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NONLINEAR BOUNDARY VALUE PROBLEMS DESCRIBING MOBILE CARRIER TRANSPORT IN SEMICONDUCTOR DEVICES

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Abstract. The present paper describes mobile carrier transport in semiconductor devices with constant densities of ionized impurities. For this purpose we use one-dimensional partial differential equations. The work gives the proofs of global existence of solutions of systems of such kind, their bifurcations and their stability under the corresponding assumptions.

Keywords: nonlinear boundary value problem, asymptotic behaviour of solutions, semiconductors, carrier transport, constant densities of ionized impurities, interior transition layer phenomena

MSC 2000: 35Q20, 35D05, 35B40

Introduction

This paper intends to extend the theory of L. Recke [1]. In his paper he considers a simple mathematical model describing the mobile carrier transport in semiconductor devices. Two functions $E(x)$ and $n(x)$ describe the electric field strength and the density of mobile electrons and satisfy under $0 < x < 1$ the system

$$\begin{gathered}\bigl(D(|E|)(n' + nE)\bigr)' = 0,\\ E' = f - n\end{gathered} \tag{1}$$

and boundary conditions

$$E(0) = E(1) = E_0, \qquad D(|E(0)|)\bigl(n'(0) + n(0)E(0)\bigr) = j_0. \tag{2}$$

Here the constant $f > 0$ represents the homogeneous density of ionized impurities, $D(|E|)$ is the diffusion coefficient, j_0 is the electron current density for $x = 0$. For sake of definiteness we shall presume $E_0 > 0$. The paper [1] proposes

$$j_0 = D(E_0)E_0 f. \tag{3}$$

Under such conditions the problem (1)–(3) has a trivial solution $E(x) = E_0$, $n(x) = f$. If $K(E_0) < 0$ (where $K(E_0) = 1 + D'(E_0)E_0D^{-1}(E_0)$), then there exists a denumerable set of points $f_k(E_0) = -K(E_0)^{-1}(E_0^2/4 + \pi^2k^2)$, $k = 1, 2, \ldots$, in the neighbourhood of which small bifurcational solutions of the problem (1)–(3) appear [1]. In this paper we prove that the condition $K(E_0) < 0$ implies that the diffusion coefficient D as a function of the field strength has an N-shaped form and contains an interval (E_1, E_2), in which this function has a negative derivative and $D(E_0) + E_0D'(E_0) < 0$, i.e. the so called condition of negative differential conductivity (NDC) is valid. In Section 1 we prove that the NDC-condition is necessary and sufficient for the existence of bifurcational solutions of the problem (1)–(3). Then in Section 2 the extendibility of the bifurcation branch for large parameters is demonstrated. The corresponding bifurcational problem can be considered as an eigenvalue nonlinear problem; in Section 3 we discuss forms of eigenfunctions of this problem. It appears that the asymptotic behaviour of bifurcational solutions under large values of the parameter f depends essentially on the parameter E_0. In Section 4 we prove that in the interval (E_1, E_2) there exists a unique point E_0^* such that the so called interior transition layer phenomena [2], [3] arise in the problem (1)–(3). If $E_0 \in (E_1, E_2)$ but $E_0 \neq E_0^*$, then the asymptotic behaviour of bifurcational solutions has a completely different nature. In Section 5 the existence and uniqueness of solutions of an initial-boundary value problem for the nonstationary version of (1)–(3) for every $t > 0$ is discussed; this theory is applied to investigate stability and instability of bifurcational solutions. It appears that the stability of the first (positive) and the second (negative) eigenfunctions depends also on the parameter E_0. In Section 6 it is proved: if $E_0 = E_0^*$, then the pair eigenfunctions are stable; if $E_1 < E_0 < E_0^*$, then only the negative function is stable, and if $E_0^* < E_0 < E_2$ or vice versa, only the positive one is stable. Other eigenfunctions are unstable for any $E_0 \in (E_1, E_2)$ for those values of the parameter f for which they exist. In the last Section 7 the existence of a parabolic travelling wave for $E_0 = E_0^*$ and its stability for sufficiently large f are proved.

1. Existence of bifurcational solutions dependent on the parameter E_0

The problem (1)–(3) is equivalent to the boundary value problem

$$
\begin{gathered}
E'' + E'E = fH(E, E_0), \qquad 0 < x < 1, \\
E(0) = E(1) = E_0,
\end{gathered} \tag{4}
$$

where $H(E, E_0) = E - E_0 D(E_0) D^{-1}(|E|)$. This problem has a trivial solution $E(x) = E_0$ for any f. Let the diffusion coefficient satisfy the conditions

1) $D(y) \in C^{(2)}(\mathbb{R}_+)$, $\quad D\colon \mathbb{R}_+ \to \mathbb{R}_+$;
2) $D(y)$ has a unique local maximum and a unique point of inflection for $y > 0$;
3) $\lim\limits_{y \to +\infty} D(y) = D_0 > 0$.

Let $E(x) = E_0 + u(x)$ and $g(u) = D(E_0)D^{-1}(|E_0 + u|) - 1$. The function $g(u)$ is continuously differentiable for $u \neq -E_0$ and

$$0 < g_i = \sup_u \bigl|g^{(i)}(u)\bigr| < +\infty, \quad i = 0, 1.$$

Proposition 1.1. *If the inequality $f < \frac{\pi^2}{E_0 g_1}$ is valid, then the problem* (4) *has a unique (trivial) solution.*

P r o o f. Let us write the problem (4) in the form

$$
\begin{gathered}
-u'' - E_0 u' + fu - u'u = E_0 f g(u), \\
u(0) = u(1) = 0
\end{gathered} \tag{5}
$$

and let u be a nontrivial solution of this problem. Multiplying the equation (5) by $u(x)$ and integrating by parts we obtain

$$\int_0^1 u'^2(x)\,\mathrm{d}x \leqslant E_0 f g_1 \int_0^1 u^2(x)\,\mathrm{d}x,$$

and since

$$\pi^2 \int_0^1 u^2(x)\,\mathrm{d}x \leqslant \int_0^1 u'^2(x)\,\mathrm{d}x, \tag{6}$$

the estimate $f \geqslant \frac{\pi^2}{E_0 g_1}$ holds, which gives Proposition 1.1. $\square$

Let $C_0^{(2)}([0,1])$ be the space of functions $u(x)$ which are continuous on $[0,1]$, have continuous second derivatives on $(0,1)$ and satisfy the conditions $u(0)=u(1)=0$; let $C([0,1])$ be the space of continuous functions on $[0,1]$. We will consider the problem (5) as a nonlinear eigenvalue problem:

$$
\begin{gathered}
Lu + fK(E_0)u + N(E_0, f, u) = 0, \\
u(0) = u(1) = 0,
\end{gathered}
\tag{7}
$$

where $Lu = -u'' - E_0u'$ is a linear operator mapping from $X = C_0^{(2)}([0,1])$ into $Y = C([0,1])$, $K(E_0) = 1 + D'(E_0)E_0D^{-1}(E_0)$, and

$$
N(E_0, f, u) = -u'u - E_0 f\left(\frac{D(E_0)}{D(|E_0+u|)} + \frac{D'(E_0)u}{D(E_0)} - 1\right)
$$

is a nonlinear mapping from $\mathbb{R}^2 \times X$ into Y. By S we denote the closure of the set of all nontrivial solutions $(f,u) \in \mathbb{R} \times X$ to (7) with $u \neq 0$, and let S_k be the maximal connected component of S containing $(f_k, 0)$, $f_k = -K(E_0)^{-1}(E_0^2/4 + \pi^2 k^2)$, $k = 1, 2, \ldots$

Theorem 1.1 [1]. *Suppose $K(E_0) < 0$. Then the following holds:*

(i) *S_k is unbounded;*

(ii) *suppose $(f,u) \in S_k$ and $u \not\equiv 0$. Then $u(x)$ has exactly $(k+1)$ zeros in $[0,1]$, and all zeros are simple;*

(iii) *for all $k \in \mathbb{N}$ there exists a constant $s_k > 0$, a neighbourhood $U_k \subset \mathbb{R} \times X$ of $(f_k, 0)$ and two C^1-mappings $\hat{f}_k \colon (-s_k, s_k) \to \mathbb{R}$, $\hat{u}_k \colon (-s_k, s_k) \to X$ such that $\hat{f}_k(s) = f_k + O(s)$, $\hat{u}_k(s) = su_k(x) + O(s^2)$ for $s \to 0$ and $S \cap U_k = \{(\hat{f}_k(s), \hat{u}_k(s)) \colon |s| < s_k\}$, where $u_k(x) = \mathrm{e}^{-E_0 x/2} \sin(\pi k x)$.*

These solutions are called the bifurcational ones [4].

It is fairly evident that the condition $K(E_0) < 0$ is equivalent to the condition of negative differential conductivity (NDC) (see Introduction).

Proposition 1.2 [5]. *Let $D(|E|)$ satisfy the* NDC. *Then there exists a unique E_0^* such that*

a) *$0 < G(E_{\min}) < G(E_0^*) < G(E_{\max})$, where $E_{\max}$, $E_{\min}$ are the local extrema of the function $G(E) = ED(E)$ for $E > 0$;*

b) *the derivative $G'(E_0) < 0$ for $E_0 \in (E_{\max}, E_{\min})$, which is equivalent to the condition $K(E_0) < 0$;*

c) *the equation $H(E, E_0) = 0$ for $E_0 \in (E_{\max}, E_{\min})$ has only three positive solutions $0 < E_1(E_0) < E_0 < E_2(E_0)$, moreover $H'_E(E_i(E_0), E_0) > 0$, $i = 1, 2$;*

d) $$\int_{E_1(E_0^*)}^{E} H(s,E_0^*)\,\mathrm{d}s \begin{cases} >0 & \textit{for } E\in\bigl(E_1(E_0^*),E_2(E_0^*)\bigr),\\ =0 & \textit{for } E=E_2(E_0^*).\end{cases}$$

Let us analyze for which values of the parameter E_0 nontrivial solutions of problem (4) exist.

Proposition 1.3. *If $0 < E_0 \leqslant E_{\max}$ or $E_0 \geqslant E_{\min}$ then the problem* (4) *has only the trivial solution $E(x) = E_0$. If $E_0 \in (E_{\max}, E_{\min})$ then the problem* (4) *has nontrivial (bifurcational) solutions.*

Proof. If $E_0 \in (E_{\max}, E_{\min})$ then $G'(E_0) < 0$, i.e. $K(E_0) < 0$ and by virtue of Theorem 1.1 the problem (4) has nontrivial solutions. If $0 < E_0 \leqslant E_{\max}$ or $E_0 \geqslant E_{\min}$ then $K(E_0) \geqslant 0$. Let us consider the problem (7). Further we prove that if $K(E_0) \geqslant 0$, then this problem has no small nontrivial solutions; from this and from Rabinowitz's results (see Theorem 2.3 [6]) it follows that the problem (7) cannot have any nontrivial solutions. The problem (7) can be linearized in the neighbourhood of a zero solution

$$\begin{aligned} Lu &= -fK(E_0)u,\\ u(0) &= u(1) = 0.\end{aligned}$$

Since the last form of the problem has only a zero solution, the problem (7) has no nontrivial solutions. □

2. Extendibility of bifurcational solutions with respect to parameter f

We show in this section that the bifurcation branch is extendible for parameters $f > f_k$, where $f_k = -K(E_0)^{-1}(E_0^2/4 + \pi^2k^2)$, $k = 1, 2, \ldots$ Let us prove the following proposition.

Proposition 2.1. *Let $E_0 \in (E_{\max}, E_{\min})$. Then there exists a positive continuous function $\varphi(f)\colon \mathbb{R}_+ \to \mathbb{R}_+$ that for any solution (u, f) of the problem* (7) *the inequality*

$$\|u\|_X(f) \leqslant \varphi(f) \tag{8}$$

is fulfilled.

P r o o f. From the form of the differential equation and the inequality (6) we have an estimate

$$\|u'\|_{L_2} \leqslant \pi^{-1} E_0 f g_0. \tag{9}$$

This estimate gives an analogous estimate in the norm of $C^0([0,1])$. The estimate of $u''(x)$ is based on (7) and on inequalities (6) and (9). As a result we have

$$\|u''\|_{L_2} \leqslant c_1 f^2 + c_2 f^3 + c_3 f^4,$$

where constants c_i depend only on E_0 and g_0. The same estimate is valid for $u(x)$ in the norm of $C^{(1)}([0,1])$. To estimate the uniform norm of $u''(x)$ the equation (7) can be used; it gives the estimate (8).

Now let us return to the problem (7). If $E_0 \in (E_{\max}, E_{\min})$, then it follows from the proposition a) of Theorem 1.1 and Propositions 1.1, 2.1 that the bifurcational solutions, which were obtained in the proposition b) of Theorem 1.1, are extendable with respect to the parameter f for any $f > f_k$, $k = 1, 2, \ldots$

To synthesize these results we denote by U_k^+ the set of $u(x) \in X$ which have $(k+1)$ simple zeros and $\operatorname*{sign}\limits_{x\to 0+} u(x) = 1$, $U_k^- = -U_k^+$, $k \in \mathbb{N}$. □

Theorem 2.1. *Given $E_0 \in (E_{\max}, E_{\min})$, then for every $k \in \mathbb{N}$, every $\nu = +$ or $-$ and for every $f > f_k$ there exists at least one solution $u(x)$ of the boundary value problem* (7) *such that $u \in U_k^\nu$.*

Theorem 2.1 is proved analogously to Theorem 2.3 [6], because the linear operator L from (7) satisfies the maximum principle. Note that if (f, u) is a solution of (7) and u has a double zero, then the behaviour of operator N near the double zero and the linearity of L and $fK(E_0)u$ imply that $u = 0$ on $[0,1]$. Therefore, in particular, any solution (f, u) of (7) with $u_k^\nu(x) \in \partial U_k^\nu$ satisfies $u = 0$. In the sequel the functions $u_k^\nu(x) \in U_k^\nu$ will be called eigenfunctions of the nonlinear operator of the problem (7).

3. Forms of eigenfunctions

To investigate the forms of these eigenfunctions let the problem (4) be reformulated as

$$\begin{gathered} -E'' - E'E = f\big(G(E_0) - G(E)\big)D^{-1}(|E|), \\ E(0) = E(1) = E_0. \end{gathered} \tag{10}$$

If $E_0 \in (E_{\max}, E_{\min})$ and (f, E) is a nontrivial solution of the problem (10), then for each $x \in [0,1]$ the inequality $E_1(E_0) < E(x) < E_2(E_0)$ is fulfilled. In this inequality $E_i(E_0)$ $(i = 1,2)$ is the solution of the equation $H(E, E_0) = 0$, which was mentioned in Proposition 1.2. This result follows from the fact that the constants E_0, $E_1(E_0)$ and $E_2(E_0)$ are solutions of the differential equation (10).

The solution $E(x)$ of the problem (10) will be called a positive one if the corresponding function $u(x) = E(x) - E_0$ is positive on $(0,1)$.

Let $E(x)$ be a positive solution of the problem (10). Let us prove that $E(x)$ has only one maximum. If it is not so, then there exists such $x_0 \in (0,1)$ that $E(x_0)$ is a local minimum. However then

$$0 \geqslant -E''(x_0) = f\big(G(E_0) - G(E(x_0))\big)D^{-1}(E(x_0)) > 0,$$

which is impossible.

Theorem 1.1 shows that each solution from S_k has $(k+1)$ simple zeros on $[0,1]$; similar reasoning gives the next result.

Proposition 3.1. *Let $E(x) = u(x) + E_0$ be such a solution of the boundary value problem* (10) *that $(f, u) \in S_k$, $k = 1, 2, \ldots$. Then*

a) *if $k = 2n$, then $E(x)$ has n maxima and n minima;*

b) *if $k = 2n+1$, then the numbers of minima and maxima differ by* 1.

4. Asymptotic behaviour of the eigenfunctions for large values of parameter f

The case of large concentrations of ionized impurities is of great interest for physical applications. Mathematically this fact can be associated with the asymptotical behaviour of the eigenfunctions for large values of the parameter f. Let the problem (4) be formulated as

$$\begin{gathered} \varepsilon E'' + \varepsilon E' E = H(E, E_0), \\ E(0) = E(1) = E_0, \end{gathered} \tag{11}$$

where $\varepsilon = f^{-1}$, $H(E, E_0) = E - E_0 D(E_0) D^{-1}(|E|)$.

Let $u_k^\nu(x, f)$ be the bifurcational solutions of the problem (7), $f > f_k$, $k = 1, 2, \ldots$, $\nu = +, -$. Then $E_k^\nu(x, \varepsilon) = u_k^\nu(x, \varepsilon^{-1}) + E_0$ are solutions of the problem (11), which are defined for $0 < \varepsilon < \varepsilon_k$, where $\varepsilon_k = f_k^{-1}$. We will call them the eigenfunctions of the problem (11). Let $E_0 \in (E_{\max}, E_{\min})$ and $E_i(E_0)$, $i = 1, 2$ be two positive solutions of the equation $H(E, E_0) = 0$ from Proposition 1.2. The main results of this section are the next three theorems.

Theorem 4.1. *Given* $E_0 \in (E_{\max}, E_0^*)$, *the families of solutions* $E_1^-(x,\varepsilon)$, $E_k^{\pm}(x,\varepsilon)$, $k = 2, 3, \ldots$, *have the following properties:*

$\lim_{\varepsilon\to 0+} E_1^-(x,\varepsilon) = E_1(E_0)$ *uniformly for every compact set from* $(0,1)$;
$\lim_{\varepsilon\to 0+} E_k^{\pm}(x,\varepsilon) = E_1(E_0)$ *almost everywhere on* $(0,1)$.

Theorem 4.2. *Given* $E_0 \in (E_0^*, E_{\min})$, *the families of solutions* $E_1^+(x,\varepsilon)$, $E_k^{\pm}(x,\varepsilon)$, $k = 2, 3, \ldots$, *have the following properties:*

$\lim_{\varepsilon\to 0+} E_1^+(x,\varepsilon) = E_2(E_0)$ *uniformly for every compact set from* $(0,1)$;
$\lim_{\varepsilon\to 0+} E_k^{\pm}(x,\varepsilon) = E_2(E_0)$ *almost everywhere on* $(0,1)$.

The asymptotic behaviour of the eigenfunctions of the problem (11) sharply changes when $E_0 = E_0^*$. In this case the families of solutions have interior transition points.

Definition [2]. Let $E(x,\varepsilon)$ be the family of solutions for the problem (10), defined for sufficiently small $\varepsilon > 0$. A point $x_0 \in (0,1)$ is called a transition point (interior transition one), if for some $\delta > 0$ the condition

$$\lim_{\varepsilon\to 0+} E(x,\varepsilon) = \begin{cases} E_1(E_0), & x_0 - \delta < x < x_0, \\ E_2(E_0), & x_0 < x < x_0 + \delta \end{cases}$$

is fulfilled (or an analogous condition where $E_2(E_0)$ and $E_1(E_0)$ are inserted in place of $E_1(E_0)$ and $E_2(E_0)$).

Theorem 4.3 states that there exists a family of solutions of the problem (11) with an arbitrary large number of transition points of such solutions.

Theorem 4.3. *Given* $E_0 = E_0^*$, *the problem* (11) *has families of solutions* $E_k^{\pm}(x,\varepsilon)$, $k = 1, 2, \ldots$ *defined for sufficiently small* ε; *moreover, every family* $E_k^{\pm}(x,\varepsilon)$ *has exactly* $k-1$ *transition points in the interval* $(0,1)$.

The proofs of these theorems follow from some results of R. O'Malley [7] and the next three propositions. It appears that the asymptotic behaviour of the solutions of the problem (11) depends essentially on the properties of such differential equations as

$$\ddot{y} + \sqrt{\varepsilon}\dot{y}y - H(y, E_0) = 0, \tag{12}$$

$$\ddot{u} - H(u, E_0) = 0. \tag{13}$$

Proposition 4.1. *Given* $E_0 \in (E_{\max}, E_0^*)$, *then for sufficiently small* $\varepsilon > 0$ *the differential equation* (12) *has a solution* $y(t, \varepsilon)$ *such that* $\lim\limits_{t\to+\infty} y(t,\varepsilon) = \lim\limits_{t\to-\infty} y(t,\varepsilon) = E_1(E_0)$, *and*

$$\lim_{\varepsilon\to 0+} y(t,\varepsilon) = y_0(t), \quad t \in \mathbb{R},$$

where $y_0(t)$ *is the solution of the equation* (13); *for this solution we have*

$$\lim_{t\to+\infty} y_0(t) = \lim_{t\to-\infty} y_0(t) = E_1(E_0).$$

Proof. Since $E_0 \in (E_{\max}, E_0^*)$, Proposition 1.2 gives $\int_{E_1(E_0)}^{E_2(E_0)} H(s, E_0)\,\mathrm{d}s < 0$. Therefore it can be obtained from Proposition 5 (see theorem from [7]) that the equation (13) has a solution $y_0(t)$ with the properties mentioned above.

Rewrite for convenience the equation (12) as

$$G_1(\ddot{y}, \dot{y}, y, \varepsilon) = \ddot{y} + \varepsilon\dot{y}y - H(y, E_0) = 0. \tag{14}$$

Substitution $y - y_0 = v$ gives

$$G_1(\ddot{y}_0 + \ddot{v}, \dot{y}_0 + \dot{v}, y_0 + v, \varepsilon) = \ddot{y}_0 + \ddot{v} + \varepsilon(\dot{y}_0 + \dot{v})(y_0 + v) - H(y_0 + v, E_0) = 0, \tag{15}$$

where $\lim\limits_{|t|\to\infty} v(t,\varepsilon) = 0$.

Lemma 4.2 [3] states that the left-hand side of (14) defines an operator $\tilde{G}_1(v,\varepsilon)$ from $X \times \mathbb{R}^1$ into Y, where $X = H_2 \cap C^2$, $Y = H_0 \cap C^0$ are equipped with the norms

$$\|v\|_X = |v|_2 + \left(\sum_{k=0}^{2} \int_{-\infty}^{+\infty} |v^{(k)}(\eta)|^2\,\mathrm{d}\eta\right)^{1/2}, \quad \|v\|_Y = |v|_0 + \left(\int_{-\infty}^{+\infty} v^2(\eta)\,\mathrm{d}\eta\right)^{1/2}.$$

Let us verify the conditions of Lemma 3.1 [3]:

(i) $M \equiv \tilde{G}_1,\ m(v,\varepsilon) \equiv v(0),\ \ \tilde{G}_1(0,0) = 0,\ m(0,0) = 0$;

(ii) $\Phi = \dot{y}_0 \in X$, $\langle \Phi^*, v\rangle = \int_{-\infty}^{+\infty} \dot{y}_0 v\,\mathrm{d}\eta$, $R(M_1(0,0)) = \{v \in Y\colon \langle\Phi^*, v\rangle = 0\}$, where $M_1(0,0)w = \ddot{w} - H'(y_0)w$;

(iii) $\langle \Phi^*, M_2(0,0;1)\rangle = \int_{-\infty}^{+\infty} \dot{y}_0\dot{y}_0 y_0\,\mathrm{d}\eta \neq 0$;

(iv) $m_1(0,0;\Phi) = \Phi(0) \neq 0$.

This lemma implies Proposition 4.1. The next proposition is proved in the same way. □

Proposition 4.2. *Given $E_0 \in (E_0^*, E_{\min})$, then for sufficiently small $\varepsilon > 0$ the differential equation* (12) *has a solution $y(t,\varepsilon)$ such that $\lim\limits_{t\to+\infty} y(t,\varepsilon) = \lim\limits_{t\to-\infty} y(t,\varepsilon) = E_2(E_0)$ and $\lim\limits_{\varepsilon\to 0+} y(t,\varepsilon) = y_0(t)$, $t \in \mathbb{R}$, where $y_0(t)$ is the solution of the equation* (13)*; for this solution we have $\lim\limits_{t\to+\infty} y_0(t) = \lim\limits_{t\to-\infty} y_0(t) = E_2(E_0)$.*

We have now the situation where $E_0 = E_0^*$.

Proposition 4.3. *Given $E_0 = E_0^*$, then for sufficiently small $\varepsilon > 0$ the differential equation* (12) *has a solution $y(t,\varepsilon)$ such that $\lim\limits_{t\to+\infty} y(t,\varepsilon) = E_2(E_0^*)$, $\lim\limits_{t\to-\infty} y(t,\varepsilon) = E_1(E_0^*)$ and $\lim\limits_{\varepsilon\to 0+} y(t,\varepsilon) = y_0(t)$, $t \in \mathbb{R}$, where $y_0(t)$ is the solution of* (13)*, for which $\lim\limits_{t\to+\infty} y_0(t) = E_2(E_0^*)$, $\lim\limits_{t\to-\infty} y_0(t) = E_1(E_0^*)$.*

P r o o f. We will use the results of P. Fife [3]. The equation $\varepsilon E'' + \varepsilon E' E - H(E, E_0^*) = 0$ can be rewritten as $F(\varepsilon E'', \sqrt{\varepsilon} E', E, \varepsilon) = 0$. Substitution $t = x - c/\sqrt{\varepsilon}$, where c is an arbitrary constant, gives

$$F(\varepsilon E'', \sqrt{\varepsilon} E', E, \varepsilon) \equiv G(\ddot{y}, \dot{y}, y, \varepsilon) = \ddot{y} + \sqrt{\varepsilon}\dot{y}y - H(y, E_0^*) = 0. \tag{16}$$

The function $y_0(t) = y(t, 0)$ satisfies the equation

$$G(\ddot{y}_0, \dot{y}_0, y_0, 0) = \ddot{y}_0 - H(y, E_0^*) = 0. \tag{17}$$

The properties c) and d) of Proposition 1.2 are equivalent to the fact that the equation (17) has a solution $y_0(t)$ satisfying the conditions $y_0(-\infty) = E_1(E_0^*)$, $y_0(+\infty) = E_2(E_0^*)$. It appears that the conditions c) and d) from Proposition 1.2 are sufficient for the existence of a solution $y(t,\varepsilon)$ of the equation (16). This solution is defined for sufficiently small $\varepsilon > 0$ and satisfies $y(-\infty, \varepsilon) = E_1(E_0^*)$, $y(+\infty, \varepsilon) = E_2(E_0^*)$. This fact can be obtained from the following special case of Theorem 4.1 [3]. $\square$

Proposition 4.4 [3]. *The conditions*

1) $H'_E\big(E_i(E_0^*), E_0^*\big) > 0$, $i = 1, 2$;

2) $$\int_{E_1(E_0^*)}^{E} H(s, E_0^*)\,\mathrm{d}s \begin{cases} > 0, & E \in \big(E_1(E_0^*), E_2(E_0^*)\big), \\ = 0, & E = E_2(E_0^*) \end{cases}$$

are sufficient for the existence of a family of solutions $y(t,\varepsilon)$ of the differential equation (16)*; these solutions are defined for $t \in \mathbb{R}$ and for sufficiently small $\varepsilon > 0$ and are uniformly continuous on ε; they satisfy $y(-\infty,\varepsilon) = E_1(E_0^*)$, $y(+\infty,\varepsilon) = E_2(E_0^*)$.*

Let us return to the proof of Theorem 4.1. It is rather easy to prove that the rest point $(E_0, 0)$ is a stable focus for sufficiently small $\varepsilon > 0$. By C_ε we denote a closed loop which circles the point $(E_0, 0)$, coming out from the point $(E_1(E_0), 0)$ and returning to it. Then there exists a trajectory $y(t, \varepsilon)$ within the domain which is formed by the loop C_ε; this trajectory has the following properties: $\omega(y(t,\varepsilon)) = (E_0, 0)$, $\alpha(y(t,\varepsilon)) = C_\varepsilon$, where $\alpha(y)$ and $\omega(y)$ are the α-limit set and the ω-limit set of $y(t,\varepsilon)$ [8]. Consider an arbitrary solution $E(x,\varepsilon)$ of the problem (11). The function $y(t,\varepsilon) = E(\sqrt{\varepsilon}t + 1, \varepsilon)$ is a solution of the boundary value problem

$$\begin{gathered} \ddot{y} + \sqrt{\varepsilon}\dot{y}y - H(y, E_0) = 0, \\ y(0) = E_0, \qquad y\Big(-\frac{1}{\sqrt{\varepsilon}}\Big) = E_0. \end{gathered} \tag{18}$$

Since $\alpha(y(t,\varepsilon)) = C_\varepsilon \supset (E_1(E_0), 0)$ and the point $(E_1(E_0), 0)$ is a unique saddle point with a separatrix C_ε, Theorem 4.1 can be obtained by the reasoning used in the 5-th proposition of R. O'Malley's theorem [7].

Proof of Theorem 4.2 completely repeats that of Theorem 4.1.

The closed loop C_ε is an α-limit set for any solution of the problem (18). When $E_0 = E_0^*$ the loop C_ε has two saddle points $(E_1(E_0^*), 0)$ and $(E_2(E_0^*), 0)$; this fact and the 4-th proposition of O'Malley's theorem [7] give Theorem 4.3.

5. Nonstationary initial-boundary value problem. Existence and uniqueness of solutions for $t > 0$

It is easy to prove that the problem (4) is a stationary problem for the following nonstationary problem [8]:

$$\begin{gathered} D(|E|)^{-1}\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial E}{\partial x}E - fH(E, E_0), \\ E(0,t) = E(1,t) = E_0, \\ E(x,0) = \tilde{E}(x). \end{gathered} \tag{19}$$

The problem (19) is equivalent to the boundary value problem

$$\begin{gathered} D(|u + E_0|)^{-1}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + E_0\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}u - fh(u, E_0), \\ u(0,t) = u(1,t) = 0, \\ u(x,0) = u_0(x), \end{gathered} \tag{20}$$

where $h(u, E_0) = H(u + E_0, E_0)$, $u_0(x) = \tilde{E}(x) - E_0$.

Consider the space $X = L_2(0,1)$ and the operator $A = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - E_0\frac{\mathrm{d}}{\mathrm{d}x}$ with the domain of definition $D(A) = H^2(0,1) \cap H_0^1(0,1)$ and $D(A^{1/2}) = X^{1/2} = H_0^1(0,1)$. It is easy to prove that for the operator $F\colon\ H_0^1(0,1) \to L_2(0,1)$, defined by the formula

$$F(\varphi)(x) = \varphi(x)\varphi'(x) - fh(\varphi(x), E_0), \qquad 0 < x < 1,$$

the conditions of Theorems 3.3.3 and 3.3.4 [8], are fulfilled. To this end it is sufficient to prove that

1) $\|F(\varphi)\|_{L_2} \leqslant c\|\varphi\|^2_{H_0^1}$, i.e. F maps bounded subsets of $H_0^1(0,1)$ to bounded subsets of $L_2(0,1)$;
2) F is locally Lipschitzian.

Using the Theorems 3.3.3 and 3.3.4 [8], we can formulate the following proposition.

Proposition 5.1. *A unique solution $u(x,t)$ of Cauchy problem* (20) *exists on some maximal interval $0 \leqslant t \leqslant \bar{t}$. Moreover, this solution exists for any initial condition and either $\bar{t} = +\infty$ or $\|u(x,t)\|_{H_0^1} \to +\infty$ for $t \to \bar{t}$.*

To prove the following proposition we need Theorem 3.5.2 [8] on a smoothing differential operator.

Proposition 5.2. *The solution $u(x,t)$ of the nonstationary problem* (20) *is a classical solution.*

P r o o f. $u(t;u_0) \in D(A)$ for $t > 0$. The function $t \mapsto \frac{\mathrm{d}u}{\mathrm{d}t} \in H_0^1(0,1)$ is locally Gölderian (see theorem 3.5.2 [8]). Therefore the functions

$$(x,t) \mapsto u(x,t;u_0), \quad \frac{\partial u}{\partial t}(x,t;u_0)$$

are continuous for $t_0 < t < \bar{t}$ and $x \in [0,1]$. Since $u \in D(A)$, we have $u' \in W_2^1(0,1) \subset C(0,1)$. There exists $\delta > 0$ such that $F(u) \in C^\delta(0,1)$, $u(\cdot,t) \in C^{2+\delta}(0,1)$. Thus, for $t > 0$ the function $(x,t) \mapsto u(x,t;u_0)$ is continuously differentiable by t and twice continuously differentiable by x; therefore it is a classical solution of (20). □

The main item of this section is the following proposition, based on the concept of the dynamical system for parabolic equations [8].

Proposition 5.3. *The nonstationary problem* (20) *defines a dynamical system in the set $C = \{u \in H_0^1(0,1) \big| \sigma_1(E_0) \leqslant u(x) \leqslant \sigma_2(E_0)$ almost everywhere on $[0,1]\}$, where $\sigma_i(E_0) = E_i(E_0) - E_0$, $i = 1,2$.*

P r o o f. First consider the fact that the solution $u(x,t)$ of the problem (20) with the initial condition $u_0 \in C$ cannot leave the set C on the interval of its existence. For this we use a version of the maximum principle. Let t_1 be a minimal value $t_1 \in (0,\bar{t})$, so that the solution $u(x, t_1)$ of the problem (20) has a local maximum $\sigma = u(x_1, t_1)$, where $x_1 \in (0, 1)$ and $\sigma > \sigma_2(E_0) > 0$, i.e. $\frac{\partial u}{\partial x}(x_1, t_1) = 0$, $\frac{\partial^2 u}{\partial x^2}(x_1, t_1) \leqslant 0$. Then from the differential equation (20) we obtain that since $(-h(\sigma, E_0)) < 0$, we have $\dot{u}(x_1, t_1) < 0$.

Furthermore, the solution $u(x, t)$ of the problem (20) cannot have a local minimum whose value is less than $\sigma_1(E_0) < 0$ (the proof is quite similar). Prove that the solution of (20) exists for any $t \geqslant 0$. Let it be not so. Then Proposition 5.1 gives that $\int_0^1 u'^2(x, t)\,\mathrm{d}x$ is unbounded for $t \to \bar{t}$. Multiplying the equation (20) by u and integrating its parts over $[0, 1]$ gives

$$\int_0^1 \frac{\dot{u}u}{D(|u + E_0|)}\,\mathrm{d}x = -\int_0^1 u'^2\,\mathrm{d}x - f\int_0^1 h(u, E_0)u\,\mathrm{d}x.$$

The integral $\int_0^1 \frac{\dot{u}u}{D(|u+E_0|)}\,\mathrm{d}x \to -\infty$ for $t \to \bar{t}$, because $\left(-f\int_0^1 h(u, E_0)u\,\mathrm{d}x\right)$ is uniformly bounded for t. But the latter statement is impossible because Gronwall's inequality $\dot{\varphi} \leqslant -c_1\varphi + c_2$ ($c_i > 0$, $i = 1, 2$) is valid for the function $\varphi(t) = \int_0^1 g\big(u(x, t)\big)\,\mathrm{d}x$, where $g(u) = \int_0^u sD^{-1}(|s + E_0|)\,\mathrm{d}s$. □

6. Stability and instability of eigenfunctions

This section deals with stability of solutions for the stationary problem (11) viewed as stationary solutions to the corresponding nonstationary problem by the linear approximation. To this aim we need two theorems (5.1.1 and 5.1.3) from [8], which give sufficient conditions of stability and instability of such solutions. Let $E_k^\nu(x, \varepsilon)$ be the solutions of the problem (11) from Section 4 defined for sufficiently small ε, $\nu = +, -$; $k = 1, 2, \ldots$ The two following theorems form the main subject of this section.

Theorem 6.1.

1) *If $E_{\max} < E_0 < E_0^*$, then the solution $E_1^-(x, \varepsilon)$ of the problem (11) is stable for sufficiently small ε;*
2) *if $E_0^* < E_0 < E_{\min}$, then the solution $E_1^+(x, \varepsilon)$ of the problem (11) is stable for sufficiently small ε;*
3) *if $E_0 = E_0^*$, then both solutions $E_1^\pm(x, \varepsilon)$ of the problem (11) are stable for sufficiently small ε.*

Theorem 6.2. *The solutions $E_k^\pm(x, \varepsilon)$ are unstable for $0 < \varepsilon < \varepsilon_k$ for every $E_0 \in (E_{\max}, E_{\min})$ and for each $k = 2, 3, \ldots$*

R e m a r k. It is easy to prove, similarly to Theorems 6.1, 6.2, that for $\varepsilon > \varepsilon_1$ the trivial solution $E(x) = E_0$ is stable, for $0 < \varepsilon < \varepsilon_1$ it is unstable. Theorem 6.1 states that the solutions are stable only for sufficiently small $\varepsilon > 0$. Apparently this fact is not accidental; it is quite possible for the solutions $E_1^{\pm}(x,\varepsilon)$ to be unstable when ε is not small.

Proofs of Theorems 6.1 and 6.2 must be preceded by a rather special preamble. First let us discuss the case when $E_{\max} < E_0 < E_0^*$. For such E_0 we have $\int_{E_1(E_0)}^{E_2(E_0)} H(s,E_0)\,\mathrm{d}s < 0$ and the problem

$$\hat{z}_0'' = H(E_1(E_0) + \hat{z}_0, E_0),\qquad \hat{z}_0(0) = E_0 - E_1(E_0), \qquad \hat{z}_0(+\infty) = 0 \tag{21}$$

has a unique solution $\hat{z}_0(t)$ which is a strongly monotonous function [7]. Let $\zeta(y)$ be a cut-off function from the C^∞-class and let the conditions $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ for $0 \leqslant y \leqslant 1/4$, $\zeta \equiv 0$ for $y \geqslant \frac{1}{2}$, be fulfilled for it. Let $z_0(x,\varepsilon) = \hat{z}_0\big(\frac{x}{\sqrt{\varepsilon}}\big)\zeta(x)$, $z_1(x,\varepsilon) = \hat{z}_0\big(\frac{1-x}{\sqrt{\varepsilon}}\big)\zeta(1-x)$, $0 \leqslant x \leqslant 1$. Consider the function

$$U_0(x,\varepsilon) = E_1(E_0) + z_0(x,\varepsilon) + z_1(x,\varepsilon).$$

This function (see Theorem 4.1) is a first approximation of $E_1^-(x,\varepsilon)$ with respect to ε. When $E_0^* < E_0 < E_{\max}$ the function $U_0(x,\varepsilon)$ can be formed quite analogously ($E_2(E_0)$ stands at the place of $E_1(E_0)$) and it is a first approximation of $E_1^+(x,\varepsilon)$ with respect to ε.

We have now to deal with the case $E_0 = E_0^*$. Since $\int_{E_1(E_0^*)}^{E_2(E_0^*)} H(s,E_0^*)\,\mathrm{d}s = 0$ only for this condition, each of the problems

$$\hat{z}_0'' = H(E_i(E_0^*) + \hat{z}_0, E_0^*),\qquad \hat{z}_0(0) = E_0^* - E_i(E_0^*), \qquad \hat{z}_0(+\infty) = 0, \quad i = 1,2$$

has a unique solution $\hat{z}_0^{(i)}(t)$ $(i = 1,2)$ which is a strongly monotonous function [7]. Let $z_0^{(i)}(x,\varepsilon) = \hat{z}_0^{(i)}\big(\frac{x}{\sqrt{\varepsilon}}\big)\zeta(x)$ and let $\hat{z}_1^{(i)}(t)$ be the unique strongly monotonous solution of the problem

$$\hat{z}_1'' = H(E_i(E_0^*) + \hat{z}_1, E_0^*),\qquad \hat{z}_1(0) = E_0^* - E_i(E_0^*), \qquad \hat{z}_1(+\infty) = 0, \quad i = 1,2.$$

Let $z_1^{(i)}(x,\varepsilon) = \hat{z}_1^{(i)}\big(\frac{1-x}{\sqrt{\varepsilon}}\big)\zeta(1-x)$, $0 \leqslant x \leqslant 1$. Consider the function $U_0^{(i)}(x,\varepsilon) = E_i(E_0^*) + z_0^{(i)}(x,\varepsilon) + z_1^{(i)}(x,\varepsilon)$, $i = 1,2$. Theorem 4.1 states that they are the first approximation of the solutions $E_1^{\pm}(x,\varepsilon)$ of the problem (11) with respect to ε. For $\varepsilon > 0$ and $u \in C_0^{(2)}$ the norm

$$|u|_2^{(\varepsilon)} = |u|_0 + \sqrt{\varepsilon}|u'|_0 + \varepsilon|u''|_0$$

can be introduced, where $|\cdot|_0$ is the norm in $C^{(0)}$; let the corresponding Banach space be denoted by $C_{0,\varepsilon}^{(2)}$. A linear operator mapping from $C_{0,\varepsilon}^{(2)}$ into $C^{(0)}$ can be constructed:

$$L_\varepsilon u = \varepsilon u'' + \varepsilon U_0' u + \varepsilon U_0 u' - H'(U_0, E_0)u$$

(the function $U_0(x,\varepsilon)$ is constructed in accordance with the value of E_0 by the means used above). The linear operators

$$L_\varepsilon^{(i)} u = \varepsilon u'' + \varepsilon U_0^{(i)\prime} u + \varepsilon U_0^{(i)} u' - H'(U_0^{(i)}, E_0^*)u, \quad (i = 1, 2)$$

are formed similarly.

Lemma 6.1. *The operators $L_\varepsilon^{(i)}$, $i = 1, 2$ and L_ε have the inverse ones, which are uniformly bounded with respect to sufficiently small $\varepsilon > 0$.*

P r o o f. For the proof it is sufficient to show that there exists a constant c independent of ε, such that for any continuous function F with $|F|_0 \leqslant 1$ and for any sufficiently small $\varepsilon > 0$ there exists a solution u_ε of

$$(22) \qquad \begin{gathered} L_\varepsilon u_\varepsilon = F(x), \qquad 0 \leqslant x \leqslant 1, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0, \end{gathered}$$

satisfying $|u_\varepsilon|_2^{(\varepsilon)} \leqslant c$. This can be done by constructing supersolutions $\overline{u}_\varepsilon$ and subsolutions $\underline{u}_\varepsilon$. For this supersolution the conditions $L_\varepsilon \overline{u}_\varepsilon \leqslant F$, $\overline{u}_\varepsilon(0) \geqslant 0$, $\overline{u}_\varepsilon(1) \geqslant 0$ must be valid by definition. The converse inequalities must be valid for the subsolution. If a positive supersolution can be constructed, then we merely take $\underline{u}_\varepsilon \equiv -\overline{u}_\varepsilon$. By a theorem of Nagumo [9], there exists an exact solution (22) with $|u_\varepsilon|_0 \leqslant |\overline{u}_\varepsilon|_0$. This inequality and equation (22) together with the interpolation inequality relating $|u''|_0$, $|u'|_0$ and $|u|_0$ give $|u_\varepsilon|_2^{(\varepsilon)} \leqslant \text{const}|\overline{u}_\varepsilon|_0$. A positive supersolution $\overline{u}_\varepsilon$ with $|\overline{u}_\varepsilon|_0$ uniformly bounded with respect to ε can be constructed in the same way as that in Lemma 2.1 [2]. $\square$

P r o o f of Theorem 6.1. For definiteness we shall presume that $E_{\max} < E_0 < E_0^*$ and prove stability of the solution $E_1^-(x,\varepsilon) = E_1^-(x, f^{-1}) = \tilde{E}_1^-(x, f)$. Consider the boundary value problem

$$\begin{gathered} v_0'' + \tilde{E}_1^{-\prime} v_0 + \tilde{E}_1^- v_0' - fH'(\tilde{E}_1^-, E_0)v_0 + \alpha_0 v_0 = 0, \\ v_0(0) = v_0(1) = 0, \end{gathered}$$

where v_0 is the first positive eigenfunction of the linear operator $L_f u = u'' + \tilde{E}_1^{-\prime} u + \tilde{E}_1^{-} u' - fH'(\tilde{E}_1^{-}, E_0)u$. It is easy to show that the spectrum of the operator L_f is real. Theorem 5.1.1 [8] states that, if $\alpha_0 > 0$, then the solution $\tilde{E}_1^{-}(x, f)$ is stable. Consider the problem

$$\psi'' + \tilde{E}_1^{-\prime}\psi + \tilde{E}_1^{-}\psi' - fH'(\tilde{E}_1^{-}, E_0)\psi = 0. \tag{23}$$

Lemma 6.1 implies that for sufficiently large f the problem (23) has only the trivial solution. Theorem of Nagumo [9] states that $\alpha_0 > 0$. In fact, suppose that $\alpha_0 < 0$, then we have $L_f v_0 \geqslant L_f \psi = 0$. Hence $v_0 \leqslant \psi(x) \equiv 0$, which is impossible. The cases when $E_0^* < E_0 < E_{\min}$ and $E_0 = E_0^*$ can be proved similarly. □

For the proof of Theorem 6.2 we use the following proposition.

Proposition 6.1 [8]. *Let functions $\varphi(x)$, $\psi(x) \in C^2$ satisfy the conditions $\varphi(0) = \psi(0) = 0$, $\varphi'(0) = \psi'(0) = 1$, $\varphi'' + b(x)\varphi' + a(x)\varphi > \psi'' + b(x)\psi' + a(x)\psi$ for $0 < x < x_1$ and $\psi(x) > 0$ for $0 < x < x_1$. Then $\varphi(x) > \psi(x)$ for $0 < x \leqslant x_1$.*

P r o o f. Let $\varphi_k^{\pm}(x, f) = E_k^{\pm\prime}(x, f^{-1}) = \tilde{E}_k^{\pm\prime}(x, f)$, $k = 2, 3, \ldots$ If $\psi(x)$ is the solution of the problem

$$\psi'' + \tilde{E}_k^{\nu\prime}\psi + \tilde{E}_k^{\nu}\psi' - fH'(\tilde{E}_k^{\nu}, E_0)\psi = 0,$$
$$\psi(0) = 0, \qquad \psi'(0) = 1,$$

$\nu = +, -$, $k \geqslant 2$, then

$$\psi(x)\varphi_k^{\nu\prime}(x) - \psi'(x)\varphi_k^{\nu}(x) = c e^{-\int_0^x \tilde{E}_k^{\nu}(s)\,\mathrm{d}s},$$

where $c = \text{const}$. Since $\varphi_k^{\nu}(0) > 0$, we have $c < 0$. Hence,

$$\psi(x)\varphi_k^{\nu\prime}(x) - \psi'(x)\varphi_k^{\nu}(x) < 0$$

for $x \in [0, 1]$. Let x_0 be the minimum point of the function $\tilde{E}_k^{\nu}(x)$. Consequently, $\psi(x_0)\varphi_k^{\nu\prime}(x_0) < 0$ because $\varphi_k^{\nu}(x_0) = 0$, $\varphi_k^{\nu\prime}(x_0) > 0$. Therefore, $\psi(x_0) < 0$ and $\psi(x)$ has negative values on $[0, 1]$. Consider the problem

$$v_0'' + \tilde{E}_k^{\nu\prime} v_0 + \tilde{E}_k^{\nu} v_0' - fH'(\tilde{E}_k^{\nu}, E_0)v_0 + \alpha_0 v_0 = 0,$$
$$v_0(0) = v_0(1) = 0,$$

where v_0 is the first positive eigenfunction of the linear operator $L_f u = u'' + \tilde{E}_k^{\nu\prime} u + \tilde{E}_k^{\nu} u' - fH'(\tilde{E}_k^{\nu}, E_0)u$. Suppose that $v_0'(0) = 1$. Theorem 5.1.3 [8] states that if $\alpha_0 < 0$, then the solution $\tilde{E}_k^{\nu}(x, f)$ is unstable. In fact, let $\alpha_0 > 0$, then it follows from Proposition 6.1 that $v_0(x) < \psi(x)$ for every $x \in [0, 1]$ because $L_f v_0 < L_f \psi$. This is impossible. □

7. A parabolic travelling wave. Stability of the travelling wave

Consider the parabolic equation of the problem (19) for $E_0 = E_0^*$, i.e.

$$D(|E|)^{-1}\frac{\partial E}{\partial t} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial E}{\partial x}E - fH(E, E_0^*), \quad x \in \mathbb{R}, \quad t > 0. \tag{24}$$

It is easy to show that, if $\varphi(s)$ is a solution of the equation

$$\varphi'' + \Big(\varphi - \frac{V}{D(|\varphi|)}\Big)\varphi' = fH(\varphi, E_0^*), \quad s \in \mathbb{R}, \tag{25}$$

then $E(x,t) = \varphi(x + Vt)$ is a travelling wave ($V = \text{const}$). We shall prove that for sufficiently large f, there exists a solution of (25) such that $\varphi(s) \to E_1(E_0^*)$ for $s \to -\infty$, $\varphi(s) \to E_2(E_0^*)$ for $s \to +\infty$. Let us divide the equation (25) by f, let $\varepsilon = f^{-1}$ and suppose $\tau = \frac{s}{\sqrt{\varepsilon}}$. Then we have

$$\ddot{y} + \sqrt{\varepsilon}\Big(y - \frac{V}{D(|y|)}\Big)\dot{y} = H(y, E_0^*). \tag{26}$$

Theorem 4.1 [3] states that for sufficiently small ε the equation (26) has a solution $y(\tau,\varepsilon)$ such that $y(\tau,\varepsilon) \to E_1(E_0^*)$ for $\tau \to -\infty$, $y(\tau,\varepsilon) \to E_2(E_0^*)$ for $\tau \to +\infty$. The following interesting result [8] states the stability of the parabolic travelling wave. Suppose that there exists a solution $\varphi(x)$ of

$$\varphi''(x) + f\big(\varphi(x), \varphi'(x)\big) = 0, \quad x \in \mathbb{R},$$

such that $\varphi(x) \to \alpha$ for $x \to -\infty$, $\varphi(x) \to \beta$ for $x \to +\infty$, $f(u,p) \in C^1$ and $f(\alpha, 0) = f(\beta, 0) = 0$. The linearized problem is

$$-Lv = v'' + a(x)v' + b(x)v,$$

where $a(x) = \frac{\partial f}{\partial p}\big(\varphi(x), \varphi'(x)\big)$, $b(x) = \frac{\partial f}{\partial u}\big(\varphi(x), \varphi'(x)\big)$. Let $a_\pm$, $b_\pm$ be the limits of these functions for $x \to \pm\infty$. Denote by $\sigma_e(L)$ the essential spectrum of the operator L [8].

Proposition 7.1 [8]. *The essential spectrum $\sigma_e(L)$ lies in the right halfplane if and only if $b_+ < 0$ and $b_- < 0$, i.e. when the solution $\varphi(x)$ connects two saddle points.*

Proposition 7.2 [8]. *If the solution $\varphi(x)$ connects two saddle points, then it is stable.*

Since $E_1(E_0^*)$, $E_2(E_0^*)$ are saddle points, Proposition 7.1 and Proposition 7.2 state that for sufficiently small ε the solution $y(\tau, \varepsilon)$ of the equation (26) connects two saddle points. Exercise 6 in § 5.1 [8] gives the result that the solution $E(x,t)$ exponentially approximates φ by the norm in $W_p^1(\mathbb{R})$, $p \geqslant 1$. It means that for every solution $E(x,t)$ such that the norm $\|E(\cdot,0) - \varphi\|$ in $W_p^1(\mathbb{R})$, $p \geqslant 1$ is sufficiently small, there exists a real c for which $\|E(\cdot,t) - \varphi(\cdot + c + Vt)\| = O\left(\mathrm{e}^{-\beta t}\right)$, $t > 0$, $\beta > 0$. Finally, we obtain stability of the parabolic travelling wave for sufficiently large f and for arbitrary velocity V.

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