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ON TESTING VARIANCE COMPONENTS IN UNBALANCED MIXED LINEAR MODEL*

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Abstract. The paper presents some approximate and exact tests for testing variance components in general unbalanced mixed linear model. It extends the results presented by Seifert (1992) with emphasis on the computational aspects of the problem.

Keywords: unbalanced mixed linear model, variance components, Wald test, ANOVA-like test, Bartlett-Scheffé tests

MSC 2000: 62F03, 62F10

1. Introduction

In balanced models the ANOVA-method leads to exact tests. In the general unbalanced mixed linear model, when the uniformly most powerful test does not exist, the situation becomes more demanding. Here we consider two approaches to test hypotheses on variance components in such models.

The first approach is based on the distribution of the maximal invariant with respect to the group of translations in mean. We suggest to consider the exact Wald test (if it exists) and several approximate tests, in particular, the ANOVA-like test and the Zmyślony-Michalski test. We do not consider, however, the locally best invariant test in this situation. The distribution of the test statistic of the approximate tests depends on the nuisance parameters from the composite null hypothesis, so the level and the power of such tests depends on those parameters.

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The second approach is based on the reduction of the general model to the model with only two variance components. If there exists such linear transformation that reduces the model, then for testing the significance of the given variance component we can use any of the known tests for testing the variance component in a model with two variance components. According to Seifert [17] the class of such tests is called the class of Bartlett-Scheffé tests. We note that such tests are exact.

For each of those tests we provide the formulae for calculating the critical value and the power function. Under the given assumptions the test statistics are distributed as a linear combination of independent chi-square variables. The desired critical values and the power functions are calculated numerically by applying Imhof's algorithm, see [4]. Another option would be to use the algorithm due to Davies [1].

2. Mixed linear model

We consider the general mixed linear model

$$y = X\beta + U\alpha + \varepsilon, \tag{1}$$

where y is an n-vector of observations of the response variable, X is a fixed and known $n \times p$ matrix with $\operatorname{rank}(X) = k$, $k \leqslant p$, and $U = (U_1, \ldots, U_r)$ is an $n \times m$ matrix, $m = \sum_{i=1}^{r} m_i$, $\mathcal{R}(U_i) \nsubseteq \mathcal{R}(X)$, where $\mathcal{R}(A)$ denotes the linear space spanned by the columns of the matrix A, β is a k-vector of unknown fixed effects, and α and ε are uncorrelated random m- and n-vectors. Here $\alpha = (\alpha_1', \ldots, \alpha_r')'$ represents the joint vector of r random effects and ε represents the unexplained random error.

If not otherwise stated, we consider the natural ordering of random effects, i.e. $i \leqslant j$ whenever $\mathcal{R}(U_i) \subseteq \mathcal{R}(U_j)$. Throughout this paper we assume the normal distribution of random vectors. We assume $\alpha_i \sim N(0, \sigma_i^2 I_{m_i})$, $i = 1, \ldots, r$, and $\varepsilon \sim N(0, \sigma_{r+1}^2 I_n)$; then

$$E(y) = X\beta, \qquad \operatorname{Var}(y) = \sum_{i=1}^{r+1} \sigma_i^2 V_i, \tag{2}$$

where $V_i = U_i U_i'$, $i = 1, \ldots, r$, and $V_{r+1} = I$.

We will study the tests for testing statistical significance of the variance component σ_i^2 for any $i = 1, \ldots, r$, i.e. for testing the hypothesis

$$H_0\colon\ \sigma_i^2 = 0 \text{ against the alternative } H_1\colon\ \sigma_i^2 > 0 \tag{3}$$

or equivalently for testing $H_0\colon\ \theta_i = 0$ against $H_1\colon\ \theta_i > 0$, $i = 1, \ldots, r$, where $\theta_i = \sigma_i^2 / \sigma_{r+1}^2$.

3. Tests in model with more than two variance components

In this section we consider three tests of the hypothesis (3) which are based on the maximal invariant statistic (with respect to the group of transformations $y \mapsto y + X\beta$ for all $\beta \in \mathbb{R}^p$), $t = B_X y$, where B_X is an $(n-k) \times n$ matrix such that $M_X = I - XX^+ = B_X' B_X$ and $B_X B_X' = I_{n-k}$. Then

$$E(t) = 0, \quad \mathrm{Var}(t) = \sum_{i=1}^{r+1} \sigma_i^2 W_i, \tag{4}$$

where $W_i = B_X V_i B_X'$, $i = 1, \ldots, r$, and $W_{r+1} = I_{n-k}$.

3.1. Wald test

The Wald test, if it exists, leads to an exact F-test for testing the hypothesis (3). The test was introduced by Wald [18], [19] and extended by Seely and El-Bassiouni [15]. They showed that the test exists only for a small number of hypotheses in general unbalanced ANOVA models (highest interactions, highest nested effects).

Consider model (1) in the form

$$y = X\beta + U_i\alpha_i + U^*\alpha^* + \varepsilon \tag{5}$$

with $U^* = (U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_r)$ and $\alpha^* = (\alpha_1', \ldots, \alpha_{i-1}', \alpha_{i+1}', \ldots, \alpha_r')'$. Let $t = B_X y$, then

$$t = B_X U_i \alpha_i + B_X U^* \alpha^* + \varepsilon^* \tag{6}$$

with $\varepsilon^* = B_X \varepsilon$.

Let $Z = B_X U^*$ and $q = \mathrm{rank}(Z)$. Denote by B_Z an $(n-q) \times n$ matrix such that $B_Z' B_Z = M_Z = I - ZZ^+$ and $B_Z B_Z' = I_{n-q}$. Then define $z = B_Z t$ and notice that

$$z = B_Z B_X U_i \alpha_i + \varepsilon^{**} \tag{7}$$

with $\varepsilon^{**} = B_Z B_X \varepsilon$, i.e. $E(z) = 0$ and $\mathrm{Var}(z) = \sigma_i^2 B_Z B_X V_i B_X' B_Z' + \sigma_{r+1}^2 I_{n-q}$. Let P_i denote the orthogonal projector onto the linear space $\mathcal{R}(B_Z B_X U_i)$ and let $M_i = I - P_i$. Then $P_i z$ and $M_i z$ are independent random vectors and the statistic

$$F_W = \frac{z' P_i z / f_1}{z' M_i z / f_2} = \frac{t' B_Z' P_i B_Z t / f_1}{t' B_Z' M_i B_Z t / f_2} \tag{8}$$

has under H_0 a central F-distribution with f_1 and f_2 degrees of freedom, where $f_1 = \mathrm{rank}(P_i) = \mathrm{rank}(B_Z B_X U_i)$ and $f_2 = \mathrm{rank}(M_i) = n - q - f_1$. The Wald test exists if $B_X U_i$ and $B_Z(B_X U_i)$ are nonzero matrices, i.e. $f_1 \neq 0$ and $f_2 \neq 0$.

Denote $A_W = B'_Z P_i B_Z/f_1 - F_{f_1,f_2}(\alpha) B'_Z M_i B_Z/f_2$ where $F_{f_1,f_2}(\alpha)$ is the critical value of the F_{f_1,f_2} random variable such that $P(F_{f_1,f_2} > F_{f_1,f_2}(\alpha)) = \alpha$. Notice that $P(t'A_W t > 0) = P\Big(\sum_{i=1}^{h} \lambda_i \chi^2_{\nu_i} > 0\Big)$ where λ_i are the distinct non-zero eigenvalues of $A_W\Sigma$, where $\Sigma = \mathrm{Var}(t)$, ν_i are their respective multiplicities and $\chi^2_{\nu_i}$ are independent χ^2 variables with ν_i degrees of freedom. This, together with Imhof's procedure, allows to calculate the power of the test at different points in the alternative H_1.

3.2. Seifert's ANOVA-like test

Seifert [16] and Kleffe and Seifert [6] suggested the ANOVA-like test for variance components. It is an approximate test on significance level α. As the author noticed the test is heuristically motivated, leads to the optimal ANOVA-test or to Satterthwaite's approximate test in balanced situations and is asymptotically correct and optimal.

ANOVA-like test statistic is based on MINQE(U,I), the minimum norm quadratic estimator (unbiased and invariant), of the linear function of the variance components. For more details see e.g. Rao [13] and [14].

Let $\sigma^2 = (\sigma_1^2, \ldots, \sigma_r^2, \sigma_{r+1}^2)' \in \Theta$, Θ representing the parameter space, denote the vector of variance components. For a fixed prior choice σ_0^2 of σ^2, the MINQE(U,I) of the linear function $g'\sigma^2$ (g is a fixed known vector such that $g \in \mathcal{R}(K_{UI})$, i.e. there exists a vector λ such that $g = K_{UI}\lambda$) is given by

$$\widehat{g'\sigma^2} = g' K_{UI}^- q = \lambda' q, \tag{9}$$

where K_{UI}^- is a g-inverse of the MINQE(U,I) criteria matrix K_{UI} defined by the elements

$$\{K_{UI}\}_{ij} = \mathrm{tr}\big(\Sigma_0^{-1} W_i \Sigma_0^{-1} W_j\big), \qquad i,j = 1,\ldots,r+1, \tag{10}$$

and $q = (q_1, \ldots, q_{r+1})'$ is the MINQE(U,I) vector of quadratics with

$$q_i = t' A_i t, \qquad i = 1,\ldots,r+1, \tag{11}$$

where $A_i = \Sigma_0^{-1} W_i \Sigma_0^{-1}$. The symbol $\mathrm{tr}(A)$ denotes the trace of the matrix A and $\Sigma_0 = \Sigma(\sigma_0^2) = \sum_{i=1}^{r+1} \sigma_{i\,0}^2 W_i$. Here, we implicitly assume that the inverse Σ_0^{-1} exists.

Notice that $\widehat{g'\sigma^2}$ is an unbiased estimator of $g'\sigma^2$, and under normality assumptions we have

$$\mathrm{Var}(\widehat{g'\sigma^2}) = 2 g' K_{UI}^- g = 2\lambda' K_{UI} \lambda \tag{12}$$

locally at $\sigma^2 = \sigma_0^2$.

Let $\sigma_0^2 \in H_0$ be a fixed vector of priors, i.e. $\sigma^2_{i\,0} = 0$. Assume for simplicity that the inverse matrix K_{UI}^{-1} exists. Then $z = L\hat{\sigma}^2$ denotes the vector of locally uncorrelated linear combinations of $\hat{\sigma}^2$—the MINQE(U,I) of σ^2. Here L is an upper triangular matrix with unit main diagonal and such that locally at σ_0^2

$$\operatorname{Var}(z) = L\operatorname{Var}(\hat{\sigma}^2)L' = 2LK_{UI}^{-1}L' = D, \tag{13}$$

where $D = \operatorname{Diag}(D_{ii})$, $i = 1, \ldots, r+1$, is a diagonal matrix. We note that L could be obtained by Cholesky decomposition of K_{UI}.

For testing $H_0\colon\ \sigma_i^2 = 0$, Seifert [16] proposed the test statistic based on the ratio of locally uncorellated functions of $\hat{\sigma}^2$:

$$F_S = \frac{z_i}{z_i - \hat{\sigma}_i^2} = \frac{\hat{\sigma}_i^2 + \sum_{j=i+1}^{r+1} L_{ij}\hat{\sigma}_j^2}{\sum_{j=i+1}^{r+1} L_{ij}\hat{\sigma}_j^2}. \tag{14}$$

By construction the local covariance of the numerator and denominator is zero. Seifert suggested to reject H_0 for large values of F_S. Witkovský [21] suggested the critical region defined by

$$z_i - c_\alpha(z_i - \hat{\sigma}_i^2) > 0, \tag{15}$$

where c_α is the critical value of the distribution such that $P(z_i - c_\alpha(z_i - \hat{\sigma}_i^2) > 0) = \alpha$, under the assumption that true $\sigma^2 = \sigma_0^2 \in H_0$.

Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)'$, then $z_i = e_i'LK_{UI}^{-1}q$, $z_i - \hat{\sigma}_i^2 = e_i'(L - I)K_{UI}^{-1}q$ and $z_i - c_\alpha(z_i - \hat{\sigma}_i^2) = e_i'(L - c_\alpha(L - I))K_{UI}^{-1}q$. If moreover $A_S = \sum_{j=1}^{r+1} \kappa_j A_j$ with $\kappa = e_i'(L - c_\alpha(L - I))K_{UI}^{-1}$ and $A_j = \Sigma_0^{-1}W_j\Sigma_0^{-1}$, $j = 1, \ldots, r+1$, then the critical region is given by the inequality $t'A_St > 0$. Further, $P(t'A_St > 0) = P\Big(\sum_{i=1}^{h} \lambda_i\chi^2_{\nu_i} > 0\Big)$, where λ_i denote distinct non-zero eigenvalues of $A_S\Sigma$, $\Sigma = \operatorname{Var}(t)$, ν_i are their respective multiplicities, and $\chi^2_{\nu_i}$ are independent χ^2 variables with ν_i degrees of freedom. Imhof's procedure allows to calculate the critical value c_α, the level of the test for any fixed point in H_0, as well as the power of the test at an arbitrary point from the alternative H_1.

3.3. Zmyślony-Michalski test

Michalski and Zmyślony [10] proposed a test of the hypothesis (3) based on the decomposition of an unbiased and invariant estimator of σ_i^2, $i = 1, \ldots, r$. Let $t'At$

be such estimator (e.g. the MINQE(U,I) of σ_i^2). Let A be decomposed as $A_+ - A_-$, where A_+ and A_- are both nonnegative definite and nonzero matrices. Since $t'At$ is an unbiased estimator, then

$$E(t'A_+t) = E(t'A_-t) + \sigma_i^2, \tag{16}$$

and under $H_0\colon\ \sigma_i^2 = 0$ we get $E(t'A_+t) = E(t'A_-t)$. The Zmyślony-Michalski test rejects the null hypothesis for large values of

$$F_{ZM} = \frac{t'A_+t}{t'A_-t}. \tag{17}$$

The authors have derived the distribution of F_{ZM} under the null hypothesis under the specific condition that the matrices W_i commute and are linearly independent. In general, we can derive the critical region of an approximate test which is locally on a significance level α, based on F_{ZM}. Let $\Sigma_0 = \Sigma(\sigma_0^2)$ be a fixed matrix such that $\sigma_0^2 \in H_0$, then reject H_0 for

$$t'A_+t - c_\alpha t'A_-t > 0 \tag{18}$$

where c_α is such that under the assumption $\mathrm{Var}(t) = \Sigma_0$ we get

$$P(t'A_+t - c_\alpha t'A_-t > 0) = \alpha.$$

Denote $A_{ZM} = A_+ - c_\alpha A_-$, then $P(t'A_{ZM}t > 0) = P\Big(\sum_{i=1}^{h} \lambda_i \chi^2_{\nu_i} > 0\Big)$ where λ_i are the distinct non-zero eigenvalues of $A_{ZM}\Sigma$, $\Sigma = \mathrm{Var}(t)$, ν_i are the respective multiplicities and $\chi^2_{\nu_i}$ are independent χ^2 variables with ν_i degrees of freedom. Imhof's procedure allows to calculate the critical value c_α, the level of the test for any fixed point in H_0, as well as the power of the test at an arbitrary point from the alternative H_1.

4. Tests in model with two variance components

The model with two variance components is a special case of the general model (1) with $r = 1$. In particular, we consider a model

$$y = X\beta + U\alpha + \varepsilon \tag{19}$$

with independent random vectors $\alpha \sim N(0, \sigma_1^2 I_m)$ and $\varepsilon \sim N(0, \sigma^2 I_n)$. The maximal invariant $t = B_X y$ is distributed as $t \sim N(0, \sigma_1^2 W + \sigma^2 I_{n-k})$, where $W = B_X V B_X'$, $V = UU'$. In this setup, we are interested in testing the hypothesis

$$H_0\colon\ \theta = 0 \quad \text{against} \quad H_1\colon\ \theta > 0, \tag{20}$$

where $\theta = \sigma_1^2/\sigma^2$. Notice that this is equivalent with testing $H_0\colon \sigma_1^2 = 0$ against $H_1\colon \sigma_1^2 > 0$. We note that Lin and Harville [8] considered testing a generalized hypothesis $H_0\colon \theta = \theta_0$ against $H_1\colon \theta \in \Theta_*$, where Θ_* is a general interval $\langle \theta_l, \theta_u)$.

Let $\lambda_1 > \lambda_2 > \ldots > \lambda_h \geqslant 0$ be h distinct eigenvalues of the matrix W with their respective multiplicities $\nu_1, \ldots, \nu_h$. The spectral decomposition of W is $W = \sum_{i=1}^{h} \lambda_i Q_i$, where $Q_i = E_i E_i'$, and E_i is a matrix of orthonormal eigenvectors corresponding to the eigenvalue λ_i.

Olsen, Seely and Birkes [12] derived the minimal sufficient statistic for the family of distributions of the maximal invariant t: It is a set of h independent quadratics $Z_i = t'Q_i t/\nu_i$ such that $\nu_i Z_i/(\sigma_1^2\lambda_i + \sigma^2) \sim \chi^2_{\nu_i}$. Moreover, $Z = (Z_1, \ldots, Z_h)'$ is a complete statistic if and only if $h = 2$.

The tests suggested for testing variance components in the general model are valid also in the model with two variance components. Moreover, there are other tests and theoretical results which are valid only for the model with two variance components. The present section gives a brief overview of the tests and their distributions. We note that the tests are exact. For more details see [7], [9], [20], [8], [2] and [5].

4.1. Neyman-Pearson test

The Neyman-Pearson test is the optimum test (most powerful test) for testing $H_0\colon \theta = 0$ against the simple alternative $H_1\colon \theta = \theta_*$, $\theta_* > 0$. The NP test is based on the test statistic

$$F_{NP}(\theta_*) = \frac{t't}{t'(I + \theta_* W)^{-1}t} = \frac{\sum_{i=1}^{h} \nu_i Z_i}{\sum_{i=1}^{h} \nu_i Z_i/(\theta_*\lambda_i + 1)}. \tag{21}$$

The NP test rejects the null hypothesis if $F_{NP}(\theta_*) > c_\alpha(\theta_*)$, where $c_\alpha(\theta_*)$ is a critical value such that

$$P\left(\sum_{i=1}^{h} \left(1 - \frac{c_\alpha(\theta_*)}{(\theta_*\lambda_i + 1)}\right) \chi^2_{\nu_i} > 0\right) = \alpha, \tag{22}$$

and $\chi^2_{\nu_i}$ are independent chi-square variables. Under the alternative $H_1\colon \theta > 0$ the power $\beta_{NP}(\theta|\theta_*)$ of the test is

$$\beta_{NP}(\theta|\theta_*) = P\left(\sum_{i=1}^{h} \left(1 - \frac{c_\alpha(\theta_*)}{\theta_*\lambda_i + 1}\right) (\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\right). \tag{23}$$

The limiting properties of the NP test are of some interest:

$$\lim_{\theta_* \to \theta_0 = 0} F_{NP}(\theta_*) = \frac{\sum_{i=1}^{h} \lambda_i \nu_i Z_i}{\sum_{i=1}^{h} \nu_i Z_i}, \tag{24}$$

and, moreover,

$$\lim_{\theta_* \to \infty} F_{NP}(\theta_*) = 1 + \frac{\sum_{i=1}^{h-1} \nu_i Z_i}{\nu_h Z_h} \qquad \text{if } \lambda_h = 0, \tag{25}$$

$$\lim_{\theta_* \to \infty} F_{NP}(\theta_*)/\theta_* = \frac{\sum_{i=1}^{h} \nu_i Z_i}{\sum_{i=1}^{h} \nu_i Z_i/\lambda_i} \qquad \text{if } \lambda_h \neq 0. \tag{26}$$

For more details see [20] and [8].

4.2. UMPI and LBI tests

Mathew [9] and Westfall [20] derived an optimum test for testing (20). They noticed that the testing problem is invariant under the group of transformations $y \mapsto c(y + X\beta)$ for arbitrary $c > 0$ and β, and the maximal invariant is $t/\|t\|$. Gnot, Jankowiak-Rosłanowska and Michalski [2] proved that a necessary and sufficient condition that guarantees the existence of the UMPI test (uniformly most powerful invariant test) is that $h = 2$, h being the number of different eigenvalues of W.

There are two important cases to distinguish, see [5, Theorem 6.2.2]:

a) the nonzero eigenvalues of W are all equal, and

b) the nonzero eigenvalues of W are not equal.

Let $\nu = \text{rank}(W)$ and suppose $0 < \nu < n - k$, i.e. W is singular. If the nonzero eigenvalues of W are all equal to $\lambda_1 > 0$ then the UMPI test rejects H_0 for large values of

$$F_{UMPI} = \frac{n-k-\nu}{\nu} \frac{t'Q_1 t}{t'(I - Q_1)t}. \tag{27}$$

Under H_0, $F_{UMPI} \sim F_{\nu,n-k-\nu}$. The test rejects H_0 on a significance level α if $F_{UMPI} > F_{\nu,n-k-\nu}(\alpha)$, where $F_{\nu,n-k-\nu}(\alpha)$ is the critical value of the $F_{\nu,n-k-\nu}$ distribution and $P(F_{\nu,n-k-\nu} > F_{\nu,n-k-\nu}(\alpha)) = \alpha$. Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_{UMPI}(\theta)$ of the test is

$$\beta_{UMPI}(\theta) = P\big((n-k-\nu)(\theta\lambda_1 + 1)\chi^2_\nu - \nu F_{\nu,n-k-\nu}(\alpha)\chi^2_{n-k-\nu} > 0\big), \tag{28}$$

where χ^2_ν and $\chi^2_{n-k-\nu}$ are independent chi-square variables.

If the nonzero eigenvalues of W are not all equal then the LBI test (locally best invariant test) rejects H_0 for large values of

$$(29)\qquad F_{LBI} = \frac{t'Wt}{t't} = \frac{t'(\sum_{i=1}^{h} \lambda_i Q_i)t}{t'(\sum_{i=1}^{h} Q_i)t} = \frac{\sum_{i=1}^{h} \lambda_i \nu_i Z_i}{\sum_{i=1}^{h} \nu_i Z_i}.$$

The LBI test rejects the null hypothesis if $F_{LBI} > c_\alpha$, where c_α is a critical value such that

$$(30)\qquad P\left(\sum_{i=1}^{h} (\lambda_i - c_\alpha)\chi^2_{\nu_i} > 0\right) = \alpha,$$

and $\chi^2_{\nu_i}$ denote independent chi-square variables. Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_{LBI}(\theta)$ of the test is

$$(31)\qquad \beta_{LBI}(\theta) = P\left(\sum_{i=1}^{h} (\lambda_i - c_\alpha)(\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\right).$$

Westfall [20] proved that the NP test is equivalent to the LBI test as θ_* approaches 0, see (24).

4.3. Wald test

In the model with two variance components (19) and under the assumption that W is a singular matrix, i.e. $\lambda_h = 0$, the Wald test statistic (8) becomes

$$(32)\qquad F_W = \frac{t'Pt/f_1}{t'Mt/f_2} = \frac{f_2}{f_1} \frac{\sum_{i=1}^{h-1} \nu_i Z_i}{\nu_h Z_h},$$

$P = B_X U(U'B'_X B_X U)^- U'B'_X$ is the orthogonal projector onto $\mathcal{R}(B_X U) = \mathcal{R}(W)$ and $M = I - P$, $f_1 = \operatorname{rank}(P) = \operatorname{rank}(W) = \sum_{i=1}^{h-1} \nu_i$ and $f_2 = n - k - f_1 = \nu_h$.

The Wald test rejects the null hypothesis if $F_W > F_{f_1,f_2}(\alpha)$. Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_W(\theta)$ of the test is

$$(33)\qquad \beta_W(\theta) = P\left(f_2 \sum_{i=1}^{h-1} (\theta\lambda_i + 1)\chi^2_{\nu_i} - F_{f_1,f_2}(\alpha) f_1 \chi^2_{\nu_h} > 0\right).$$

Notice that according to (25) the Wald test is equivalent to the limit case of the NP test for $\theta_* \to \infty$. Mathew [9] noticed that the Wald test is equivalent to the UMPI test if $h = 2$.

4.4. Modified Wald tests

4.4.1. Lin-Harville test. According to (26), it is natural to consider the test based on the statistic

$$F_{LH} = \frac{\sum_{i=1}^{h} \nu_i Z_i}{\sum_{i=1}^{h} \nu_i Z_i / \lambda_i} \tag{34}$$

as a modification of the Wald test provided $\lambda_h > 0$. In such a case the test rejects the null hypothesis if $F_{LH} > c_\alpha$, and c_α is a critical value such that

$$P\left(\sum_{i=1}^{h} (1 - c_\alpha/\lambda_i)\chi^2_{\nu_i} > 0\right) = \alpha \tag{35}$$

and under the alternative $H_1\colon\ \theta > 0$ the power $\beta_{LH}(\theta)$ of the test is

$$\beta_{LH}(\theta) = P\left(\sum_{i=1}^{h} (1 - c_\alpha/\lambda_i)(\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\right). \tag{36}$$

4.4.2. LaMotte-McWhorter test. LaMotte, McWhorter and Prasad [7] suggested a modification of the Wald test based on the test statistic

$$F_{LM} = \frac{\sum_{i=h^*+1}^{h} \nu_i}{\sum_{i=1}^{h^*} \nu_i} \, \frac{\sum_{i=1}^{h^*} \nu_i Z_i}{\sum_{i=h^*+1}^{h} \nu_i Z_i}, \tag{37}$$

where h^* is a chosen number from $1, \ldots, h-1$. Notice that by choosing $h^* = h - 1$ the test statistic coincides with the F_W statistic. The test is well defined for both cases: $\lambda_h = 0$ and also for $\lambda_h > 0$.

Under $H_0\colon\ \theta = 0$ the LM test rejects the null hypothesis if $F_{LM} > F_{f_1,f_2}(\alpha)$, with $f_1 = \sum_{i=1}^{h^*} \nu_i$ and $f_2 = \sum_{i=h^*+1}^{h} \nu_i$. Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_{LM}(\theta)$ of the test is

$$\beta_{LM}(\theta) = P\left(f_2 \sum_{i=1}^{h^*} (\theta\lambda_i + 1)\chi^2_{\nu_i} - F_{f_1,f_2}(\alpha) f_1 \sum_{i=h^*+1}^{h} (\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\right). \tag{38}$$

4.4.3. Gnot-Michalski test. Gnot and Michalski [3] suggested a modified Wald test which is based on the ratio of the non-negative admissible invariant quadratic and unbiased estimators of $\varrho_U \sigma_1^2 + \sigma^2$ and $\varrho_L \sigma_1^2 + \sigma^2$, where

$$\varrho_L = \begin{cases} \dfrac{(\lambda_1 \operatorname{tr}(W^+) - \operatorname{rank}(W))}{(\lambda_1 \operatorname{tr}(W^+ W^+) - \operatorname{tr}(W^+))} & \text{for } \lambda_h > 0, \\ 0 & \text{for } \lambda_h = 0, \end{cases} \tag{39}$$

and

$$\varrho_U = \frac{(\operatorname{tr}(W^2) - \lambda_h \operatorname{tr}(W))}{(\operatorname{tr}(W) - \lambda_h \operatorname{rank}(W))}. \tag{40}$$

Notice that such a test exists even for W nonsingular. The test statistic is then

$$F_{GM} = \begin{cases} \dfrac{\sum_{i=1}^{h} (\lambda_i - \lambda_h)\nu_i Z_i}{\sum_{i=1}^{h} (\lambda_1 - \lambda_i)\nu_i Z_i} & \text{for } \lambda_h > 0, \\ \\ \dfrac{\sum_{i=1}^{h-1} \lambda_i \nu_i Z_i}{\nu_h Z_h} & \text{for } \lambda_h = 0. \end{cases} \tag{41}$$

The test rejects the null hypothesis for $F_{GM} > c_\alpha$, where c_α is such that under H_0 we get

$$P\left(\sum_{i=1}^{h} \left((\lambda_i - \lambda_h) - c_\alpha (\lambda_1 - \lambda_i) \right) \chi^2_{\nu_i} > 0 \right) = \alpha \tag{42}$$

if $\lambda_h > 0$, and

$$P\left(\sum_{i=1}^{h-1} \lambda_i \chi^2_{\nu_i} - c_\alpha \chi^2_{\nu_h} > 0 \right) = \alpha \tag{43}$$

if $\lambda_h = 0$.

Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_{GM}(\theta)$ of the test is

$$\beta_{GM}(\theta) = P\left(\sum_{i=1}^{h} \left((\lambda_i - \lambda_h) - c_\alpha (\lambda_1 - \lambda_i) \lambda_i^{-2} \right) (\theta \lambda_i + 1) \chi^2_{\nu_i} > 0 \right) \tag{44}$$

for $\lambda_h > 0$, and

$$\beta_{GM}(\theta) = P\left(\sum_{i=1}^{h-1} \lambda_i (\theta \lambda_i + 1) \chi^2_{\nu_i} - c_\alpha \chi^2_{\nu_h} > 0 \right) \tag{45}$$

for $\lambda_h = 0$.

4.5. ANOVA-like test

In the model with two variance components, the MINQE(U,I) criteria matrix (10) has the form

$$K_{UI} = \frac{1}{\sigma_0^4}\begin{pmatrix} \operatorname{tr}(W^2) & \operatorname{tr}(W) \\ \operatorname{tr}(W) & n-k \end{pmatrix}, \tag{46}$$

where $\operatorname{tr}(W^2) = \operatorname{tr}\Big(\sum\limits_{i=1}^{h} \lambda_i^2 Q_i\Big) = \sum\limits_{i=1}^{h} \lambda_i^2 \nu_i$ and $\operatorname{tr}(W) = \sum\limits_{i=1}^{h} \lambda_i \nu_i$. Then the matrix

$$L = \begin{pmatrix} 1 & \dfrac{\operatorname{tr}(W)}{\operatorname{tr}(W^2)} \\ 0 & 1 \end{pmatrix} \tag{47}$$

fulfils the required condition $2LK_{UI}^{-1}L' = D$, where D is a diagonal matrix. Notice that L does not depend on σ_0^2. Then, considering MINQE(U,I) of $(\sigma_1^2, \sigma^2)'$,

$$(\hat\sigma_1^2, \hat\sigma^2)' = K_{UI}^{-1} q = \frac{1}{\mathrm{Det}}\begin{pmatrix} n-k & -\operatorname{tr}(W) \\ -\operatorname{tr}(W) & \operatorname{tr}(W^2) \end{pmatrix}\begin{pmatrix} t'Wt \\ t't \end{pmatrix}, \tag{48}$$

where $\mathrm{Det} = (n-k)\operatorname{tr}(W^2) - \operatorname{tr}(W)^2$. By solving $z = L(\hat\sigma_1^2, \hat\sigma^2)'$ we get

$$\begin{aligned} z_1 &= \frac{1}{\operatorname{tr}(W^2)} t'Wt, \\ z_1 - \hat\sigma_1^2 &= \frac{1}{\mathrm{Det}}\left(\operatorname{tr}(W)t't - \frac{\operatorname{tr}(W)^2}{\operatorname{tr}(W^2)} t'Wt\right). \end{aligned} \tag{49}$$

Notice that z_1 is a nonnegative definite quadratic form in t. The ANOVA-like test statistic for testing $H_0\colon\ \theta = 0$ against $H_1\colon\ \theta > 0$, $\theta = \sigma_1^2/\sigma^2$, is then given by

$$F_S = \frac{z_1}{z_1 - \hat\sigma_1^2} = \frac{\sum\limits_{i=1}^{h} a\lambda_i\nu_i Z_i}{\sum\limits_{i=1}^{h} (b - c\lambda_i)\nu_i Z_i}, \tag{50}$$

where

$$\begin{aligned} a &= \mathrm{Det} = (n-k)\sum_{i=1}^{h} \lambda_i^2\nu_i - \left(\sum_{i=1}^{h} \lambda_i\nu_i\right)^2, \\ b &= \operatorname{tr}(W)\operatorname{tr}(W^2) = \left(\sum_{i=1}^{h} \lambda_i\nu_i\right) \times \left(\sum_{i=1}^{h} \lambda_i^2\nu_i\right), \\ c &= \left(\sum_{i=1}^{h} \lambda_i\nu_i\right)^2. \end{aligned}$$

The test rejects the null hypothesis for $F_S > c_\alpha$, where c_α is such that under H_0 we get

$$P\Big(\sum_{i=1}^{h}\big(a\lambda_i - c_\alpha(b - c\lambda_i)\big)\chi^2_{\nu_i} > 0\Big) = \alpha. \tag{51}$$

Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_S(\theta)$ of the test is

$$\beta_S(\theta) = P\Big(\sum_{i=1}^{h}\big(a\lambda_i - c_\alpha(b - c\lambda_i)\big)(\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\Big). \tag{52}$$

Witkovský [21] proved that the ANOVA-like test with the critical region $z_1 - c_\alpha(z_1 - \hat{\sigma}_1^2) > 0$ is equivalent to the optimum test (UMPI test if it exists or LBI test, otherwise) for testing (20).

4.6. Zmyślony-Michalski test

Let $t'At$ be a MINQE(U,I) of σ_1^2, see (48). Let $A = A_+ - A_-$. Then the ZM test statistic is

$$F_{ZM} = \frac{t'A_+t}{t'A_-t} = \frac{\sum_{\lambda_i^* > 0}\lambda_i^*\nu_i Z_i}{\sum_{\lambda_i^* < 0} -\lambda_i^*\nu_i Z_i}, \tag{53}$$

where $\lambda_i^* = \lambda_i - \operatorname{tr}(W)/\operatorname{rank}(W) = \lambda_i - \Big(\sum_{i=1}^{h}\lambda_i\nu_i\Big)\Big/\Big(\sum_{i=1}^{h}\nu_i\Big)$. The test rejects the null hypothesis for $F_{ZM} > c_\alpha$, where c_α is such that under H_0 we get

$$P\Big(\sum_{\lambda_i^* > 0}\lambda_i^*\chi^2_{\nu_i} + c_\alpha\sum_{\lambda_i^* < 0}\lambda_i^*\chi^2_{\nu_i} > 0\Big) = \alpha. \tag{54}$$

Under the alternative $H_1\colon\ \theta > 0$ the power $\beta_S(\theta)$ of the test is

$$\beta_{ZM}(\theta) = P\Big(\sum_{\lambda_i^* > 0}\lambda_i^*(\theta\lambda_i + 1)\chi^2_{\nu_i} + c_\alpha\sum_{\lambda_i^* < 0}\lambda_i^*(\theta\lambda_i + 1)\chi^2_{\nu_i} > 0\Big). \tag{55}$$

Seifert [17] proposed a new class of exact Bartlett-Scheffé tests for variance components. The basic idea is to find a linear transformation of the general model to a model with just two variance components. Then, standard techniques for testing in the model with two variance components can be applied.

Consider the general model (1). Suppose that we want to test $H_0\colon\ \sigma_1^2 = 0$ against $H_1\colon\ \sigma_1^2 > 0$, otherwise reorganize the ordering of the effects. Let T be a matrix such that

$$w = Ty = \tilde{U}_1\alpha_1 + \tilde{\alpha}, \tag{56}$$

where $\alpha_1 \sim N(0, \sigma_1^2 I)$ and $\tilde{\alpha} \sim N(0, \tilde{\sigma}^2 I)$ are independent random effects and $\tilde{\sigma}^2 = c_2\sigma_2^2 + \ldots + c_{r+1}\sigma_{r+1}^2$ for some coefficients $c_2, \ldots, c_{r+1}$. Notice that $w \sim N(0, \sigma_1^2 W + \tilde{\sigma}^2 I)$, where $W = \tilde{U}_1\tilde{U}_1'$.

We can use the above tests for testing the hypothesis $H_0\colon\ \tilde{\theta} = 0$ against $H_1\colon\ \tilde{\theta} > 0$, where $\tilde{\theta} = \sigma_1^2/\tilde{\sigma}^2$, in the model (56). Those tests are exact Bartlett-Scheffé tests for testing $H_0\colon\ \sigma_1^2 = 0$ against $H_1\colon\ \sigma_1^2 > 0$ in the original model (1).

5.1. Algorithm for reduction of the model

The following stepwise procedure reduces the model by one variance component in each step.

Consider model (1), $y = X\beta + U_1\alpha_1 + \ldots + U_r\alpha_r + \varepsilon$, with $r+1$ variance components. Let us introduce a step-counter m, and set $m = 1$. The algorithm starts with an $n^{(m)}$-dimensional maximal invariant $t^{(m)} = B_X y$, where B_X is a full rank matrix such that $B_X' B_X = M_X = I - XX^+$ and $B_X B_X' = I_{n^{(m)}}$, $n^{(m)} = n - k$, $k = \mathrm{rank}(X)$.

Denote $T^{(m)} = B_X$ and further $U_i^{(m)} = B_X U_i$, $V_i^{(m)} = U_i^{(m)} U_i^{(m)\prime}$ and $\sigma_i^{2(m)} = \sigma_i^2$ for $i = 1, \ldots, \kappa^{(m)}$, where $\kappa^{(m)} = r + 1$ is the number of variance components; then

$$t^{(m)} = U_1^{(m)}\alpha_1 + \ldots + U_{\kappa^{(m)}-1}^{(m)}\alpha_{\kappa^{(m)}-1} + U_{\kappa^{(m)}}^{(m)}\varepsilon \tag{57}$$

and $t^{(m)} \sim N\big(0, \sigma_1^{2(m)} V_1^{(m)} + \ldots + \sigma_{\kappa^{(m)}-1}^{2(m)} V_{\kappa^{(m)}-1}^{(m)} + \sigma^{2(m)} I_{n^{(m)}}\big)$ with $m = 1$.

After m steps the algorithm proceeds as follows:

Step 0

Compute and remember the vector of *'residuals'* which could be useful later in Step 5 of the algorithm: Let

$$M^{(m)} = I_{n^{(m)}} - P_{\left[U_1^{(m)}, \ldots, U_{\kappa^{(m)}-1}^{(m)}\right]}, \tag{58}$$

and let $B^{(m)}$ be a matrix such that $M^{(m)} = B^{(m)\prime} B^{(m)}$ and $B^{(m)} B^{(m)\prime} = I_{f^{(m)}}$, $f^{(m)} = \operatorname{rank}\big(M^{(m)}\big)$. Then the vector of residuals is defined as

$$\gamma^{(m)} = B^{(m)} t^{(m)}, \tag{59}$$

and we have $\gamma^{(m)} \sim N\left(0, \sigma^{2(m)}_{\kappa^{(m)}} I_{f^{(m)}}\right)$.

Step 1

The algorithm succeeded if the number of variance components $\kappa^{(m)} = 2$ and if $\operatorname{rank}\left(U_1^{(m)}\right) > 0$.

We note that if $\operatorname{rank}\left(U_1^{(m)}\right) = n^{(m)}$, the Wald test does not exist.

Step 2

The algorithm failed if $\operatorname{rank}\left(U_1^{(m)}\right) = 0$.

Step 3

If there is such i_0, $i_0 \in \{2, \ldots, \kappa^{(m)} - 1\}$ that $\mathcal{R}\big(U_{i_0}^{(m)}\big) \subseteq \mathcal{R}\big(U_1^{(m)}\big)$, then use the $n^{(m+1)}$-dimensional vector

$$t^{(m+1)} = B_{U_{i_0}^{(m)}} t^{(m)}, \tag{60}$$

where $B_{U_{i_0}^{(m)}}$ is a full rank matrix such that $M_{U_{i_0}^{(m)}} = B'_{U_{i_0}^{(m)}} B_{U_{i_0}^{(m)}}$ and $B_{U_{i_0}^{(m)}} B'_{U_{i_0}^{(m)}} = I_{n^{(m+1)}}$. Reduce the number of variance components to $\kappa^{(m+1)} = \kappa^{(m)} - 1$ and denote by $\sigma_i^{2(m+1)}$ the remaining variance components for $i = 1, \ldots, \kappa^{(m+1)}$, and $U_i^{(m+1)} = B_{U_{i_0}^{(m)}} U_i^{(m)}$, $V_i^{(m+1)} = U_i^{(m+1)} U_i^{(m+1)\prime}$; then

$$t^{(m+1)} \sim N\left(0, \sum_{i=1}^{\kappa^{(m+1)}} \sigma_i^{2(m+1)} V_i^{(m+1)}\right). \tag{61}$$

Notice that $\sigma_1^{2(m+1)} = \sigma_1^2$ and $V^{(m+1)}_{\kappa^{(m+1)}} = I_{n^{(m+1)}}$.

Finally, denote

$$T^{(m+1)} = B_{U_{i_0}^{(m)}} T^{(m)}, \tag{62}$$

set $m := m + 1$ and restart the algorithm.

Step 4

If there is such i_0, $i_0 \in \{2, \ldots, \kappa^{(m)} - 1\}$ that $U_{i_0}^{(m)}$ is not comparable with $U_1^{(m)}$, i.e. neither $\mathcal{R}\big(U_{i_0}^{(m)}\big) \subset \mathcal{R}\big(U_1^{(m)}\big)$ nor $\mathcal{R}\big(U_{i_0}^{(m)}\big) \supset \mathcal{R}\big(U_1^{(m)}\big)$, use an $n^{(m+1)}$-dimensional vector

$$t^{(m+1)} = B_{U_{i_0}^{(m)}} t^{(m)}, \tag{63}$$

where $B_{U_{i_0}^{(m)}}$ is a full rank matrix such that $M_{U_{i_0}^{(m)}} = B'_{U_{i_0}^{(m)}} B_{U_{i_0}^{(m)}}$ and $B_{U_{i_0}^{(m)}} B'_{U_{i_0}^{(m)}} = I_{n^{(m+1)}}$. Reduce the number of variance components to $\kappa^{(m+1)} = \kappa^{(m)} - 1$ and denote by $\sigma_i^{2(m+1)}$ the remaining variance components for $i = 1, \ldots, \kappa^{(m+1)}$, and $U_i^{(m+1)} = B_{U_{i_0}^{(m)}} U_i^{(m)}$, $V_i^{(m+1)} = U_i^{(m+1)} U_i^{(m+1)'}$; then

$$t^{(m+1)} \sim N\Bigl(0, \sum_{i=1}^{\kappa^{(m+1)}} \sigma_i^{2(m+1)} V_i^{(m+1)}\Bigr), \tag{64}$$

and notice that $\sigma_1^{2(m+1)} = \sigma_1^2$ and $V_{\kappa^{(m+1)}}^{(m+1)} = I_{n^{(m+1)}}$.

Finally, denote

$$T^{(m+1)} = B_{U_{i_0}^{(m)}} T^{(m)}, \tag{65}$$

set $m := m + 1$ and restart the algorithm.

Remark. There is no unique method for handling cross-classified effects. As pointed out by Seifert [17] an alternative step would be

$$t^{(m+1)} = B_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} t^{(m)}, \tag{66}$$

where $B_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]}$ is a full rank matrix such that

$$P_{U_{i_0}^{(m)}} + M_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} = B'_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} B_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]}, \tag{67}$$

where $P_{U_{i_0}}^{(m)}$ is an orthonormal projector onto $\mathcal{R}(U_{i_0}^{(m)})$ and $M_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]}$ is an orthonormal projector onto the null space of $\left[U_1^{(m)}, U_{i_0}^{(m)}\right]'$, and

$$B_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} B'_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} = I_{n^{(m+1)}}.$$

This formula uses intra-block information about the effect i_0 and makes that effect nested in the first effect. The second approach is suggested if the true $\sigma_{i_0}^{2(m)}$ is small. Here $T^{(m+1)} = B_{\left[U_1^{(m)}, U_{i_0}^{(m)}\right]} T^{(m)}$.

Step 5

If there is such i_0, $i_0 \in \{2, \ldots, \kappa^{(m)} - 1\}$ that $\mathcal{R}\bigl(U_{i_0}^{(m)}\bigr) \subseteq \mathcal{R}\bigl(U_i^{(m)}\bigr)$ holds true only for $i = \kappa^{(m)}$ (notice that $\mathcal{R}\bigl(U_{\kappa^{(m)}}^{(m)}\bigr) = \mathcal{R}(I_{n^{(m)}})$), compute

$$c_{i_0}^{(m)} = \lambda_{\max}\bigl(V_{i_0}^{(m)+}\bigr), \tag{68}$$

the maximal eigenvalue of the Moore-Penrose g-inverse of $V_{i_0}^{(m)}$. Let $P_{U_{i_0}^{(m)}}$ denote the orthogonal projector onto $\mathcal{R}\big(U_{i_0}^{(m)}\big)$ and let B_P denote a matrix such that $P_{U_{i_0}^{(m)}} = B_P' B_P$ and $B_P B_P' = I_f$, where $f = \operatorname{rank}(P_{U_{i_0}^{(m)}})$. We assume that $f^{(m)} \geqslant f$, then compute

$$t = B_P t^{(m)} + \eta^{(m)}, \tag{69}$$

where $\eta^{(m)}$ denotes the artificial vector of disturbances,

$$\eta^{(m)} = \left(B_P \left(c_{i_0}^{(m)} V_{i_0}^{(m)} - I_{n^{(m)}}\right) B_P'\right)^{\frac{1}{2}} D\gamma^{(m)} \tag{70}$$

with $D = [I_f \,\vdots\, 0_{f,f^{(m)}-f}]$, and $A^{\frac{1}{2}}$ denotes a matrix such that $A^{\frac{1}{2}} A^{\frac{1}{2}'} = A$. $B_P t^{(m)}$ and $\eta^{(m)}$ are independent random vectors with the distribution

$$B_P t^{(m)} \sim N\bigg(0, \sum_{i=1}^{\kappa^{(m)}} \sigma_i^{2(m)} B_P V_i^{(m)} B_P'\bigg), \tag{71}$$

$$\eta^{(m)} \sim N\left(0, c_{i_0}^{(m)} \sigma_{\kappa^{(m)}}^{2(m)} B_P V_{i_0}^{(m)} B_P' - \sigma_{\kappa^{(m)}}^{2(m)} I_f\right). \tag{72}$$

By adding noise to $B_P t^{(m)}$ we have reduced the number of variance components by one, and

$$\operatorname{Var}(t) = \sum_{i \neq i_0}^{\kappa^{(m)}-1} \sigma_i^{2(m)} B_P V_i^{(m)} B_P' + \left(\sigma_{i_0}^{2(m)} + c_{i_0}^{(m)} \sigma_{\kappa^{(m)}}^{2(m)}\right) B_P V_{i_0}^{(m)} B_P'. \tag{73}$$

Further, compute

$$t^{(m+1)} = Bt, \tag{74}$$

where B is such that $B B_P V_{i_0}^{(m)} B_P' B' = I_{n^{m+1}}$. Denote the new number of variance components by $\kappa^{(m+1)} = \kappa^{(m)} - 1$ and rename and denote by $\sigma_i^{2(m+1)}$, $i = 1, \dots, \kappa^{(m+1)}$, the remaining variance components. In particular, denote $\sigma_{\kappa^{(m+1)}}^{2(m+1)} = \left(\sigma_{i_0}^{2(m)} + c_{i_0}^{(m)} \sigma_{\kappa^{(m)}}^{2(m)}\right)$. Further, $U_i^{(m+1)} = B B_P U_i^{(m)}$, $V_i^{(m+1)} = U_i^{(m+1)} U_i^{(m+1)'}$ for $i = 1, \dots, \kappa^{(m+1)}$, and $V_{\kappa^{(m+1)}} = I_{n^{(m+1)}}$.
Then

$$t^{(m+1)} \sim N\bigg(0, \sum_{i=1}^{\kappa^{(m+1)}} \sigma_i^{2(m+1)} V_i^{(m+1)}\bigg), \tag{75}$$

and notice that $\sigma_1^{2(m+1)} = \sigma_1^2$ and $V_{\kappa^{(m+1)}}^{(m+1)} = I_{n^{(m+1)}}$.

Finally, denote

$$T^{(m+1)} = B\Big\{B_P + \big(B_P\big(c_{i_0}^{(m)} V_{i_0}^{(m)} - I_{n^{(m)}}\big)B_P'\big)^{\frac{1}{2}} DB^{(m)}\Big\}T^{(m)}, \tag{76}$$

set $m := m + 1$ and restart the algorithm.

If the algorithm succeeds, denote $T = T^{(m)}$ and compute $w = Ty$. According to (56), the distribution of w depends only on two variance components.

6. Example

We consider an unbalanced random two-way cross-classification model with interactions and with some empty cells

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk} \tag{77}$$

with $i = 1, \ldots, 3$, $j = 1, \ldots, 4$ and $k = 1, \ldots, n_{ij}$, where n_{ij} are given by the following incidence matrix:

		j			
		1	2	3	4
	1	4	0	0	0
i	2	5	5	4	0
	3	6	5	4	3

We will assume that μ is an unknown constant, $\alpha \sim N(0, \sigma_1^2 I_3)$, $\beta \sim N(0, \sigma_2^2 I_4)$, $\gamma \sim N(0, \sigma_3^2 I_8)$ and $\varepsilon \sim N(0, \sigma_4^2 I_{36})$ are independent random vectors. The present model was considered in [6], [17] and [5].

Let us assume that the hypothesis of interest is

$$H_0\colon\ \sigma_1^2 = 0 \quad \text{against} \quad H_1\colon\ \sigma_1^2 > 0. \tag{78}$$

We note that there is no uniformly optimum test for testing H_0 and the Wald test based on (8) does not exist, either.

6.1. ANOVA-like test

Let us assume that $\sigma_0^2 = (0, 1, 1, 1)'$ denotes a chosen prior value of the parameter, $\sigma_0^2 \in H_0$. Then, according to (15), the modified ANOVA-like test rejects the null hypothesis for $z_1 - c_{0.05}(z_1 - \hat{\sigma}_1^2) > 0$, where the critical value is $c_{0.05} = 15.5150$.

Tab. 1 reports the significance levels of the test, calculated under different values of the true parameter $\sigma^2 = (0, \sigma_2^2, \sigma_3^2, 1)'$, $\sigma^2 \in H_0$. Assuming that the true parameter

		σ_2^2						
		0	0.1	0.5	1	5	10	100
σ_3^2	0	0.0426	0.0457	0.0554	0.0640	0.0977	0.1168	0.1685
	0.1	0.0431	0.0455	0.0530	0.0600	0.0889	0.1066	0.1630
	0.5	0.0436	0.0448	0.0489	0.0531	0.0730	0.0869	0.1469
	1	0.0437	0.0444	0.0471	0.0500	0.0650	0.0763	0.1336
	5	0.0438	0.0440	0.0447	0.0455	0.0512	0.0565	0.0948
	10	0.0438	0.0439	0.0443	0.0447	0.0480	0.0514	0.0801
	100	0.0438	0.0438	0.0439	0.0439	0.0443	0.0448	0.0515
	Power	0.0500	0.0651	0.1195	0.1732	0.3629	0.4405	0.5731

Table 1. ANOVA-like test. The levels of significance $P(z_1 - 15.5150\,(z_1 - \hat{\sigma}_1^2) > 0)$ for different values of the true parameter $\sigma^2 \in H_0$, where $\sigma^2 = (0, \sigma_2^2, \sigma_3^2, 1)'$. The last row reports the power of the test for different alternatives $\sigma^2 \in H_1$, where $\sigma^2 = (\sigma_1^2, 1, 1, 1)'$ and $\sigma_1^2 = 0, 0.1, 0.5, 1, 5, 10, 100$.

coincides with σ_0^2, the last row reports the power of the test for alternatives $\sigma^2 \in H_1$, where $\sigma^2 = (\sigma_1^2, 1, 1, 1)'$ and $\sigma_1^2 = 0, 0.1, 0.5, 1, 5, 10, 100$.

6.2. Zmyślony-Michalski test

Let us assume that $\sigma_0^2 = (0, 1, 1, 1)'$ denotes a chosen prior value of the parameter, $\sigma_0^2 \in H_0$. Then calculate MINQE(U,I) of σ_1^2 and, according to (18), the Zmyślony-Michalski test rejects the null hypothesis for $t'A_+t - c_\alpha t'A_-t > 0$ where the critical value is $c_{0.05} = 7.2442$.

Tab. 2 reports the significance levels of the test, calculated under different values of the true parameter $\sigma^2 = (0, \sigma_2^2, \sigma_3^2, 1)'$, $\sigma^2 \in H_0$. The last row reports the power of the test for alternatives $\sigma^2 \in H_1$, where $\sigma^2 = (\sigma_1^2, 1, 1, 1)'$ and $\sigma_1^2 = 0, 0.1, 0.5, 1, 5, 10, 100$.

		σ_2^2						
		0	0.1	0.5	1	5	10	100
σ_3^2	0	0.0605	0.0517	0.0321	0.0211	0.0043	0.0018	0.0001
	0.1	0.0625	0.0560	0.0391	0.0280	0.0070	0.0030	0.0001
	0.5	0.0649	0.0616	0.0513	0.0424	0.0162	0.0083	0.0004
	1	0.0656	0.0636	0.0568	0.0500	0.0247	0.0142	0.0009
	5	0.0665	0.0660	0.0641	0.0619	0.0486	0.0381	0.0059
	10	0.0666	0.0663	0.0654	0.0642	0.0561	0.0484	0.0121
	100	0.0667	0.0667	0.0666	0.0665	0.0655	0.0643	0.0482
	Power	0.0500	0.0687	0.1404	0.2154	0.4987	0.6258	0.8953

Table 2. Zmyślony-Michalski test. The levels of significance $P(t'A_+t - 7.2442\,t'A_-t > 0)$ for different values of the true parameter $\sigma^2 \in H_0$, where $\sigma^2 = (0, \sigma_2^2, \sigma_3^2, 1)'$. The last row reports the power of the test for different alternatives $\sigma^2 \in H_1$, where $\sigma^2 = (\sigma_1^2, 1, 1, 1)'$ and $\sigma_1^2 = 0, 0.1, 0.5, 1, 5, 10, 100$.

6.3. Bartlett-Scheffé tests

By applying the algorithm for the reduction of the general model to the model with two variance components, we get a 4-dimensional vector $w = Ty$ such that

$$w \sim N(0, \sigma_1^2 W + (\sigma_3^2 + 0.25\,\sigma_4^2) I_4), \tag{79}$$

where

$$W = \begin{pmatrix} 0.2823 & -0.4468 & -0.1704 & 0.0508 \\ -0.4468 & 0.8415 & 0.2373 & -0.6891 \\ -0.1704 & 0.2373 & 0.1106 & 0.1157 \\ 0.0508 & -0.6891 & 0.1157 & 2.7656 \end{pmatrix}. \tag{80}$$

The matrix W has three distinct eigenvalues: 3, 1, and 0 with their respective multiplicities 1, 1, and 2.

Now we can apply the results from Section 3. Denote $\theta = \sigma_1^2 / \left(\sigma_3^2 + 0.25\,\sigma_4^2\right)$; then the hypothesis of interest is

$$H_0\colon\ \theta = 0 \quad \text{against} \quad H_1\colon\ \theta > 0. \tag{81}$$

Tab. 3 reports the critical values of Bartlett-Scheffé tests calculated on the significance level 0.05.

Test	$c_{0.05}$	Formula	Test statistic	Power
Neyman-Pearson test ($\theta_* = 1$)	2.7199	(22)	(21)	(23)
Locally Best Invariant test	2.4019	(30)	(29)	(31)
Wald test	19.000	$F_{2,2}$	(32)	(33)
Gnot-Michalski test	37.762	(43)	(41)	(45)
Zmyślony-Michalski test	2.1054	(54)	(53)	(55)

Table 3. Critical values of Bartlett-Scheffé tests calculated on the significance level $\alpha = 0.05$.

Figs. 1 and 2 plot the powers of the tests for the alternatives $\theta \in (0, 10)$ and $\theta \in (0, 100)$.

All critical values and powers were calculated numerically by Imhof's procedure. The Matlab code of Imhof's procedure is available on request from the authors.

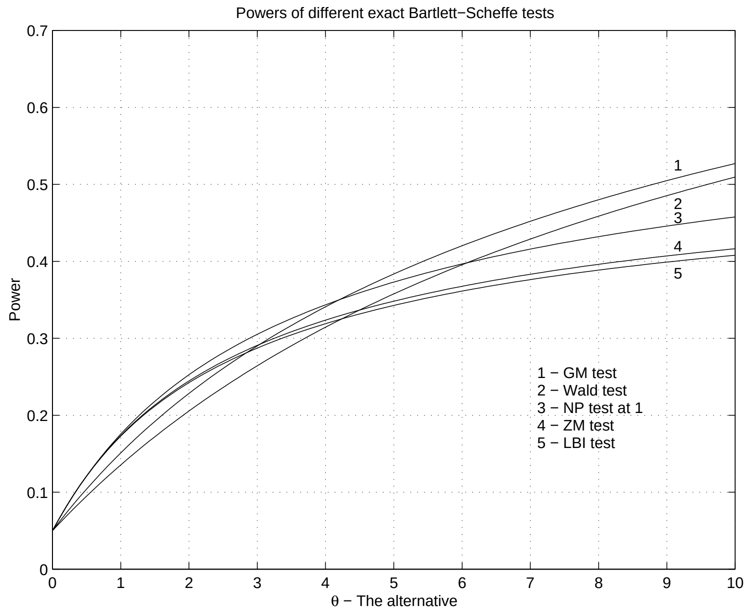

Fig. 1. Powers of the tests for the alternatives $\theta \in (0, 10)$. 1—Gnot-Michalski test, 2—Wald test, 3—Neyman-Pearson test ($\theta_* = 1$), 4—Zmyślony-Michalski test, 5—Locally Best Invariant test.

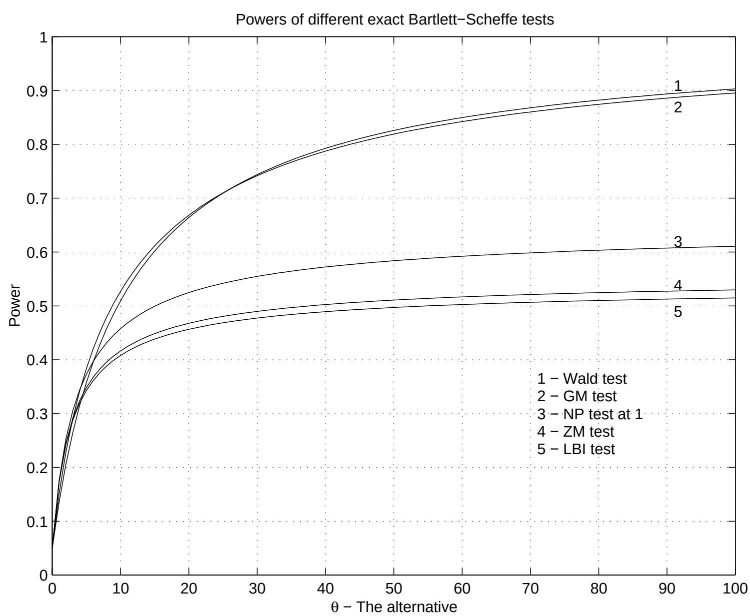

Fig. 2. Powers of the tests for the alternatives $\theta \in (0, 100)$. 1—Wald test, 2—Gnot-Michalski test, 3—Neyman-Pearson test ($\theta_* = 1$), 4—Zmyślony-Michalski test, 5—Locally Best Invariant test.

References

[1] *R. B. Davies*: The distribution of a linear combination of chi-square random variables. J. Roy. Statist. Soc. Ser. C *29* (1980), 323–333.

[2] *S. Gnot, M. Jankowiak-Rosłanowska and A. Michalski*: Testing for hypothesis in mixed linear models with two variance components. Listy Biometryczne—Biometrical Letters *29* (1992), 13–31.

[3] *S. Gnot, A. Michalski*: Tests based on admissible estimators in two variance components models. Statistics *25* (1994), 213–223.

[4] *J. P. Imhof*: Computing the distribution of quadratic forms in normal variables. Biometrics *48* (1961), 419–426.

[5] *A. I. Khuri, T. Mathew and B. K. Sinha*: Statistical Tests for Mixed Linear Models. J. Wiley, New York, 1998.

[6] *J. Kleffe, B. Seifert*: On the role of MINQUE in testing of hypotheses under mixed linear models. Comm. Statist. Theory Methods *17* (1988), 1287–1309.

[7] *L. R. LaMotte, A. McWhorter and R. A. Prasad*: Confidence intervals and tests on the ratio in random models with two variance components. Comm. Statist. Theory Methods *17* (1988), 1135–1164.

[8] *T. H. Lin, D. Harville*: Some alternatives to Wald's confidence interval and test. J. Amer. Statist. Assoc. *86* (1991), 179–187.

[9] *T. Mathew*: Optimum invariant tests in mixed linear models with two variance components. In: Statistical Data Analysis and Inference (Y. Dodge, ed.). North-Holland, Amsterdam, 1989, pp. 381–388.

[10] *A. Michalski, R. Zmyślony*: Testing hypotheses for variance components in mixed linear models. Statistics *27* (1996), 297–310.

[11] *T. Mathew, B. K. Sinha*: Optimum tests in unbalanced two-way models without interactions. Ann. Statist. *16* (1988), 1727–1740.

[12] *A. Olsen, J. Seely and D. Birkes*: Invariant quadratic estimation for two variance components. Ann. Statist. *4* (1976), 878–890.

[13] *C. R. Rao*: Estimation of variance components—MINQUE theory. J. Multivariate Anal. *1* (1971), 257–275.

[14] *C. R. Rao, J. Kleffe*: Estimation of Variance Components and Applications. North-Holland Publishing Company, Amsterdam, 1988.

[15] *J. F. Seely, Y. El-Bassiouni*: Applying Wald's variance component test. Ann. Statist. *11* (1983), 197–201.

[16] *B. Seifert*: Estimation and test of variance components using the MINQUE-method. Statistics *16* (1985), 621–635.

[17] *B. Seifert*: Exact tests in unbalanced mixed analysis of variance. J. Statist. Plann. Inference *30* (1992), 257–266.

[18] *A. Wald*: A note on the analysis of variance with unequal class frequencies. Ann. Statist. *11* (1940), 96–100.

[19] *A. Wald*: A note on regression analysis. Ann. Statist. *18* (1947), 586–589.

[20] *P. H. Westfall*: Power comparisons for invariant variance ratio tests in mixed ANOVA models. Ann. Statist. *17* (1989), 318–326.

[21] *V. Witkovský*: Optimality of the ANOVA-like test in model with two variance components. In: MEASUREMENT 99. Proceedings of the International Conference on Measurement, Smolenice, Slovak Republic, 26–29 April 1999 (I. Frollo, A. Plačková, eds.). 1999, pp. 28–31.

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