) \sim 20\,^\circ\mathrm{C}$, $T \sim 6000\,\mathrm{s}$.

Our aim is to formulate a mathematical model of the furnace. The model should predict both the displacements and the stresses of the medium at each position and time. The medium (i.e. the initially melted glass) is subject to a transition from the "liquid" state to the "glassy" state.

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We will consider a 1-D model of the reality assuming that the thickness $2a$ of the sheet of glass is small in comparison with its width and length. Data of our model are the cooling schedule and the initial temperature of the glass which is assumed to be a constant ϑ_0.

The basic governing equation is a linear model of viscoelasticity, see e.g. [1]. We assume u to be a 1-D field of displacements on the interval $[-a,a]$. The field depends on time $t \in [0,T]$, i.e., $u = u(t)$. In particular, for all $0 \leqslant t \leqslant T$ we assume $u(t) \in \mathcal{V}$, where $\mathcal{V} \equiv H_0^1(\Omega)$, $\Omega = (-a,a)$. In a weak formulation, see e.g. [6], we seek for $u \in L^\infty([0,T],\mathcal{V})$ such that

$$a\big(u(t),v\big) + \int_0^t \psi(t,s)\,a\big(u(s),v\big)\,\mathrm{d}s = \big(f(t),v\big) \tag{1}$$

for any test function $v \in \mathcal{V}$ and for almost any time $t \in [0,T]$. Here $a(\cdot\,,\cdot)$ denotes the bilinear form $a(v,w) = D\int_{-a}^{a} \frac{\partial v(x)}{\partial x}\frac{\partial w(x)}{\partial x}\,\mathrm{d}x$ and $\psi(t,s) = -\frac{\partial M(t,s)}{\partial s}\frac{1}{M(t,t)}$ is the kernel of the relaxation operator; D is a positive constant and $M(t,s) > 0$ is the relaxation module from the Bolzmann superposition principle for viscolelastic materials. Finally, $(\cdot\,,\cdot)$ is the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$ in the Hilbert triplet $\mathcal{V} \subset L^2(\Omega) \subset \mathcal{V}'$.

The kernel $\psi(t,s)$ models a synchronous material with a fading memory, see e.g. [2]. We will specify it in Section 3.

The right-hand side of (1), i.e. the load, will be defined in the next section. In Section 3, we consider (1) and suggest a representation of the solution. The numerical solution will be discussed in Section 4. We do not provide a convergence analysis.

The contribution of the paper: We have applied a well known technique (see e.g. [2]) to a solution of a particular technical problem. From this point of view, Sections 2 and 3 represent the main result.

2. The load

The linear functional $v \mapsto \big(f(t),v\big)$ in (1) can be interpreted as a load. In our model, $f(t)$ will be a linear combination of *temperature gradients* and *structural strains*. Let us elaborate.

In the model under consideration a 1-D temperature field $(x,t) \mapsto \vartheta(x,t)$ is defined for $-a \leqslant x \leqslant a$, $0 \leqslant t \leqslant T$. It satisfies a nonlinear heat convection equation

$$\frac{\partial \vartheta(x,t)}{\partial t} = \frac{1}{c\varrho}\frac{\partial}{\partial x}\left(\lambda(\vartheta)\,\frac{\partial \vartheta(x,t)}{\partial x}\right) \tag{2}$$

where $\lambda(\vartheta) = \lambda_0 + p\vartheta(x,t) + q\big(\vartheta(x,t)\big)^m$. Moreover, the Newton condition

$$\frac{\partial\vartheta(x,t)}{\partial x} = -\omega\big(\vartheta(x,t) - \vartheta_e(t)\big) \tag{3}$$

at $x = \pm a$ and the constant initial condition

$$\vartheta(x,0) = \vartheta_0 \tag{4}$$

at $t = 0$ are considered. The function $\vartheta_e(t)$ is the given cooling schedule. For the particular values of the constants c, ϱ, p, q and ω, see Section 4.

In classical *thermoelasticity*,

$$\big(f(t), v\big) \equiv -D\int_{-a}^{a} \alpha_g\big(\vartheta(x,t) - \vartheta_d\big)\frac{\partial v(x)}{\partial x}\,\mathrm{d}x \tag{5}$$

for $v \in \mathcal{V}$; D, α_g and ϑ_d are constants. As a consequence, $f(t)$ is proportional to the gradient $\frac{\partial}{\partial x}\vartheta(x,t)$.

Some authors claim, see e.g. [5], that we cannot get by with the classical thermoelasticity in the problems which concern annealing models. The error is claimed to amount to thirty percents. One should take into account the *structural relaxation.* It will result in a correction term in (5).

Following [5], the notion of the *fictive temperature* is introduced. The fictive temperature $\vartheta_f(x,t)$ is defined at each fixed $-a \leqslant x \leqslant a$ by means of an integral average of the temperatute $\vartheta(x,\cdot)$ as

$$\vartheta_f(x,t) = \vartheta(x,t) - \int_0^t \varphi\big(\xi(x,t) - \xi(x,s)\big)\frac{\partial\vartheta(x,s)}{\partial s}\,\mathrm{d}s \tag{6}$$

where

$$\varphi(y) = \alpha_1\mathrm{e}^{\beta_1\,y} + \alpha_2\mathrm{e}^{\beta_2\,y}.$$

Moreover, $\alpha_1 + \alpha_2 = 1$. The function $\xi(x,s)$ is called the *reduced time.* It is defined as

$$\xi(x,t) = \int_0^t \exp\bigg\{B\bigg(\frac{2}{\vartheta_d} - \frac{1}{\vartheta(x,s)} - \frac{1}{\vartheta_f(x,s)}\bigg)\bigg\}\,\mathrm{d}s. \tag{7}$$

The fictive temperature can be computed by complicated quadrature rules, see [3]. An alternative was pointed out: The evolution of $\vartheta_f(x,\cdot)$ is governed by a system of two ordinary differential equations. Let us elaborate the idea:

Differentiating (6) twice with respect to t yields after some manipulations the following equivalent definition of $\vartheta_f = \vartheta_f(x,t)$: Given a fixed $-a \leqslant x \leqslant a$, let us solve on the interval $0 \leqslant t \leqslant T$ the system

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ f(y_1, y_2, t) \end{pmatrix} \tag{8}$$

with the initial condition

$$y_1(0) = \vartheta_0, \quad y_2(0) = 0. \tag{9}$$

Then the first component of the solution $y_1(t)$ is the fictive temperature $\vartheta_f(x,t)$. Therefore, $\vartheta_f(x,t) = y_1(t)$ for $-a \leqslant x \leqslant a$, $0 \leqslant t \leqslant T$.

Definition of the right-hand side of (8), namely of $f(y_1, y_2, t)$, is quite complicated: Let us define an auxiliary function

$$\eta(y_1, t) = \exp\biggl(B\Bigl(\frac{2}{\vartheta_d} - \frac{1}{\vartheta(x,t)} - \frac{1}{y_1}\Bigr)\biggr);$$

it is related to (7). Then

$$\begin{aligned} f(y_1, y_2, t) = {} & \biggl((\beta_1 + \beta_2)\eta(y_1, t) + B\Bigl(\frac{1}{\vartheta^2(x,t)}\,\frac{\partial\vartheta(x,t)}{\partial t} + \frac{y_2}{(y_1)^2}\Bigr)\biggr) y_2 \\ & + \beta_1\beta_2\eta^2(y_1,t)\bigl(\vartheta(x,t) - y_1\bigr) - (\alpha_1\beta_1 + \alpha_2\beta_2)\eta(y_1,t)\,\frac{\partial\vartheta(x,t)}{\partial t}; \end{aligned}$$

B, α_i and β_i are constants, see Section 4.

The final modification of the load reads as follows:

$$\begin{aligned} \bigl(f(t), v\bigr) \equiv -D \int_{-a}^{a} \Bigl(& \alpha_g\bigl(\vartheta(x,t) - \vartheta_d\bigr) \\ & + (\alpha_l - \alpha_g)\bigl(\vartheta_f(x,t) - \vartheta_d\bigr)\Bigr)\frac{\partial v(x)}{\partial x}\,\mathrm{d}x \end{aligned} \tag{10}$$

for each $v \in \mathcal{V}$; D, α_g, α_l and ϑ_d are positive constants. The gradient $\frac{\partial}{\partial x}\vartheta_f(x,t)$ is called the *structural strain.*

Performing integration by parts in (10), we may conclude that the right-hand side of (1) is a linear combination of the temperature gradient and the structural strain.

We consider (1) under the assumption that the kernel $\psi(t,s)$ represents a *synchronous material with fading memory*, see e.g. [7]. We assume

$$\psi(t,s) = \Phi'(t-s) \tag{11}$$

where $\Phi = \Phi(\tau)$ is a positive, exponentially damped *relaxation function*, $\Phi(0) = 1$. In particular, we assume

$$\Phi'(t-s) = \sum_{j=1}^{k} P_j(t) Q_j(s), \tag{12}$$

where

$$P_j(t) = \mathrm{e}^{-\lambda_j t}, \quad Q_j(s) = -c_j \mathrm{e}^{\lambda_j s}, \quad \lambda_j > 0, \quad c_j \neq 0$$

for $j = 1, \ldots, k$. We shall also assume the ordering $0 < \lambda_1 < \lambda_2 \ldots < \lambda_k$. The resulting model of the relaxation function Φ,

$$\Phi(\tau) = c_0 + \sum_{j=1}^{k} c_j \lambda_j^{-1} \mathrm{e}^{-\lambda_j \tau}, \quad c_0 \equiv 1 - \sum_{j=1}^{k} c_j \lambda_j^{-1},$$

is called the *discrete spectral model*, see [1]. The reciprocal values of λ_j can be interpreted as the *decay times*.

We solve (1) under the assumptions (11) and (12). Essentially, we will follow [2].

Let $U = U(t)$, $U \in L^\infty([0,T], \mathcal{V})$, satisfy

$$a\big(U(t), v\big) = (f(t), v) \tag{13}$$

for any test function $v \in \mathcal{V}$ and for almost any time $t \in [0,T]$. Therefore, $U(t)$ can be interpreted as the elastic deformation which is due to the load $f(t)$. We shall consider the solution $u(t)$ of (1) in the form

$$u(t) = U(t) + \sum_{j=1}^{k} P_j(t) y_j(t), \tag{14}$$

$y_j \in L^\infty([0,T], \mathcal{V})$, where the functions y_j will be specified in the sequel.

Let us note that

$$\begin{aligned}\int_0^t \psi(t,s) a\big(u(s), v\big)\,\mathrm{d}s &= \sum_{i=1}^{k} P_i(t) \int_0^t Q_i(s) a\Big(U(s) + \sum_{j=1}^{k} P_j(s) y_j(s), v\Big)\,\mathrm{d}s \\ &= a\Big(\sum_{i=1}^{k} P_i(t) \int_0^t Q_i(s)\Big(U(s) + \sum_{j=1}^{k} P_j(s) y_j(s)\Big)\,\mathrm{d}s, v\Big).\end{aligned}$$

Now, we define $y_i \in L^\infty([0,T],\mathcal{V})$ to satisfy

$$y_i(t) = -\int_0^t Q_i(s)\Big(U(s) + \sum_{j=1}^k P_j(s)y_j(s)\Big)\,\mathrm{d}s,$$

$i = 1,\dots,k$. In other words, $y_i(t)$, $i = 1,\dots,k$, are the solutions of the linear system

$$\dot{y}_i(t) = -Q_i(t)\sum_{j=1}^k P_j(t)y_j(t) - Q_i(t)U(t) \tag{15}$$

with the initial condition

$$y_i(0) = 0. \tag{16}$$

One should have in mind that the function value $y_i(t)$ is in $\mathcal{V}$. Therefore, the system (15), (16) describes an evolution at each fixed point $x \in [-a,a]$.

Let us *define* $u(t)$ by (14), where $U(t)$ solves (13) and $y_i(t)$, $i = 1,\dots,k$, are the solutions of the system (15), (16). We easily conclude that

$$\begin{aligned}\int_0^t \psi(t,s)a\big(u(s),v\big)\,\mathrm{d}s &= a\Big(\sum_{i=1}^k P_i(t)y_i(t),v\Big)\\ &= -a\big(u(t),v\big) + a\big(U(t),v\big)\\ &= -a\big(u(t),v\big) + \big(f(t),v\big)\end{aligned}$$

for each $v \in \mathcal{V}$. Therefore, $u(t)$ solves (1).

Finally, let us define

$$w_i(t) = P_i(t)y_i(t), \tag{17}$$

$i = 1,\dots,k$; we call them the *creep modes*. Due to (15), (16), the creep modes $w_i(t)$, $i = 1,\dots,k$, satisfy

$$\dot{w}_i(t) = -\lambda_i w_i(t) + c_i\sum_{j=1}^k w_j(s) - c_iU(t) \tag{18}$$

with the initial condition

$$w_i(0) = 0. \tag{19}$$

Again, $w_i(t)$ are functions with values in $\mathcal{V}$. Therefore, the system (18), (19) defines evolution at each fixed $x \in [-a, a]$. The displacement $u(t)$ comes out as the sum $u(t) = U(t) + \sum_{j=1}^{k} w_j(t)$. Pointwise,

$$u(x,t) = U(x,t) + \sum_{j=1}^{k} w_j(x,t), \tag{20}$$

$x \in [-a, a]$, $0 \leqslant t \leqslant T$.

4. Numerical solution

The structure of the solver is as follows:

Step 1. Find the temperature field $\vartheta = \vartheta(x,t)$ for $x \in [-a,a]$, $0 \leqslant t \leqslant T$ solving (2) with (3) and (4).

Step 2. Find the fictive temperature field $\vartheta_f = \vartheta_f(x,t)$ for $x \in [-a,a]$, $0 \leqslant t \leqslant T$. It consists in solving the ODE system (8), (9) at each fixed $x \in [-a,a]$.

Step 3. Define the load-functional (10).

Step 4. Find the elastic displacements $U(x,t)$ on $x \in [-a,a]$ at each time level $0 \leqslant t \leqslant T$. This represents solving the boundary value problem (13) for each fixed time.

Step 5. Evaluate the creep modes $w_i(x,t)$, $i = 1,\dots,k$. We need to solve the ODE system (18), (19) for $0 \leqslant t \leqslant T$ at each $x \in [-a,a]$ fixed.

Step 6. The resulting displacement $u(x,t)$ is just the superposition (20).

The solution steps can be performed canonically (no coupling). It is due to the fact that the model is linear.

In order to approximate the fields ϑ, ϑ_f, U and w_i, $i = 1,\dots,k$, we considered an approximation on the same equidistant grid in the space variable $x \in [-a,a]$. In particular, we used the method of lines to approximate ϑ in **Step 1**. The aim was to use adaptive time-stepping. We have to use an A-stable solver (the resulting ODE's are very stiff). In **Step 2**, we need to solve a nonautonomous system of two ordinary equations (the initial value problem) at each particular grid point. The system is moderately stiff. The elastic displacements (at each fixed time) are approximated by finite elements (piecewise linear test functions on the grid under consideration). This verbal description represents **Step 3** and **Step 4** of the algorithm. In **Step 5**, we considered $k = 2$, i.e. just two creep modes. Then this step just mimics **Step 2**. In the numerical realization we used different time-steppings in **Step 1**, **Step 2** and **Step 5**. We need to communicate between different time-meshes. In our particular implementation, we used just a linear interpolation. For the details, see [3]. The algorithm was coded in MATLAB version 5.3.

In the numerical experiments reported below, we used the following settings:

- geometry: $a = 0.57 \times 10^{-2}\,\mathrm{m}$
- discretisation: number of grid points = 241
- mesh size in space: $h = 2.375 \times 10^{-5}\,\mathrm{m}$
- ad Heat equation: $c = 999.6\,\mathrm{J\,kg^{-1}\,K^{-1}}$, $\varrho = 2492\,\mathrm{kg\,m^{-3}}$, $\lambda_0 = .903$, $p = 4.94 \times 10^{-4}$, $q = 1.5861 \times 10^{-19}$, $m = 3.486$, $\omega = 39.9225$
- ad Fictive temperature: $\vartheta_d = 756\,\mathrm{K}$, $B = 2.7661 \times 10^4\,\mathrm{K}$, $\alpha_1 = 0.43$, $\beta_1 = -1.773 \times 10^{-3}$, $\alpha_2 = 0.57$, $\beta_2 = -3.307 \times 10^{-4}$
- ad Load: $\alpha_g = 8.80 \times 10^{-6}$, $\alpha_l = 6.16 \times 10^{-5}$
- ad Elastic deformation: $D = \frac{E}{1-\mu}$, $E = 72480\,\mathrm{MPa}$, $\mu = 0.218$
- ad Stress relaxation function: $c_1 = .59$, $c_2 = .41$, $\lambda_1 = -1.280 \times 10^{-3}$, $\lambda_2 = -6.927 \times 10^{-3}$

4.1. Example 1.

In the first numerical experiment, we considered the cooling schedule depicted in Fig. 1 (dashed line, ϑ_e). This figure shows also one computed temperature profile of ϑ and ϑ_f at $x = a$. Let us note that for this particular ϑ_e there were some experimental data available, see [5].

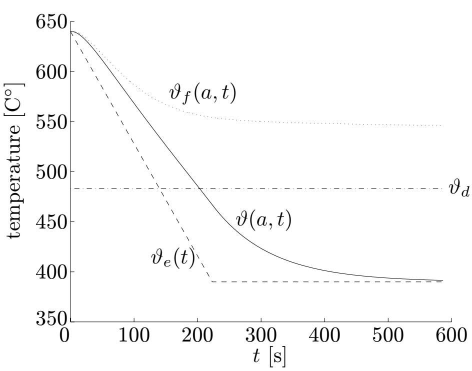

Fig. 1. Cooling schedule $\vartheta_e(t)$, temperature profiles $\vartheta(a,t)$, $\vartheta_f(a,t)$.

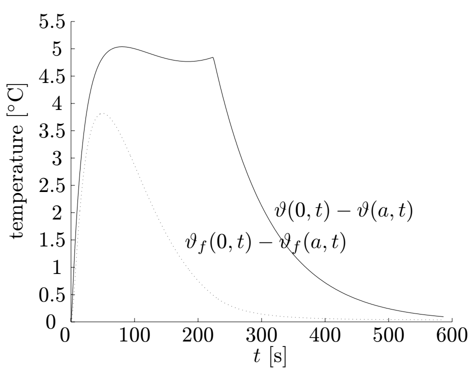

Fig. 2. Temperature profile differences.

Fig. 2 shows the evolution of the differences $\vartheta(0,t) - \vartheta(a,t)$ and $\vartheta_f(0,t) - \vartheta_f(a,t)$. In Fig. 3, we plot the evolution of U, $w_1 + w_2$ and u as $x = a/2$ is fixed.

Our model (1) was fomulated in displacements. Nevertheless, a crude computation of stresses $\sigma(x,t)$ is also possible: Let u_{xt} and u_{xtt} be the relevant partial derivatives of u. We fix $x \in [-a, a]$ and consider the system

$$\begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \frac{E}{1-\mu} \begin{pmatrix} 0 \\ f(x,t) \end{pmatrix}$$

with the initial conditions $y_1(0) = y_2(0) = 0$. The function f is defined as

$$f(x,t) = (-\lambda_1 c_2 - \lambda_2 c_1) u_{xt}(x,t) + u_{xtt}(x,t).$$

The solution components $y_1(t)$ and $y_2(t)$ can be interpreted as

$$y_1(t) = \sigma(x,t), \quad y_2(t) = \dot{\sigma}(x,t).$$

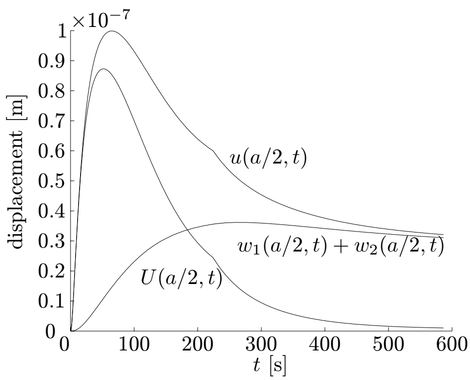

Fig. 3. Displacement evolution at $x = a/2$.

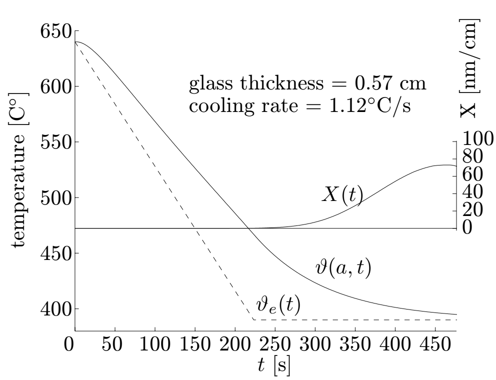

Fig. 4. Stress-induced birefringence $X(t)$, cooling schedule $\vartheta_e(t)$, temperature profile $\vartheta(a,t)$.

Obviously, while constructing f, we have to get by with just an approximation of the relevant partials of u. Such computation of σ is not very reliable. In [5] in Fig. 2, a plot of *stress-induced birefringence* $X(t)$ versus time t was shown. Note that the cooling schedule was the same as was assumed in Example 1 (see Fig. 1). We *computed* the stress-induced birefringence for the current example. The definition of $X(t)$ is as follows:

$$\bar{\sigma}(t) = \sigma(a,t) - \sigma(0,t), \quad X(t) = \mathsf{B}\bar{\sigma}(t)$$

where $\mathsf{B} = 26.0\,\mathrm{nm\,cm^{-1}\,MPa^{-1}}$ is the photoelastic constant. The plot of X is shown in Fig. 4: The agreement with Fig. 2 in [5] is not quite convincing. The effects of the structural relaxation seem to be delayed.

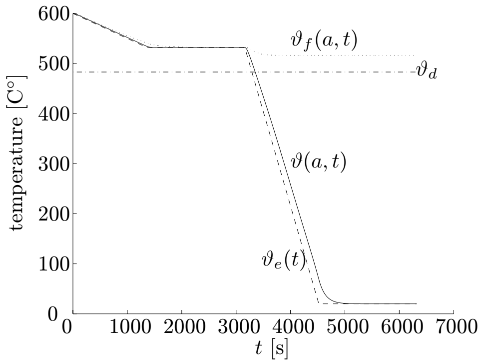

Fig.5. Cooling schedule $\vartheta_e(t)$, temperature profiles $\vartheta(a,t)$, $\vartheta_f(a,t)$.

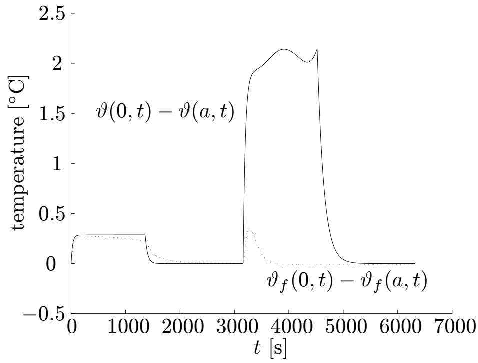

Fig. 6. Temperature profile differences.

4.2. Example 2.

In this example we considered a cooling schedule in practical use, see Fig. 5.

Fig. 6 depicts the evolution of the differences $\vartheta(0,t)-\vartheta(a,t)$ and $\vartheta_f(0,t)-\vartheta_f(a,t)$. Fig. 7 shows the plots of $U(a/2,t)$, $w_1(a/2,t)+w_2(a/2,t)$ and $u(a/2,t)$ versus time t. An attempt to compute σ failed.

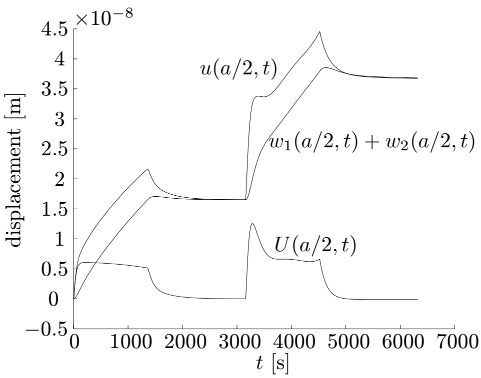

Fig. 7. Displacement evolution at $x=a/2$.

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Authors' addresses: *V. Janovský*, Charles University of Prague, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic, e-mail: `janovsky@karlin.mff.cuni.cz`; *D. Just*, Anenská 274, 258 06 Sázava, e-mail: `justdavid@email.cz`.