Applications of Mathematics

Shu Hua Zhang; Tao Lin; Yan Ping Lin; Ming Rao

Defect correction and a posteriori error estimation of Petrov-Galerkin methods for nonlinear Volterra integro-differential equations

Applications of Mathematics, Vol. 45 (2000), No. 4, 241–263

Persistent URL: http://dml.cz/dmlcz/134438

Terms of use:

© Institute of Mathematics AS CR, 2000

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

DEFECT CORRECTION AND A POSTERIORI ERROR ESTIMATION OF PETROV-GALERKIN METHODS FOR NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

Shuhua Zhang, Edmonton, Tao Lin, Blacksburg,
Yanping Lin and Ming Rao, Edmonton

(Received August 27, 1998, in revised form May 29, 1999)

Abstract. We present two defect correction schemes to accelerate the Petrov-Galerkin finite element methods [19] for nonlinear Volterra integro-differential equations. Using asymptotic expansions of the errors, we show that the defect correction schemes can yield higher order approximations to either the exact solution or its derivative. One of these schemes even does not impose any extra regularity requirement on the exact solution. As by-products, all of these higher order numerical methods can also be used to form a posteriori error estimators for accessing actual errors of the Petrov-Galerkin finite element solutions. Numerical examples are also provided to illustrate the theoretical results obtained in this paper.

Keywords: Volterra integro-differential equations, Petrov-Galerkin methods, asymptotic expansions, defect correction, a posteriori error estimators

MSC 2000: 65R20, 65B05, 65N30

1. Introduction

In this paper we continue our study of the Petrov-Galerkin finite element (PGFE) methods [19] for the initial value problem of a nonlinear Volterra integro-differential equation (VIDE): Find $y = y(t)$ such that

$$y'(t) = f\big(t, y(t)\big) + \int_0^t k\big(t, s, y(s)\big)\,\mathrm{d}s, \quad t \in I := [0, T], \quad y(0) = 0, \tag{1.1}$$

* This work is supported partially by SRF for ROCS, SEM, the NSF grant DMS-9704621 and NSERC (Canada).

where $f = f(t,y)\colon\ I\times\mathbb{R}\to\mathbb{R}$ and $k = k(t,s,y)\colon\ D\times\mathbb{R}\to\mathbb{R}$ (with $D := \{(t,s)\colon\ 0\leqslant s\leqslant t\leqslant T\}$) denote given functions.

The nonlinear Volterra integro-differential equation (1.1) plays an important role in the mathematical modeling of many physical and biological phenomena in which it is necessary to take into account the effect of past history. Particularly in such fields as heat transfer, nuclear reactor dynamics and thermoelasticity, there is often a need to have mathematical models which reflect the effects of the "memory" of the system. For example, the partial VIDE

$$u_t = \Delta u + \int_0^t a(t-s)g\bigl(u(x,s)\bigr)\,\mathrm{d}s + f$$

has been used in the feedback heat control of some heat-conducting medium, where the control mechanism possesses some inertia. A similar control situation for a reaction-diffusion problem can be seen in [20].

Mathematically, partial VIDEs like the one given above can be reformulated as abstract VIDEs of the type (1.1) in suitable function spaces. For the details of formulations of (1.1) and their physical interpretations we refer readers to [6] and [21]. On the other hand, the problem (1.1) can be viewed as a system of VIDEs obtained from the semi-spatial discretizations [10] or the methods of the lines [12].

In recent years, various aspects of numerical methods for VIDEs have been studied. See, for example, [1]–[5], [9], [11], [19] and [22]–[23]. At the same time, the superconvergence of finite element methods has received considerable attention. The literature on this subject is now quite extensive. The most recent survey paper by Křížek and Neittaanmäki [14] and the references cited therein convey a good picture on this topic. As our contribution to these researches, we present in this paper two (interpolation and iterative) defect correction schemes that can be used to improve the PGFE solutions. Using asymptotic expansions of the error in a PGFE solution, we will show that the defect correction schemes can yield higher order approximations to either the exact solution or its derivative. In particular, the approximation generated by applying the interpolation defect correction to a linear PGFE solution/derivative can have a convergence rate which is twice as high as that of the linear PGFE solution/derivative itself. Moreover, the iterative defect correction works even without imposing any extra regularity requirement on the exact solution.

Throughout the paper, it will always be assumed that the problem (1.1) possesses a unique solution $y\in C^1(I)$, namely, the given functions $f(t,y)$ and $k(t,s,y)$, which are, respectively, continuous for $t\in I$ and $(t,s)\in D$, will be subject to the following (uniform) Lipschitz conditions [6]:

$$(V1)\quad |f(t,y_1)-f(t,y_2)|\leqslant L_1|y_1-y_2|,$$
$$(V2)\quad |k(t,s,y_1)-k(t,s,y_2)|\leqslant L_2|y_1-y_2|$$

for all $t \in I$, $(t,s) \in D$, and $|y_i| < \infty$ $(i = 1, 2)$.

This paper is organized in the following way. In Section 2, we recall the Petrov-Galerkin finite element methods for (1.1) and some of their fundamental error estimates [19]. In Section 3, we discuss an interpolation defect correction that can be used to treat both the PGFE solution and the iterated PGFE derivative. In Section 4, we discuss an iterative defect correction scheme that can enhance the iterated PGFE derivative without extra regularity. At the end of both Sections 3 and 4, a posteriori error estimators based on these higher order approximations are developed. Numerical examples are provided in Section 5 to illustrate our theoretical results.

2. The PGFE solutions and their global convergence

In this section we will introduce the Petrov-Galerkin finite element (PGFE) method and recall the basic global convergence results obtained in [19]. First we define a nonlinear integral operator $G\colon\ C(I) \to C(I)$ by

$$(G\varphi)(t) := f\big(t, \varphi(t)\big) + \int_0^t k\big(t, s, \varphi(s)\big)\, \mathrm{d}s.$$

Then, the problem (1.1) is reduced to: Find $y = y(t)$ such that

$$y'(t) = (Gy)(t), \quad t \in I, \tag{2.1}$$

and its Petrov-Galerkin weak form becomes: Find $y \in H_0^1(I)$ (and then $y' \in L^2(I)$) such that

$$(y', v) = (Gy, v) \quad \forall v \in L^2(I), \tag{2.2}$$

where $(\cdot,\cdot)$ denotes the usual inner product in the $L^2(I)$-space and $H_0^1(I) := \{v \in H^1(I)\colon\ v(0) = 0\}$ is the standard Sobolev space.

Let $T_h\colon\ 0 = t_0 < t_1 < \ldots < t_N = T$ be a given mesh for the interval I, and denote the finite element trial and test function spaces, respectively, by

$$S_m^{(0)}(T_h) := \{v \in H_0^1(I)\colon\ v|_{\sigma_k} \in P_m,\ 0 \leqslant k \leqslant N-1\}$$

and

$$S_{m-1}^{(-1)}(T_h) := \{v \in L^2(I)\colon\ v|_{\sigma_k} \in P_{m-1},\ 0 \leqslant k \leqslant N-1\} \quad \text{with} \quad m \geqslant 1,$$

where P_r denotes the space of (real) polynomials of degree not exceeding r, $\sigma_k := [t_k, t_{k+1}]$ $(0 \leqslant k \leqslant N-1)$, $h_k := t_{k+1} - t_k$ and $h := \max\limits_{(k)}\{h_k\}$. Clearly, the dimensions of $S_m^{(0)}(T_h)$ and $S_{m-1}^{(-1)}(T_h)$ are equal to Nm. While $S_m^{(0)}(T_h)$ is a subspace of $H^1(I)$ whose elements therefore have to be continuous, $S_{m-1}^{(-1)}(T_h)$ is not a subspace of $C(I)$, and the superscript (-1) in $S_{m-1}^{(-1)}(T_h)$ emphasizes the fact that its elements may be discontinuous at the mesh points of T_h.

The Petrov-Galerkin finite element approximation of (2.2) considered in this paper is defined as in [19]: Find $u \in S_m^{(0)}(T_h)$ (and then $u' \in S_{m-1}^{(-1)}(T_h)$) such that

$$(u', v) = (Gu, v) \quad \forall v \in S_{m-1}^{(-1)}(T_h). \tag{2.3}$$

Let $P_h\colon\ L^2(I) \to S_{m-1}^{(-1)}(T_h)$ be the L^2-projection operator defined by

$$(\varphi, v) = (P_h\varphi, v) \quad \forall v \in S_{m-1}^{(-1)}(T_h). \tag{2.4}$$

Then the problem (2.3) can be equivalently written as follows: Find $u \in S_m^{(0)}(T_h)$ (and then $u' \in S_{m-1}^{(-1)}(T_h)$) such that

$$u' = P_h G u. \tag{2.5}$$

We have proved in [19] that if the conditions $(V1)$ and $(V2)$ are fulfilled, then the problem (2.3) (or (2.5)) is uniquely solvable whenever the mesh size h is sufficiently small. To approximate the derivative of the exact solution, we also introduce the iterated PGFE solution of (1.1)

$$u_{it}(t) := (LGu)(t), \tag{2.6}$$

where L is the integral operator defined by $(Lf)(t) := \int_0^t f(s)\,\mathrm{d}s$. As for the accuracy, we call $e = u - y$ the PGFE error and call $e_{it} := u_{it} - y$ the iterated PGFE error. Then the convergence properties of u and u_{it} can be summarized in the following theorem [19]:

Theorem 2.1. *Assume that $f \in C^m(I \times \mathbb{R})$ and $k \in C^m(D \times \mathbb{R})$. Then the PGFE error $e = u - y$ and the iterated PGFE error $e_{it} = u_{it} - y$ satisfy*

$$\|e'\|_{0,\infty} := \sup\{|e'(t)|\colon\ t \in \sigma_j,\ 0 \leqslant j \leqslant N-1\} \leqslant Ch^m \|y\|_{m+1,\infty},$$
$$\|e\|_{0,\infty} \leqslant Ch^{m+1}\|y\|_{m+1,\infty} \quad \textit{and} \quad \|e'_{it}\|_{0,\infty} \leqslant Ch^{m+1}\|y\|_{m+1,\infty}.$$

The projection operator P_h plays an important role in the investigation of the PGFE methods. For any $v \in S_{m-1}^{(-1)}(T_h)$, taking

$$\bar{v} := \begin{cases} v|_{\sigma_k}, & t \in \sigma_k, \\ 0, & t \in I - \sigma_k, \end{cases}$$

where $A - B := \{x\colon\ x \in A \text{ and } x \notin B\}$, we have $\bar{v} \in S_{m-1}^{(-1)}(T_h)$ since $S_{m-1}^{(-1)}(T_h)$ is a discontinuous piecewise-polynomial space of degree not exceeding $m-1$. Thus, substituting $\bar{v} \in S_{m-1}^{(-1)}(T_h)$ into (2.4) we obtain that

$$\int_{\sigma_k} v P_h \varphi \,\mathrm{d}t = \int_{\sigma_k} v\varphi \,\mathrm{d}t \quad \forall v \in S_{m-1}^{(-1)}(T_h), \tag{2.7}$$

with

$$\|P_h\varphi - \varphi\|_{0,\infty} \leqslant Ch^m \|\varphi\|_{m,\infty},$$

where, for any nonnegative integer r, $\|v\|_{r,\infty} := \max\limits_{0 \leqslant k \leqslant r} \{\|v^{(k)}\|_\infty\}$. In this case, P_h is defined on each element of the mesh T_h, and it can be regarded as an interpolation operator of degree $m-1$ (it is a kind of interpolation in average which is different from the standard Lagrange interpolation) associated with the mesh T_h.

Here and hereafter, C denotes a generic positive constant, independent of the PGFE solution u of (1.1) and the mesh size h, whose particular meaning will become clear by the context in which it arises.

3. Interpolation defect correction

In this section we propose and investigate an interpolation correction scheme [18] (also compare [8] and [15]) that can be applied to the PGFE solution $u \in S_m^{(0)}(T_h)$ and the iterated PGFE derivative u'_{it} to obtain approximations with higher convergence rates. In addition, these new approximations are naturally used to form a posteriori error estimators that can be used to access the actual error of a PGFE solution.

First, we need to define an interpolation operator that forms a piecewise polynomial with a degree higher than the PGFE solution. For ease of exposition, we demonstrate our idea mainly for the interpolation operator of degree 3. Let the number of elements N for the mesh T_h be a multiple of 3 and let $e_k := \sigma_{k-1} \cup \sigma_k \cup \sigma_{k+1}$ (σ_{k-1}, σ_k and $\sigma_{k+1} \in T_h$, $1 \leqslant k \leqslant N-2$) be an arbitrary element of the mesh T_{3h} with mesh size $3h$ (i.e., each element of T_{3h} is a combination of 3 adjacent elements in T_h), such that we can define a Lagrange interpolation operator I_{3h}^3 of degree 3 associated with T_{3h} as follows:

$$I_{3h}^3 u|_{e_k} \in P_3, \quad k = 3l + 1, \quad l = 0, 1, \ldots, \frac{N}{3} - 1$$

and

$$I_{3h}^3 u(t_i) = u(t_i), \quad i = k-1, k, k+1, k+2 \quad (1 \leqslant k \leqslant N-2),$$

where $t_i \in e_k$ are all endpoints of σ_{k-1}, σ_k and σ_{k+1}.

Similarly, we can also define a Lagrange interpolation operator $I^{2m}_{(2m)h}$ of degree $2m$ associated with the mesh $T_{(2m)h}$.

In addition, we also need the following theorem [19]:

Theorem 3.1. *Assume that $f \in C^{m+2}(I \times R)$ and $k \in C^{m+2}(D \times R)$. Then, for the PGFE error $e = u - y$ where $u \in S^{(0)}_m(T_h)$, we have the following asymptotic expansions at the points t_n $(1 \leqslant n \leqslant N)$ of the mesh T_h:*

$$e(t_n) = \begin{cases} \alpha(t_n)h^2 + O(h^4), & m = 1, \\ \alpha(t_n)h^{2m} + O(h^{2m+1}), & m \geqslant 2, \end{cases}$$

where $\alpha \in C^4(I)$ is invariable when the mesh is refined uniformly.

And now we can obtain

Theorem 3.2. *Suppose that the conditions of Theorem 3.1 hold. Then, for the PGFE solution $u \in S^{(0)}_m(T_h)$ and each $t \in I$, we have the following global asymptotic expansions:*

$$I^3_{3h}u(t) - y(t) = h^2\alpha(t) + O(h^4), \quad m = 1, \tag{3.1}$$

$$I^{2m}_{(2m)h}u(t) - y(t) = h^{2m}\alpha(t) + O(h^{2m+1}), \quad m \geqslant 2, \tag{3.2}$$

where $\alpha \in C^4(I)$.

Proof. For any $t \in e_k$ $(1 \leqslant k \leqslant N - 2)$, denoting the basis function corresponding to $\{t_j\}$ by $\{\varphi_j\}$ $(k - 1 \leqslant j \leqslant k + 2)$, we have

$$I^3_{3h}(u - y - h^2\alpha)(t) = \sum_{j=k-1}^{k+2} (u - y - h^2\alpha)(t_j)\varphi_j(t)$$

which, together with Theorem 3.1 and the uniform boundedness of $\{\varphi_j\}^{k+2}_{k-1}$, yields

$$\|I^3_{3h}(u - y - h^2\alpha)\|_{0,\infty} \leqslant \sum_{j=k-1}^{k+2} Ch^4\|\varphi_j\|_{0,\infty} \leqslant Ch^4.$$

This leads to the global expansion

$$\begin{aligned} I^3_{3h}u - y &= h^2 I^3_{3h}\alpha + (I^3_{3h}y - y) + O(h^4) \\ &= h^2\alpha + h^2(I^3_{3h}\alpha - \alpha) + O(h^4) \\ &= h^2\alpha + O(h^4), \end{aligned}$$

since $\|I^3_{3h}\alpha - \alpha\|_{0,\infty} \leqslant Ch^4\|\alpha\|_{4,\infty}$.

Analogously, we can also obtain (3.2). $\square$

Let $B_h\colon\ H_0^1(I)\to S_m^{(0)}(T_h)$ be the Petrov-Galerkin finite element projection operator defined by

$$((B_hy)'-GB_hy,v)=(y'-Gy,v)\quad\forall v\in S_{m-1}^{(-1)}(T_h). \tag{3.3}$$

Then B_hy is a solution of (2.3) if y is a solution of (1.1). Note that $y'-Gy$ on the right-hand side is the residual or the defect in y.

By means of the L^2-projection operator $P_h\colon\ L^2(I)\to S_{m-1}^{(-1)}(T_h)$, the problem (3.3) can be equivalently written as the operator equation

$$(B_hy)'=P_hGB_hy+P_h(y'-Gy)$$

or

$$B_hy=LP_hGB_hy+LP_h(y'-Gy) \tag{3.4}$$

since $(B_hy)(0)=0$, where L is the integral operator defined in Section 2.

Lemma 3.1. *If the conditions* $(V1)$ *and* $(V2)$ *are fulfilled, then there exists a unique* $B_hy\in S_m^{(0)}(T_h)$ *satisfying* (3.4) *whenever the mesh size* h *is sufficiently small.*

Proof. Define $g:=LP_h(y'-Gy)\in S_m^{(0)}(T_h)$ for any $y\in H_0^1(I)$, and two operators $E\colon\ S_m^{(0)}(T_h)\to S_m^{(0)}(T_h)$ as well as $E_*\colon\ S_m^{(0)}(T_h)\to S_m^{(0)}(T_h)$ by $E:=LP_hG$ and $E_*u:=Eu+g\ \forall u\in S_m^{(0)}(T_h)$. Thus, in order to prove Lemma 3.1, it is sufficient to show that the operator E_* has a unique fixed point $u_*:=B_hy\in S_m^{(0)}(T_h)$. To this end, by the standard contraction mapping principle, we need only to prove that the operator $E_*^n\colon\ S_m^{(0)}(T_h)\to S_m^{(0)}(T_h)$ is a contraction mapping as n and h are respectively sufficiently large and small since E_* and E_*^n have the same fixed points.

Decompose the operator E into

$$E=L(P_h-I)G+LG:=E_1+E_2,$$

where I is the identity operator. For the operator E_1, it follows from (2.7) and the conditions $(V1)$ and $(V2)$ that for any $u_1,u_2\in S_m^{(0)}(T_h)$ and any $t\in\sigma_k$ $(0\leqslant k\leqslant N-1)$ we have

$$\int_0^{t_k}(P_h-I)(Gu_1-Gu_2)(s)\,\mathrm{d}s=\sum_{j=0}^{k-1}\int_{\sigma_j}(P_h-I)(Gu_1-Gu_2)(s)\,\mathrm{d}s=0$$

and

$$
\begin{aligned}
(3.5)\qquad |(E_1u_1)(t)-(E_1u_2)(t)| &= \left|\int_0^{t_k}(P_h-I)(Gu_1-Gu_2)(s)\,\mathrm{d}s\right.\\
&\qquad\left.+\int_{t_k}^{t}(P_h-I)(Gu_1-Gu_2)(s)\,\mathrm{d}s\right|\\
&= \left|\int_{t_k}^{t}(P_h-I)(Gu_1-Gu_2)(s)\,\mathrm{d}s\right|\\
&\leqslant C(t-t_k)\|Gu_1-Gu_2\|_{0,\infty}\\
&\leqslant C(L_1+L_2T)h\|u_1-u_2\|_{0,\infty}.
\end{aligned}
$$

For the operator E_2, from the conditions $(V1)$ and $(V2)$ we obtain that for any $u_1,u_2\in S_m^{(0)}(T_h)$ we have

$$
\begin{aligned}
|(E_2u_1)(t)-(E_2u_2)(t)| &\leqslant \int_0^t |(Gu_1)(s)-(Gu_2)(s)|\,\mathrm{d}s\\
&\leqslant L_1\int_0^t|u_1(s)-u_2(s)|\,\mathrm{d}s+L_2\int_0^t\left(\int_0^s|u_1(\tau)-u_2(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\leqslant (L_1+L_2t)\int_0^t|u_1(s)-u_2(s)|\,\mathrm{d}s,
\end{aligned}
$$

which, together with (3.5), yields that for any $u_1,u_2\in S_m^{(0)}(T_h)$ we have

$$
\begin{aligned}
|E_*u_1-E_*u_2| &= |Eu_1-Eu_2|\\
&\leqslant Ch\|u_1-u_2\|_{0,\infty}+(L_1+L_2t)\int_0^t|u_1(s)-u_2(s)|\,\mathrm{d}s.
\end{aligned}
$$

And now,

$$
\begin{aligned}
\left|E_*^2u_1-E_*^2u_2\right| &= |E_*(E_*u_1)-E_*(E_*u_2)|\\
&\leqslant Ch\|E_*u_1-E_*u_2\|_{0,\infty}+(L_1+L_2t)\int_0^t|(E_*u_1)(s)-(E_*u_2)(s)|\,\mathrm{d}s\\
&\leqslant Ch\|u_1-u_2\|_{0,\infty}+(L_1+L_2T)\\
&\qquad\times\int_0^t\left((L_1+L_2s)\int_0^s|u_1(\tau)-u_2(\tau)|\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\leqslant Ch\|u_1-u_2\|_{0,\infty}+(L_1+L_2T)\frac{(L_1+L_2t)^2-L_1^2}{2L_2}\\
&\qquad\times\int_0^t|u_1(s)-u_2(s)|\,\mathrm{d}s\\
&\leqslant Ch\|u_1-u_2\|_{0,\infty}+\frac{(L_1+L_2T)}{L_2}\frac{(L_1+L_2t)^2}{2!}\int_0^t|u_1(s)-u_2(s)|\,\mathrm{d}s.
\end{aligned}
$$

This recurrently leads to

$$|(E_*^n u_1)(t) - (E_*^n u_2)(t)| \leqslant Ch\|u_1 - u_2\|_{0,\infty} + \left(\frac{L_1 + L_2 T}{L_2}\right)^{n-1} \frac{(L_1 + L_2 t)^n}{n!} \int_0^t |u_1(s) - u_2(s)| \,\mathrm{d}s$$

or

$$\|E_*^n u_1 - E_*^n u_2\|_{0,\infty} \leqslant Ch\|u_1 - u_2\|_{0,\infty} + \frac{T(L_1 + L_2 T)^{2n-1}}{L_2^{n-1} n!} \|u_1 - u_2\|_{0,\infty},$$

which yields that there exists a positive integer N_0 such that

$$\|E_*^{N_0} u_1 - E_*^{N_0} u_2\|_{0,\infty} \leqslant \alpha\|u_1 - u_2\|_{0,\infty}$$

with $\alpha \in (0,1)$ whenever the mesh size h is sufficiently small; that is, $E_*^{N_0}\colon S_m^{(0)}(T_h) \to S_m^{(0)}(T_h)$ is a contraction mapping subject to the smallness of h. Thus, we have completed the proof of Lemma 3.1. $\square$

Now, for each PGFE solution $u \in S_m^{(0)}(T_h)$, we define its interpolation defect correction as follows:

$$\begin{aligned} u_1^{(c)} &:= I_{3h}^3 u + u - B_h I_{3h}^3 u, && \text{when } m = 1,\\ u_m^{(c)} &:= I_{(2m)h}^{2m} u + u - B_h I_{(2m)h}^{2m} u, && \text{when } m \geqslant 2. \end{aligned}$$

Then the error estimates of these new approximations are given in the following theorem.

Theorem 3.3. *Assume that the conditions of Theorem* 3.1 *hold. Then, for the PGFE solution $u \in S_m^{(0)}(T_h)$, its interpolation defect correction satisfies*

$$\|y - u_1^{(c)}\|_{0,\infty} \leqslant Ch^4\|y\|_{4,\infty}, \qquad m = 1, \tag{3.6}$$

$$\|y - u_m^{(c)}\|_{0,\infty} \leqslant Ch^{2m+1}\|y\|_{2m+1,\infty}, \qquad m \geqslant 2. \tag{3.7}$$

Proof. By means of (3.1), we derive from the boundedness of the operator $(I - B_h)$ that

$$(I - B_h)(I_{3h}^3 u - y) = h^2\alpha - B_h(h^2\alpha) + O(h^4),$$

which, together with the global convergence of the linear PGFE solution in Theorem 2.1 that yields

$$\|h^2\alpha - B_h(h^2\alpha)\|_{0,\infty} \leqslant Ch^2\|h^2\alpha\|_{2,\infty} = Ch^4\|\alpha\|_{2,\infty},$$

leads to

$$(I - B_h)(I_{3h}^3 u - y) = O(h^4),$$

where the left-hand side is simply

$$(I - B_h)(I_{3h}^3 u - y) = u_1^{(c)} - y.$$

And hence, we obtain (3.6). Analogously, we can also obtain (3.7) by means of (3.2). $\square$

Next, we proceed to discuss the interpolation defect correction for the iterated PGFE derivative u'_{it}. To start, we recall a basic error expansion for the iterated PGFE derivative from [19].

Theorem 3.4. *Assume that $f \in C^{m+2}(I \times R)$ and $k \in C^{m+2}(D \times R)$. Then, for the iterated PGFE derivative error $e'_{it} := u'_{it} - y'$ produced by the PGFE solution $u \in S_m^{(0)}(T_h)$ of (1.1), we have the following asymptotic expansions at the points t_n $(1 \leqslant n \leqslant N)$ of the mesh T_h:*

$$e'_{it}(t_n) = \begin{cases} \beta(t_n)h^2 + O(h^4), & m = 1, \\ \beta(t_n)h^{2m} + O(h^{2m+1}), & m \geqslant 2, \end{cases}$$

where $\beta \in C^3(I)$ is invariable when the mesh is refined uniformly.

In parallel to Theorem 3.2, by virtue of Theorem 3.4 we can also obtain the following theorem.

Theorem 3.5. *Suppose that the conditions of Theorem 3.4 hold. Then, for the iterated PGFE derivative u'_{it} corresponding to the PGFE solution $u \in S_m^{(0)}(T_h)$ and each $t \in I$, we have the global asymptotic expansions*

$$I_{3h}^3 u'_{it}(t) - y'(t) = h^2\beta(t) + O(h^4), \qquad m = 1, \tag{3.8}$$

$$I_{(2m)h}^{2m} u'_{it}(t) - y'(t) = h^{2m}\beta(t) + O(h^{2m+1}), \qquad m \geqslant 2, \tag{3.9}$$

where $\beta \in C^3(I)$.

From (2.6) and the definition of the PGFE projection operator B_h we derive that

$$u'_{it} = GB_h L y'.$$

Thus, we define the iterated PGFE derivative projection operator $Q_h\colon\ L^2(I) \to C(I)$ by setting

$$Q_h := GB_h L.$$

Then $Q_h y'$ is the iterated PGFE derivative of the problem (1.1) if y is its exact solution. In addition, from Lemma 3.1 we know that for any $y \in L^2(I)$, $Q_h y \in C(I)$ exists uniquely. Using this operator, for an iterated PGFE derivative we can similarly define its interpolation defect correction as

$$\begin{aligned} u_{it,1}^{(c)}(t) &:= I_{3h}^3 u'_{it}(t) + u'_{it}(t) - Q_h I_{3h}^3 u'_{it}(t), && \text{when } m = 1, \\ u_{it,m}^{(c)}(t) &:= I_{(2m)h}^{2m} u'_{it}(t) + u'_{it}(t) - Q_h I_{(2m)h}^{2m} u'_{it}(t), && \text{when } m \geqslant 2. \end{aligned}$$

The following theorem shows the effects of the interpolation defect correction on the iterated PGFE derivative.

Theorem 3.6. *Assume that the conditions of Theorem 3.4 hold. Then, for the iterated PGFE derivative u'_{it} produced by the PGFE solution $u \in S_m^{(0)}(T_h)$, its interpolation defect correction satisfies*

$$\|y' - u_{it,1}^{(c)}\|_{0,\infty} \leqslant Ch^4 \|y\|_{5,\infty}, \qquad m = 1, \tag{3.10}$$

$$\|y' - u_{it,2}^{(c)}\|_{0,\infty} \leqslant Ch^{2m+1} \|y\|_{2m+2}, \qquad m \geqslant 2. \tag{3.11}$$

Proof. For the function $\beta \in C^3(I)$ in (3.8), let $\hat{y}(t) := h^2 \int_0^t \beta(s)\,\mathrm{d}s$. Then it follows from the global superconvergence of the iterated PGFE derivative error e'_{it} in Theorem 2.1 corresponding to the linear PGFE solution $u \in S_1^{(0)}(T_h)$ that

$$\|\hat{y}' - Q_h \hat{y}'\|_{0,\infty} \leqslant Ch^2 \|\hat{y}\|_{2,\infty} \leqslant Ch^4 \|\beta\|_{1,\infty}. \tag{3.12}$$

Thus, we obtain from (3.8), (3.12) and from the boundedness of the operator $(I - Q_h)$ that

$$\begin{aligned} (I - Q_h)(I_{3h}^3 u'_{it} - y') &= (I - Q_h)(h^2 \beta) + O(h^4) \\ &= (\hat{y}' - Q_h \hat{y}') + O(h^4) \\ &= O(h^4), \end{aligned}$$

where the left-hand side is just

$$(I - Q_h)(I_{3h}^3 u'_{it} - y') = u_{it,1}^{(c)} - y'.$$

Hence we complete the proof of (3.10). Similarly, we can also get (3.11). □

As usual, the new approximations with higher convergence rates after the interpolation defect correction can be used to form a posteriori estimators for the PGFE methods by following the procedure of obtaining Theorem 2.4 in [19]. In fact, by Theorem 3.3, for a PGFE solution $u \in S_m^{(0)}(T_h)$ we can use $(I - B_h)I_{3h}^3 u$ or $(I - B_h)I_{(2m)h}^{2m} u$ to estimate its actual accuracy because

$$\|y - u\|_{0,\infty} = \begin{cases} \|(I - B_h)I_{3h}^3 u\|_{0,\infty} + O(h^4), & m = 1, \\ \|(I - B_h)I_{(2m)h}^{2m} u\|_{0,\infty} + O(h^{2m+1}), & m \geqslant 2. \end{cases}$$

Similarly, from Theorem 3.6, the computable quantity $(I - Q_h)I_{3h}^3 u'_{it}$ or $(I - Q_h)I_{(2m)h}^{2m} u'_{it}$ can be used to access the actual error in the iterated PGFE derivative u'_{it} because

$$\|y' - u'_{it}\|_{0,\infty} = \begin{cases} \|(I - Q_h)I_{3h}^3 u'_{it}\|_{0,\infty} + O(h^4), & m = 1, \\ \|(I - Q_h)I_{(2m)h}^{2m} u'_{it}\|_{0,\infty} + O(h^{2m+1}), & m \geqslant 2. \end{cases}$$

4. Iterative defect correction

In this section, we will discuss an iterative correction ([8]) for the iterated PGFE derivative u'_{it} produced by the PGFE solution $u \in S_m^{(0)}(T_h)$ of the problem (1.1). It will be proved that the $(n-1)$-fold application of the iterative correction leads to a global convergence rate of $O(h^{m+n})$ under a rather moderate regularity requirement on the exact solution: $y \in C^{m+1}(I)$, which is independent of n. In addition, as a by-product of the iterative correction a posteriori error estimators are also obtained.

To start, we recall the following results from [6].

Lemma 4.1. *Let the functions g and K characterizing the integral equation*

$$y(t) = g(t) + \int_0^t K(t,s)y(s)\,\mathrm{d}s, \quad t \in I := [0,T],$$

be continuous on I and $D := \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant T\}$, respectively. Then this equation has a unique solution $y \in C(I)$ given by

$$y(t) = g(t) + \int_0^t R(t,s)g(s)\,\mathrm{d}s, \quad t \in I,$$

where $R \in C(D)$ is the resolvent kernel associated with the given kernel K and defined by $R(t,s) := \sum\limits_{m=1}^{\infty} K_m(t,s)$, $(t,s) \in D$ with $K_1(t,s) := K(t,s)$ and $K_n(t,s) :=$

$\int_s^t K_1(t,\tau)K_{n-1}(\tau,s)\,\mathrm{d}\tau$, $(t,s)\in D$ $(n\geqslant 2)$. *Moreover, the resolvent kernel satisfies the identities (usually called the Fredholm identities)*

$$R(t,s)=K(t,s)+\int_s^t K(t,\tau)R(\tau,s)\,\mathrm{d}\tau,\quad (t,s)\in D,$$

and

$$R(t,s)=K(t,s)+\int_s^t R(t,\tau)K(\tau,s)\,\mathrm{d}\tau,\quad (t,s)\in D.$$

Now, let $\delta(t):=u'(t)-(Gu)(t)$ $(t\in I)$ be the residual (or defect) function. Then, it is easy to see from (2.5) that

$$\delta=P_hGu-Gu=(P_h-I)Gu. \tag{4.1}$$

Subtracting (2.1) from (2.5), we have by (4.1) that

$$e'=P_hGu-Gy=\delta+(Gu-Gy),\quad t\in I, \tag{4.2}$$

with $e(0)=0$. Thus, (4.2) and Taylor's formula imply that there are functions ξ_* and η_* whose values $\xi_*(t)$ and $\eta_*(t)$ at t are between $y(t)$ and $u(t)$, such that

$$\begin{aligned} e'(t)&=\delta(t)+(Gu-Gy)(t)\\ &=\delta(t)+f_y\big(t,y(t)\big)e(t)+\int_0^t k_y\big(t,s,y(s)\big)e(s)\,\mathrm{d}s\\ &\quad+\frac{1}{2}f_{yy}\big(t,\xi_*(t)\big)e^2(t)+\frac{1}{2}\int_0^t k_{yy}\big(t,s,\eta_*(s)\big)e^2(s)\,\mathrm{d}s, \end{aligned} \tag{4.3}$$

which, together with Theorem 2.1, leads to

$$e'(t)=\delta(t)+p(t)e(t)+\int_0^t K(t,s)e(s)\,\mathrm{d}s+O(h^{2m+2}) \tag{4.4}$$

under the conditions that $f_{yy}(t,y)$ and $k_{yy}(t,s,y)$ are bounded, respectively, in a suitable region containing $D_1:=\{\big(t,y(t)\big)\colon t\in I\}$ and another proper domain containing $D_2:=\{\big(t,s,y(s)\big)\colon 0\leqslant s\leqslant t\leqslant T\}$, where $p(t):=f_y\big(t,y(t)\big)$ and $K(t,s):=k_y\big(t,s,y(s)\big)$.

By setting $e_*(t):=e(t)\exp\big(-\int_0^t p(s)\,\mathrm{d}s\big)$, it is easy to see from a simple calculation that (4.4) becomes

$$e_*'(t)=\delta_*(t)+\int_0^t K_*(t,s)e_*(s)\,\mathrm{d}s+O(h^{2m+2})\quad\text{with}\quad e_*(0)=0, \tag{4.5}$$

where

$$K_*(t,s) := K(t,s)\exp\Big(-\int_s^t p(\tau)\,\mathrm{d}\tau\Big), \quad \delta_* := \delta(t)\exp\Big(-\int_0^t p(s)\,\mathrm{d}s\Big).$$

We further get via exchanging the order of the integration with respect to s and τ that

$$\begin{aligned} e'_* &= \delta_* + \int_0^t K_*(t,s)\Big(\int_0^s e'_*(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s + O(h^{2m+2}) \\ &= \delta_* + \int_0^t K_{*1}(t,s)e'_*(s)\,\mathrm{d}s + O(h^{2m+2}), \end{aligned}$$

where the kernel function $K_{*1}(t,s) := \int_s^t K_*(t,\tau)\,\mathrm{d}\tau$. And then, setting $F(t) := \delta_*(t) + O(h^{2m+2})$, it follows from Lemma 4.1 that

$$e'_* = F + \int_0^t R_{*1}(t,s)F(s)\,\mathrm{d}s, \tag{4.6}$$

where $R_{*1}(t,s)$ is the resolvent kernel associated with the given kernel $K_{*1}(t,s)$, which inherits the smoothness of $K_{*1}(t,s)$, defined by

$$R_{*1}(t,s) = K_{*1}(t,s) + \int_s^t K_{*1}(t,\tau)R_{*1}(\tau,s)\,\mathrm{d}\tau, \quad (t,s)\in D.$$

And it is easy to see by integrating from 0 to t on both sides of (4.6) and exchanging the order of integration that

$$e_* = \int_0^t R_{*2}(t,s)F(s)\,\mathrm{d}s, \tag{4.7}$$

where $R_{*2}(t,s) := 1 + \int_s^t R_{*1}(\tau,s)\,\mathrm{d}\tau$. Now, we know from (4.7) that

$$e_* = \int_0^t R_{*2}(t,s)\delta_*(s)\,\mathrm{d}s + O(h^{2m+2}). \tag{4.8}$$

Set

$$(R_h^*\varphi)(t) := \int_0^t R_*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s, \tag{4.9}$$

where $R_*(t,s) := R_{*2}(t,s)\exp\big(-\int_0^s p(\tau)\,\mathrm{d}\tau\big)$, and let $G'\colon C(I)\to C(I)$ be the linear Volterra integral operator defined by

$$(G'\varphi)(t) := f_y\big(t,y(t)\big)\varphi(t) + \int_0^t k_y\big(t,s,y(s)\big)\varphi(s)\,\mathrm{d}s,$$

where y is the exact solution of the problem (1.1). Then one finds from (2.1), (4.1) and (4.3) that

$$\begin{aligned} e_* &= R_h^* G u + O(h^{2m+2}) \\ &= R_h^* G y + R_h^*(Gu - Gy) + O(h^{2m+2}) \\ &= R_h^* y' + R_h^* G' e + O(h^{2m+2}) \\ &= R_h^* y' + R_h^* G'_* e_* + O(h^{2m+2}), \end{aligned} \tag{4.10}$$

where $G'_*\varphi := G'\bigl(\exp\bigl(\int_0^t p(s)\,\mathrm{d}s\bigr)\varphi\bigr)$.

We derive from (2.1), (2.6) and (4.10) that

$$\begin{aligned} e'_{it} :&= u'_{it} - y' = Gu - Gy \\ &= G'e + O(h^{2m+2}) \\ &= G'_* e_* + O(h^{2m+2}). \end{aligned} \tag{4.11}$$

Then (4.10) and (4.11) yield a recurrence formula

$$e'_{it} = \sum_{i=0}^{m} G'_*(R_h^* G'_*)^i R_h^* y' + G'_*(R_h^* G'_*)^{m+1} e_* + O(h^{2m+2}). \tag{4.12}$$

Lemma 4.2. *For the operators G'_* and R_h^* we have*

$$\|G'_* R_h^* \varphi\|_{0,\infty} \leqslant C h^{m+1} \|\varphi\|_{m,\infty} \quad \text{and} \quad \|G'_* R_h^*\|_{C(I)\to C(I)} \leqslant Ch,$$

where

$$\|A\|_{C(I)\to C(I)} := \sup_{\varphi \in C(I)} \frac{\|A\varphi\|_{0,\infty}}{\|\varphi\|_{0,\infty}}.$$

P r o o f. For any $t \in \sigma_k$ $(0 \leqslant k \leqslant N-1)$, from (4.9) we know that

$$\begin{aligned} |(R_h^*\varphi)(t)| &= \left| \int_0^{t_k} R_*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s + \int_{t_k}^t R_*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s \right| \\ &= \left| \sum_{i=0}^{k-1} \int_{\sigma_k} (I - P_h) R_*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s \right. \\ &\qquad \left. + \int_{t_k}^t R_*(t,s)(P_h - I)\varphi(s)\,\mathrm{d}s \right| \\ &\leqslant C h^{2m}\|\varphi\|_{m,\infty} + C(t - t_k)h^m \|\varphi\|_{m,\infty} \leqslant C h^{m+1}\|\varphi\|_{m,\infty}, \end{aligned} \tag{4.13}$$

which leads to

$$\|R_h^*\varphi\|_{0,\infty} \leqslant C h^{m+1} \|\varphi\|_{m,\infty}. \tag{4.14}$$

In particular, we derive from (4.13) that

$$\|R_h^*\varphi\|_{0,\infty} \leqslant Ch\|\varphi\|_{0,\infty},$$

that is

$$\|R_h^*\|_{C(I)\to C(I)} \leqslant Ch. \tag{4.15}$$

From (4.14), (4.15) and the boundedness of the operator G'_* we find that

$$\|G'_*R_h^*\varphi\|_{0,\infty} \leqslant C\|R_h^*\varphi\|_{0,\infty} \leqslant Ch^{m+1}\|\varphi\|_{m,\infty}$$

and

$$\|G'_*R_h^*\|_{C(I)\to C(I)} \leqslant Ch.$$

Hence, we complete the proof of Lemma 4.2. $\square$

From (4.15) and the boundedness of the operator G'_* we have

$$\|R_h^*G'_*\|_{C(I)\to C(I)} \leqslant Ch,$$

which, together with Theorem 2.1, leads to

$$\begin{aligned}\|G'_*(R_h^*G'_*)^{m+1}e_*\|_{0,\infty} &\leqslant C\|(R_h^*G'_*)^{m+1}\|_{C(I)\to C(I)}\cdot\|e_*\|_{0,\infty}\\ &\leqslant Ch^{2m+2}\|y\|_{m+1,\infty}.\end{aligned}$$

This, together with (4.12), implies that

$$e'_{it} = \sum_{i=0}^{m} G'_*(R_h^*G'_*)^i R_h^* y' + O(h^{2m+2}). \tag{4.16}$$

Now, for an iterated PGFE derivative u'_{it}, we define its iterative defect correction as

$$\tilde{u}_{it,n}^{(c)} := \sum_{k=1}^{n}(-1)^{k-1}C_n^k Q_h^{k-1}u'_{it},$$

where Q_h is the iterated PGFE derivative projection operator defined in Section 3 and $C_n^k := \dfrac{n!}{k!(n-k)!}$ is the usual binomial coefficient. The following theorem provides an error estimate on this new approximation generated by the iterative defect correction.

Theorem 4.1. *Assume that $f \in C^m(I \times \mathbb{R}) \cap C^2(I \times \mathbb{R})$ and $k \in C^m(D \times \mathbb{R}) \cap C^2(D \times \mathbb{R})$. Then the $(n-1)$st iterative defect correction $\tilde{u}^{(c)}_{it,n}$ of the iterated PGFE derivative u'_{it} corresponding to the PGFE solution $u \in S^{(0)}_m(T_h)$ satisfies*

$$\|y' - \tilde{u}^{(c)}_{it,n}\|_{0,\infty} \leqslant Ch^{m+n}\|y\|_{m+1,\infty}, \quad 1 \leqslant n \leqslant m+2.$$

P r o o f. By definition, we have

$$\tilde{u}^{(c)}_{it,n} := \sum_{k=1}^{n} (-1)^{k-1} C^k_n Q^{k-1}_h u'_{it} = \sum_{k=1}^{n} (-1)^{k-1} C^k_n Q^k_h y'.$$

From (4.16) we derive that

$$(I - Q_h)y' = -\sum_{i=0}^{m} G'_*(R^*_h G'_*)^i R^*_h y' + O(h^{2m+2}). \tag{4.17}$$

Therefore, we obtain from the boundedness of the operator $(I - Q_h)$ that

$$(I - Q_h)^2 y' = -(I - Q_h)\left(\sum_{i=0}^{m} G'_*(R^*_h G'_*)^i R^*_h y'\right) + O(h^{2m+2}). \tag{4.18}$$

Set

$$\hat{y}_*(t) := \int_0^t \sum_{i=0}^{m} G'_*(R^*_h G'_*)^i R^*_h y'(s)\,\mathrm{d}s.$$

Then it is easy to see from (4.17) that

$$\begin{aligned}(I - Q_h)&\left(\sum_{i=0}^{m} G'_*(R^*_h G'_*)^i R^*_h y'\right) \\ &= (I - Q_h)\hat{y}'_* = -\sum_{j=0}^{m} G'_*(R^*_h G'_*)^j R^*_h \hat{y}'_* + O(h^{2m+2}) \\ &= -\sum_{j=0}^{m}\sum_{i=0}^{m} G'_*(R^*_h G'_*)^j R^*_h G'_*(R^*_h G'_*)^i R^*_h y' + O(h^{2m+2}).\end{aligned} \tag{4.19}$$

Substituting (4.19) into (4.18), we obtain

$$\begin{aligned}(I - Q_h)^2 y' &= \sum_{j=0}^{m}\sum_{i=0}^{m} G'_*(R^*_h G'_*)^j R^*_h G'_*(R^*_h G'_*)^i R^*_h y' + O(h^{2m+2}) \\ &= \sum_{j=0}^{m}\sum_{i=0}^{m} (G'_* R^*_h)^{j+i+2} y' + O(h^{2m+2}).\end{aligned} \tag{4.20}$$

From Lemma 4.2 we know that $(G'_*R_h^*)^2y'$ is the principal part of (4.20) and

$$\begin{aligned}\|(I-Q_h)^2y'\|_{0,\infty} &\leqslant C\|(G'_*R_h^*)^2y'\|_{0,\infty}\\ &\leqslant C\|G'_*R_h^*\|_{C(I)\to C(I)}\cdot\|G'_*R_h^*y'\|_{0,\infty}\\ &\leqslant Ch^{m+2}\|y\|_{m+1,\infty}.\end{aligned}$$

Inductively, we eventually obtain

$$\|(I-Q_h)^ny'\|_{0,\infty}\leqslant Ch^{m+n}\|y\|_{m+1,\infty}.$$

Note that

$$(I-Q_h)^ny'=y'-\tilde{u}^{(c)}_{it,n},$$

which completes the proof. □

Again, due to the error estimate in the above theorem, the iterative defect correction suggests that we can use $(n-1)u'_{it}+\sum\limits_{k=2}^{n}(-1)^{k-1}C_n^kQ_h^{k-1}u'_{it}$ to estimate the actual error in u'_{it} because

$$\begin{aligned}\|y'-u'_{it}\|_{0,\infty}&=\Big\|(n-1)u'_{it}+\sum_{k=2}^{n}(-1)^{k-1}C_n^kQ_h^{k-1}u'_{it}\Big\|_{0,\infty}\\ &\quad+O(h^{m+n}),\quad 1\leqslant n\leqslant m+2.\end{aligned}$$

We can also use $\tilde{u}^{(c)}_{it,n+1}-\tilde{u}^{(c)}_{it,n}$ to estimate the actual error in $\tilde{u}^{(c)}_{it,n}$ because

$$\|y'-\tilde{u}^{(c)}_{it,n}\|_{0,\infty}=\|\tilde{u}^{(c)}_{it,n+1}-\tilde{u}^{(c)}_{it,n}\|_{0,\infty}+O(h^{m+n+1}),\quad 1\leqslant n\leqslant m+2.$$

5. Numerical examples

In this section we present some numerical results which illustrate the features of the defect correction methods. Unless otherwise specified, all the numerical solutions given here are generated by the PGFE methods with the space $S_m^{(0)}(T_h)$, $m=1,2$, for the nonlinear Volterra integro-differential equation (1.1) in which

$$\begin{aligned}k(t,s,y)&=\sin(t)+2s+\cos(s)\mathrm{e}^y,\\ f(t,y)&=1-\mathrm{e}^{\sin(t)}-t^2+\cos(t)+\cos(t+2y)-\cos\big(t+2\sin(t)\big)-t\sin(t),\end{aligned}$$

so that $y(t)=\sin(t)$ is the exact solution. In all our computations, Newton's method is used to solve the nonlinear algebraic equations produced by the PGFE methods, and we have observed a quadratic convergence in Newton's iterations provided that the initial guess and the exact solution are close enough.

E x a m p l e 1. Let us first look at the numerical results generated by the interpolation defect correction. For any PGFE solution $u \in S_1^{(0)}(T_h)$, its interpolation defect correction is

$$u_1^{(c)} = I_{3h}^3 u + u - B_h I_{3h}^3 u,$$

where $B_h I_{3h}^3 u \in S_1^{(0)}(T_h)$ is generated by the PGFE equation

$$\bigl(v, (B_h I_{3h}^3 u)'\bigr) = (v, \tilde{g}) + \bigl(v, G(B_h I_{3h}^3 u)\bigr), \quad \forall v \in S_0^{(-1)}(T_h),$$

with $\tilde{g}$ as the defect of $I_{3h}^3 u(t)$:

$$\tilde{g}(t) = \bigl(I_{3h}^3 u(t)\bigr)' - f\bigl(t, I_{3h}^3 u(t)\bigr) - \int_0^t k\bigl(t, s, I_{3h}^3 u(s)\bigr)\, \mathrm{d}s.$$

The errors of the PGFE solution in the space $S_1^{(0)}(T_h)$ and the approximations generated by applying the interpolation defect correction to this PGFE solution are listed in Table 1. While the errors of the PGFE solutions in this table are obviously about $O(h^2)$, the errors of $u_1^{(c)}$ in this group of computations obey

$$\|u_1^{(c)} - y\|_\infty \approx 0.04338 h^{3.9948},$$

which is within the prediction of Theorem 3.3.

h	$\|u - y\|_\infty$	$\|u_1^{(c)} - y\|_\infty$
1/12	$0.52183924780080\times10^{-3}$	$0.00211330281763\times10^{-3}$
1/24	$0.13261082013960\times10^{-3}$	$0.00013301415924\times10^{-3}$
1/48	$0.03343465932215\times10^{-3}$	$0.00000835746927\times10^{-3}$
1/96	$0.00839474795822\times10^{-3}$	$0.00000052403404\times10^{-3}$
1/192	$0.00210325054983\times10^{-3}$	$0.00000003281464\times10^{-3}$
1/384	$0.00052638643000\times10^{-3}$	$0.00000000205236\times10^{-3}$

Table 1. Errors of the PGFE solution and those generated by the interpolation defect correction.

E x a m p l e 2. For the iterated PGFE derivative generated by the PGFE solution in $S_1^{(0)}(T_h)$, its interpolation defect correction is

$$u_{it,1}^{(c)} = I_{3h}^3 u_{it}' + u_{it}' - Q_h I_{3h}^3 u_{it}',$$

where $Q_h I_{3h}^3 u_{it}'$ is the iterated PGFE derivative produced by the PGFE solution in $S_1^{(0)}(T_h)$ for the following initial value problem of a VIDE:

$$\text{(5.1)} \qquad \begin{cases} y'(t) = r(t) + f\bigl(t, y(t)\bigr) + \displaystyle\int_0^t k\bigl(t, s, y(s)\bigr)\, \mathrm{d}s, \\ y(0) = 0 \end{cases}$$

with $r(t)$ as the defect of $\int_0^t I_{3h}^3 u_{it}'(s)\,\mathrm{d}s$:

$$r(t) = I_{3h}^3 u_{it}'(t) - f\Big(t, \int_0^t I_{3h}^3 u_{it}'(s)\,\mathrm{d}s\Big) - \int_0^t k\Big(t, s, \int_0^s I_{3h}^3 u_{it}'(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.$$

In addition to the errors of the iterated PGFE solutions, Table 2 also presents the errors of the approximations to the derivatives generated by the interpolation defect correction. The data in this table satisfy

$$\|u_{it}' - y'\|_\infty \approx 0.0929h^{2.0000},$$
$$\|u_{it,1}^{(c)} - y'\|_\infty \approx 0.02552h^{4.0016},$$

which corroborates the error estimates given in Theorem 3.6.

h	$\|u_{it}' - y'\|_\infty$	$\|u_{it,1}^{(c)} - y'\|_\infty$
1/12	$0.64620301056006\times10^{-3}$	$0.00122698146399\times10^{-3}$
1/24	$0.16116813244327\times10^{-3}$	$0.00007648996425\times10^{-3}$
1/48	$0.04034140964893\times10^{-3}$	$0.00000477555617\times10^{-3}$
1/96	$0.01008765519206\times10^{-3}$	$0.00000029803060\times10^{-3}$
1/192	$0.00252196308848\times10^{-3}$	$0.00000001861966\times10^{-3}$
1/384	$0.00063048499044\times10^{-3}$	$0.00000000116351\times10^{-3}$

Table 2. Errors of the iterated PGFE derivative and those generated by the interpolation defect correction.

Example 3. We now consider some examples for the iterative defect correction. The 2-fold iterative defect correction of the iterated PGFE derivative induced by the PGFE solution in $S_1^{(0)}(T_h)$ is given by

$$\tilde{u}_{it,2}^{(c)} = 2u_{it}' - Q_h u_{it}',$$

where $Q_h u_{it}'$ is the iterated PGFE derivative yielded by the PGFE solution in $S_1^{(0)}(T_h)$ for the initial value problem (5.1) with

$$r(t) = u_{it}'(t) - f\big(t, u_{it}(t)\big) - \int_0^t k\big(t, s, u_{it}(s)\big)\,\mathrm{d}s.$$

Table 3 lists the errors of $\tilde{u}_{it,2}^{(c)}$ for various step sizes h, from which we can see that these numerical results satisfy

$$\|\tilde{u}_{it,2}^{(c)} - y'\|_\infty \approx 0.01882h^{3.0098},$$

which is within the prediction of Theorem 4.1.

h	$\|u'_{it} - y'\|_\infty$	$\|\tilde{u}^{(c)}_{it,2} - y'\|_\infty$
1/12	$0.64620301056006\times10^{-3}$	$0.010731633355077 \times10^{-3}$
1/24	$0.16116813244327\times10^{-3}$	$0.001313080086551 \times10^{-3}$
1/48	$0.04034140964893\times10^{-3}$	$0.000162998233022 \times10^{-3}$
1/96	$0.01008765519206\times10^{-3}$	$0.000020266912149 \times10^{-3}$
1/192	$0.00252196308848\times10^{-3}$	$0.000002526510556 \times10^{-3}$
1/384	$0.00063048499044\times10^{-3}$	$0.000000315380055 \times10^{-3}$

Table 3. Errors of the iterated PGFE derivative and those generated by the iterative defect correction.

Example 4. We present this example to show the capability of the iterative defect correction for handling the VIDE whose solution has limited regularity. Specifically, we consider the initial value problem (1.1) with

$$\begin{aligned} k(t,s,y) &= \sin(t) + 2s + y^2, \\ f(t,y) &= g(t) - \cos(t+2y), \end{aligned}$$

where the function $g(t)$ is such that

$$y(t) = t\left(\left(t - \frac{1}{2}\right)^2\right)^{2.9/3}$$

is the exact solution which is in $H^2(I)$ but not in $H^3(I)$. Obviously, the PGFE methods with higher degree elements will have difficulties when applied to this problem. In fact, Table 4 lists the numerical results in the quadratic finite element trial function space $S_2^{(0)}(T_h)$ for this problem from which we have

$$\begin{aligned} \|u' - y'\|_{0,\infty} &\approx 0.0900h^{1.2602}, \\ \|u'_{it} - y'\|_{0,\infty} &\approx 0.0121h^{2.3242}, \end{aligned}$$

not up to the convergence rates given in Theorem 2.1. On the other hand, using the PGFE method with linear finite elements, we can obtain the results in Table 5. The data in these computations obey

$$\begin{aligned} \|u'_{it} - y'\|_{0,\infty} &\approx 0.6681h^{1.9884}, \\ \|\tilde{u}^{(c)}_{it,2} - y'\|_{0,\infty} &\approx 0.1223h^{2.9953}, \end{aligned}$$

which are not only within the prediction of Theorems 2.1 and 4.1, but also corroborate the fact that the iterative defect correction is a preferable choice for solving a nonlinear VIDE whose solution has limited regularity.

h	$\|u' - y'\|_\infty$	$\|u'_{it} - y'\|_\infty$
1/12	0.00476075400752	0.00004998837554
1/24	0.00153737923138	0.00000654669540
1/48	0.00059101530429	0.00000123164672
1/96	0.00025362659403	0.00000025924323
1/192	0.00011821039946	0.00000005955378
1/384	0.00005808099842	0.00000001450022

Table 4. Errors of the PGFE derivative and those generated by the iterated PGFE derivative corresponding to the PGFE solution in the space $S_2^{(0)}(T_h)$. The exact solution is in $H^2(I)$ but not in $H^3(I)$.

h	$\|u'_{it} - y'\|_\infty$	$\|\tilde{u}^{(c)}_{it,2} - y'\|_\infty$
1/12	0.00471978632177	0.00007105017907
1/24	0.00121073043422	0.00000901726331
1/48	0.00030596303644	0.00000113128302
1/96	0.00007686152747	0.00000014152404
1/192	0.00001925914051	0.00000001769297
1/384	0.00000482008819	0.00000000221162

Table 5. Errors of the iterated PGFE derivative produced by the PGFE solution in $S_1^{(0)}(T_h)$ and those generated by the iterative defect correction. The exact solution is in $H^2(I)$ but not in $H^3(I)$.

Acknowledgement. The authors are grateful to Professor Michal Křížek and an anonymous referee for their valuable comments on this paper which helped to improve the presentation of the paper.

References

[1] *H. Brunner*: A survey of recent advances in the numerical treatment of Volterra integral and integro-differential equations. J. Comput. Appl. Math. *8* (1982), 213–229.

[2] *H. Brunner*: Nonpolynomial spline collocation for Volterra equations with weakly singular kernels. SIAM J. Numer. Anal. *20* (1983), 1106–1119.

[3] *H. Brunner*: Implicit Runge-Kutta methods of optimal order for Volterra integro-differential equations. Math. Comp. *42, 165* (1984), 95–109.

[4] *H. Brunner*: Polynomial spline collocation methods for Volterra integrodifferential equations with weakly singular kernels. IMA J. Numer. Anal. *6* (1986), 221–239.

[5] *H. Brunner*: The numerical treatment of Volterra integro-differential equations with unbounded delay. J. Comput. Appl. Math. *28* (1989), 5–23.

[6] *H. Brunner, P. J. van der Houwen*: The Numerical Solution of Volterra Equations. CWI Monographs, Vol. 3, North-Holland, Amsterdam, 1986.

[7] *H. Brunner, J.-P. Kauthen and A. Ostermann*: Runge-Kutta time discretizations of parabolic Volterra integro-differential equations. J. Integral Equations Appl. *7* (1995), 1–16.
[8] *H. Brunner, Y. Lin and S. Zhang*: Higher accuracy methods for second-kind Volterra integral equations based on asymptotic expansions of iterated Galerkin methods. J. Integral Equations Appl. *10* (1998), 375–396.
[9] *H. Brunner, A. Makroglou and R. K. Miller*: Mixed interpolation collocation methods for first and second order Volterra integro-differential equations with periodic solution. Appl. Numer. Math. *23* (1997), 381–402.
[10] *J. Cannon, Y. Lin*: A priori L^2 error estimates for finite element methods for nonlinear diffusion equations with memory. SIAM J. Numer. Anal. *27* (1990), 595–607.
[11] *Q. Hu*: Stieltjes derivatives and β-polynomial spline collocation for Volterra integro-differential equations with singularities. SIAM J. Numer. Anal. *33* (1996), 208–220.
[12] *J. P. Kauthen*: The method of lines for parabolic partial integro-differential equations. J. Integral Equations Appl. *4* (1992), 69–81.
[13] *M. Křížek, P. Neittaanmäki*: Finite Element Approximation of Variational Problems and Applications. Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Essex, 1990.
[14] *M. Křížek, P. Neittaanmäki*: Bibliography on Superconvergence. In: Finite Element Methods: Superconvergence, Post-Processing, and A Posteriori Estimates (edited by M. Křížek, P. Neittaanmäki and R. Stenberg). Lecture Notes in Pure and Applied Mathematics, vol. 196. Marcel Dekker, New York, 1998, pp. 315–348.
[15] *Q. Lin, N. Yan*: The Construction and Analysis of High Efficiency Finite Element Methods. Hebei University Publishers, 1996.
[16] *Q. Lin, S. Zhang*: An immediate analysis for global superconvergence for integro-differential equations. Appl. Math. *42* (1997), 1–21.
[17] *Q. Lin, S. Zhang and N. Yan*: Methods for improving approximate accuracy for hyperbolic integro-differential equations. Systems Sci. Math. Sci. *10* (1997), 282–288.
[18] *Q. Lin, S. Zhang and N. Yan*: An acceleration method for integral equations by using interpolation post-processing. Adv. in Comput. Math. *9* (1998), 117–128.
[19] *T. Lin, Y. Lin, M. Rao and S. Zhang*: Petrov-Galerkin methods for nonlinear Volterra integro-differential equations. Submitted to J. Math. Anal. Appl.
[20] *R. K. Miller*: An integro-differential equation for rigid heat conductions with memory. J. Math. Anal. Appl. *66* (1978), 313–332.
[21] *J. Prüss*: Evolutionary Integral Equations and Applications. Birkhauser Verlag, Basel, 1993.
[22] *T. Tang*: Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations. Numer. Math. *61* (1992), 373–382.
[23] *T. Tang*: A note on collocation methods for Volterra integro-differential equations with weakly singular kernels. IMA J. Numer. Anal. *13* (1993), 93–99.

Authors' addresses: *Shuhua Zhang*, Department of Mathematics and Physics, Tianjin University of Commerce, Tianjin, 300400 P. R. China, e-mail: `shuhua@263.net`; *Tao Lin*, Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA, e-mail: `tlin@math.vt.edu`; *Yanping Lin*, Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1, e-mail: `ylin@hilbert.math.ualberta.ca`; *Ming Rao*, Department of Chemical and Materials Engineering, University of Alberta, Edmonton, Alberta, Canada T6G 2G6, e-mail: `ming.rao@ualberta.ca`.