Applications of Mathematics

Alexander Ženíšek

Green's theorem from the viewpoint of applications

Applications of Mathematics, Vol. 44 (1999), No. 1, 55–80

Persistent URL: `http://dml.cz/dmlcz/134406`

Terms of use:

© Institute of Mathematics AS CR, 1999

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

GREEN'S THEOREM FROM THE VIEWPOINT OF APPLICATIONS

Alexander Ženíšek,[1] Brno

(Received May 27, 1998)

Abstract. Making use of a line integral defined without use of the partition of unity, Green's theorem is proved in the case of two-dimensional domains with a Lipschitz-continuous boundary for functions belonging to the Sobolev spaces $W^{1,p}(\Omega) \equiv H^{1,p}(\Omega)$ $(1 \leqslant p < \infty)$.

Keywords: Green's theorem, elliptic problems, variational problems

MSC 2000: 65N99

Introduction

In [5] (and similarly in [4]) the surface and line integrals are defined with the use of the partition of unity in a rather complicated and unnatural way. This approach is then used with advantage in the proofs of the trace theorems and the Green's theorem. Using Green's theorem proved in such a way for deriving the variational formulation of the boundary value problem

$$-\Delta u = f \text{ in } \Omega, \quad u\big|_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial n}\Big|_{\Gamma_2} = q,$$

where $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, we obtain the following problem: Find

$$u \in V := \big\{v \in H^1(\Omega)\colon\ u\big|_{\Gamma_1} = 0\big\}$$

such that

$$\iint_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \iint_\Omega vf\,\mathrm{d}x\,\mathrm{d}y + \int_{\Gamma_2} vq\,\mathrm{d}s \quad \forall v \in V.$$

[1] The work was supported by the grant No. 201/97/0153 of the Grant Agency of the Czech Republic.

In this variational problem the line integral is defined with the use of the partition of unity and thus it is not suitable for practical computations.

In this paper we define a line integral in a quite natural way (this means, *without use* of partition of unity) and prove the Green's theorem and necessary trace theorems. In [8] we will generalize the results of this paper to the three-dimensional case.

1. Curves in a plane

1.1. Definition (of a simple curve). Let

$$x = \varphi(t),\ y = \psi(t), \quad t \in \langle a, b\rangle \tag{1.1}$$

be two functions *continuous* on a segment $\langle a, b\rangle$ where $a < b$. For arbitrary two different values $t_1, t_2 \in \langle a, b\rangle$ let us have

$$[\varphi(t_1), \psi(t_1)] \neq [\varphi(t_2), \psi(t_2)]. \tag{1.2}$$

Then the set Γ of all points $[\varphi(t), \psi(t)]$, $t \in \langle a, b\rangle$, is called *a simple curve.*

1.2. Definition (orientation of a simple curve). a) A simple curve Γ is oriented *in agreement* with the parametric expression (1.1), if for two arbitrary points $P_1 = [\varphi(t_1), \psi(t_1)]$, $P_2 = [\varphi(t_2), \psi(t_2)]$ of the curve Γ we say that P_1 precedes P_2 if $t_1 < t_2$.

b) A simple curve Γ is oriented *not in agreement* with the parametric expression (1.1), if for two arbitrary points $P_1 = [\varphi(t_1), \psi(t_1)]$, $P_2 = [\varphi(t_2), \psi(t_2)]$ of the curve Γ we say that P_1 precedes P_2 if $t_1 > t_2$.

1.3. Definition (of the first and last point). a) Let Γ be a simple curve oriented in agreement with the parametric expression (1.1). Then the point $A = [\varphi(a), \psi(a)]$ is called the first point of the curve Γ and the point $B = [\varphi(b), \psi(b)]$ the last point of this curve. Both points are called the end points of the curve Γ.

b) If Γ is not oriented in agreement with parametric expression (1.1) then the point $A = [\varphi(a), \psi(a)]$ is called the last point and the point $B = [\varphi(b), \psi(b)]$ the first point of the curve Γ.

1.4. Definition (of a closed simple curve). Let functions (1.1) be continuous on the segment $\langle a, b\rangle$ and let

$$[\varphi(a), \psi(a)] = [\varphi(b), \psi(b)].$$

Further, let an arbitrary pair of numbers $t_1, t_2 \in \langle a, b\rangle$ ($t_1 \neq t_2$) which is different from the pair a, b (or b, a) satisfy (1.2). Then the set Γ of points $[\varphi(t), \psi(t)]$, $t \in \langle a, b\rangle$, is called *a closed simple curve.*

1.5. Definition (orientation of a closed simple curve). Let Γ be a closed simple curve. Let us orientate it, i.e., let us choose the direction in which we go around it. Let P_1, P_2, P_3 be three arbitrary mutually different points of the curve Γ, which are denoted in such a way that in going round the curve in the sense of its orientation we go from P_1 to P_2, from P_2 to P_3 and from P_3 to P_1. Let t_1, t_2, t_3 be the parameters of the points P_1, P_2, P_3.

a) We say that the closed simple curve Γ is oriented *in agreement* with the parametric expression (1.1) if either $t_1 < t_2 < t_3$, or $t_3 < t_1 < t_2$, or $t_2 < t_3 < t_1$.

b) We say that the closed simple curve Γ *is not oriented in agreement* with the parametric expression (1.1) if either $t_3 < t_2 < t_1$, or $t_1 < t_3 < t_2$, or $t_2 < t_1 < t_3$.

1.6. Definition (of a simple smooth curve). Let Γ be a simple curve or a closed simple curve. Let there exist such a parametric expression (1.1) of Γ that functions $\varphi(t)$, $\psi(t)$ have besides the properties described in Definition 1.1 (or 1.4) the following property: The derivatives $\dot\varphi(t)$, $\dot\psi(t)$ are continuous on the segment $\langle a, b\rangle$ and

$$[\dot\varphi(t)]^2 + [\dot\psi(t)]^2 \neq 0 \quad \forall t \in \langle a, b\rangle \tag{1.3}$$

with a possible exception at the end points of the curve Γ. Let in the case of a simple closed curve the following conditions be also satisfied:

$$\dot\varphi(a) = \dot\varphi(b), \quad \dot\psi(a) = \dot\psi(b).$$

Then we say that Γ is a simple *smooth* curve (or *a simple smooth closed curve*).

1.7. Definition (of a piecewise smooth curve). Let Γ be a simple curve (or a simple closed curve), which is a union of simple smooth curves Γ_i $(i = 1, \dots, n)$. Let an arbitrary point of Γ, which is not the end point of any curve Γ_i, belong to only one of the curves $\Gamma_1, \dots, \Gamma_n$ (i.e., the curves $\Gamma_1, \dots, \Gamma_n$ have mutually disjoint interiors). Then we say that Γ is a piecewise smooth curve; the curves Γ_i are called its smooth parts.

1.8. Remark. Curves in the plane are often defined by the equation

$$y = f(x), \quad x \in \langle a, b\rangle, \tag{1.4}$$

or

$$x = g(y), \quad y \in \langle c, d\rangle. \tag{1.5}$$

Equations (1.4), (1.5) can be easily transformed to a parametric form:

$$x = t,\ y = f(t), \quad t \in \langle a, b \rangle, \tag{1.6}$$

$$x = g(t),\ y = t, \quad t \in \langle c, d \rangle. \tag{1.7}$$

1.9. R e m a r k . It should be noted that every curve can be expressed in a parametric form in infinitely many ways.

1.10. Definition. A given smooth curve is called *an elementary arc* (with respect to a given Cartesian coordinate system) if it is a segment parallel with one of the coordinate axes, or if every straight line parallel with the x-axis intersects this curve at most at one point and, similarly, if every straight line parallel with the y-axis intersects this curve at most at one point.

1.11. E x a m p l e . The half-circle $C_h = \{[x, y]\colon\ x^2 + y^2 = 1,\ y \geqslant 0\}$ is not an elementary arc but the quarter of the circle $C_q = \{[x, y]\colon\ x^2 + y^2 = 1,\ x \geqslant 0,\ y \geqslant 0\}$ is an elementary arc.

1.12. Convention. We shall consider only curves which can be divided into a finite number of elementary arcs.

1.13. R e m a r k . Convention 1.12 simplifies our considerations because it excludes all exotic curves which complicate theories and do not appear in applications. We mention some of them:

a) An infinite "saw" on the segment $\langle 0, 1 \rangle$: Let $f(0) = f(1) = 0$ and

$$f((2^k - 1)/2^k) = 0 \quad (k = 1, 2, \ldots).$$

On the subsegment $\langle 0, \frac{1}{2} \rangle$ the function $f(x)$ is such that its graph is formed by two oblique sides of the equilateral triangle whose basic side is the segment $\langle 0, \frac{1}{2} \rangle$. In general for $k \geqslant 2$: On the subsegment

$$\langle (2^{k-1} - 1)/2^{k-1}, (2^k - 1)/2^k \rangle \quad (k \geqslant 2)$$

the function $f(x)$ is such that its graph is formed by two oblique sides of the equilateral triangle whose basic side is the segment $\langle (2^{k-1} - 1)/2^{k-1}, (2^k - 1)/2^k \rangle$.

b) The function

$$f(x) = x^3 \sin \frac{1}{x}, \quad x \in \langle -1, 0) \cup (0, 1 \rangle,$$

$$f(0) = 0, \quad f'(0) = 0$$

belongs to $C^1\langle -1, 1 \rangle$ but has infinitely many relative extremes; hence, it cannot be divided into a finite number of elementary arcs.

c) Infinite spirals; they cannot be divided into a finite number of elementary arcs.

1.14. Remark. The notion "an elementary arc" is not invariant with respect to a transformation between two Cartesian coordinate systems; however, the notion "a curve which can be divided into a finite number of elementary arcs" is invariant. This fact is fundamental for our further considerations.

2. The notion of a line integral in the plane

2.1. Definition. Let Γ be a smooth plane curve with the parametric expression (1.1). Let functions $F(x,y)$, $P(x,y)$, $Q(x,y)$ be continuous in a domain $\mathcal{O}$ which contains the curve Γ. Then we define:

a) A line integral of the first kind by the relation

$$\int_\Gamma F(x,y)\,\mathrm{d}s := \int_a^b F(\varphi(t),\psi(t))\sqrt{[\dot\varphi(t)]^2+[\dot\psi(t)]^2}\,\mathrm{d}t. \tag{2.1}$$

The expression $\sqrt{\dot\varphi^2+\dot\psi^2}\,\mathrm{d}t$ is called the differential of the curve and we denote it by $\mathrm{d}s$.

b) Line integrals of the second kind by the relations

$$\int_\Gamma P(x,y)\,\mathrm{d}x+Q(x,y)\,\mathrm{d}y := \int_\Gamma P(x,y)\,\mathrm{d}x+\int_\Gamma Q(x,y)\,\mathrm{d}y, \tag{2.2}$$

where

$$\int_\Gamma P(x,y)\,\mathrm{d}x := \beta\int_a^b P(\varphi(t),\psi(t))\dot\varphi(t)\,\mathrm{d}t, \tag{2.3a}$$

$$\int_\Gamma Q(x,y)\,\mathrm{d}y := \beta\int_a^b Q(\varphi(t),\psi(t))\dot\psi(t)\,\mathrm{d}t, \tag{2.3b}$$

where $\beta=1$ (or $\beta=-1$) if the orientation of the curve Γ agrees (or disagrees) with the given parametric expression (1.1).

2.2. Remark. a) The line integral of the first kind does not depend on the orientation of the curve Γ.

b) The assumption concerning the continuity of functions $F(x,y)$, $P(x,y)$, $Q(x,y)$ in a domain $\mathcal{O}$ containing the curve Γ can be satisfied in all situations occurring in this paper. We introduce this assumption because of the proof of Theorem 2.6.

2.3. R e m a r k. In the case when the curve Γ is given by (1.4), the expressions (2.1) and (2.3) are reduced to the expressions

$$\int_\Gamma F(x,y)\,\mathrm{d}s = \int_a^b F(x,f(x))\sqrt{1+[f'(x)]^2}\,\mathrm{d}x, \tag{2.1*}$$

$$\int_\Gamma P(x,y)\,\mathrm{d}x = \beta\int_a^b P(x,f(x))\,\mathrm{d}x, \tag{2.3a*}$$

$$\int_\Gamma Q(x,y)\,\mathrm{d}y = \beta\int_a^b Q(x,f(x))f'(x)\,\mathrm{d}x; \tag{2.3b*}$$

in the case when the curve Γ is given by (1.5) the expressions (2.1) and (2.3) are reduced to the expressions

$$\int_\Gamma F(x,y)\,\mathrm{d}s = \int_a^b F(g(y),y)\sqrt{1+[g'(y)]^2}\,\mathrm{d}y, \tag{2.1**}$$

$$\int_\Gamma P(x,y)\,\mathrm{d}x = \beta\int_a^b P(g(y),y)g'(y)\,\mathrm{d}y, \tag{2.3a**}$$

$$\int_\Gamma Q(x,y)\,\mathrm{d}y = \beta\int_a^b Q(g(y),y)\,\mathrm{d}y. \tag{2.3b**}$$

2.4. Definition (of the line integral along a piecewise smooth curve). If Γ is a piecewise smooth curve which consists of smooth curves $\Gamma_1,\dots,\Gamma_n$ (see Definition 1.7), then we define

$$\int_\Gamma F(x,y)\,\mathrm{d}s = \sum_{k=1}^n \int_{\Gamma_k} F(x,y)\,\mathrm{d}s, \tag{2.4}$$

$$\int_\Gamma P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \sum_{k=1}^n \int_{\Gamma_k} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y. \tag{2.5}$$

2.5. R e m a r k. It can be easily proved that Definition 2.4 is not contradictory. This means: If

$$\Gamma = \bigcup_{j=1}^m \overline{\Gamma}_j, \quad \Gamma = \bigcup_{k=1}^n \overline{C}_k,$$

where $\Gamma_1,\dots,\Gamma_m$ and $C_1,\dots,C_n$ are simple smooth curves satisfying

$$\Gamma_i\cap\Gamma_j=\emptyset \ \ (i\neq j;\ i,j=1,\dots,m), \quad C_i\cap C_j=\emptyset \ \ (i\neq j;\ i,j=1,\dots,n),$$

then

$$\sum_{j=1}^m \int_{\Gamma_j} F(x,y)\,\mathrm{d}s = \sum_{k=1}^n \int_{C_k} F(x,y)\,\mathrm{d}s,$$

$$\sum_{j=1}^{m} \int_{\Gamma_j} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \sum_{k=1}^{n} \int_{C_k} P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y.$$

2.6. Theorem. *The line integral of the first or second kind along a smooth curve does not depend on a parametric representation of the curve.*

For the proof see, e.g., [6].

3. Green's theorem in elementary domains

3.1. Definition (of a two-dimensional elementary domain). Let x, y be a Cartesian coordinate system.

a) A two-dimensional bounded domain Ω is called *elementary with respect to the x-axis*, if every straight line parallel to the y-axis intersects the boundary $\partial\Omega$ at two different points or has with $\partial\Omega$ a common segment which can degenerate to a point.

b) A two-dimensional bounded domain Ω is called *elementary with respect to the y-axis*, if every straight line parallel to the x-axis intersects the boundary $\partial\Omega$ at two different points or has with $\partial\Omega$ a common segment which can degenerate to a point.

c) A two-dimensional bounded domain Ω is called *elementary with respect to a given Cartesian coordinate system* if it is elementary with respect to each of the coordinate axes.

3.2. Remarks. a) Every bounded convex domain is elementary with respect to every Cartesian coordinate system.

b) It follows from Definition 3.1 that every elementary domain is simply connected.

c) The notion of a domain which is elementary with respect to a given Cartesian coordinate system has a disadvantage: this notion is not invariant to a change of coordinates. For example, the domain which is bounded in the system x, y by two segments A_1A_2, A_2A_3 and the curve σ, where $A_1 = [0,1]$, $A_2 = [0,0]$, $A_3 = [1,0]$ and σ is defined by the relations

$$x = 1 + \cos t, \quad y = 1 + \sin t, \quad t \in \left\langle \pi, \tfrac{3}{2}\pi \right\rangle,$$

is not elementary with respect to other Cartesian coordinate systems, whose axes are not parallel with x and y.

3.3. Definition (of the orientation of the boundary with respect to the domain). a) Let a simply connected domain Ω be bounded by a simple closed curve $\partial\Omega$. We say that $\partial\Omega$ is oriented positively with respect to the domain Ω if Ω is on our left-hand side when we move along $\partial\Omega$ in the direction of its orientation.

b) Let Ω be a multiply connected bounded domain, i.e.,

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_m \quad (\Gamma_i \cap \Gamma_j = \emptyset,\ i \neq j),$$

where Γ_k ($k = 0, \ldots, m$) are closed simple curves such that $\Gamma_1, \ldots, \Gamma_m \subset \Omega_0$, where Ω_0 is a simply connected domain with the boundary $\partial\Omega_0 \equiv \Gamma_0$. We say that $\partial\Omega$ is oriented positively with respect to the domain Ω if Γ_k are oriented so that Ω is on our left-hand side when we move along Γ_k in the direction of its orientation.

The curve Γ_0 is called the *external curve*; the curves $\Gamma_1, \ldots, \Gamma_m$ are called *interior curves.* They are boundaries of the holes of Ω.

3.4. Definition (of a piecewise smooth boundary). Let a domain Ω satisfy the conditions of Definition 3.3 with $0 \leqslant m < \infty$. Then the (non-oriented) boundary $\partial\Omega$ is called a *piecewise smooth boundary of the domain* Ω. (In other words: If we say "Let the domain Ω have a piecewise smooth boundary $\partial\Omega$" then we assume that the assumptions of Definition 3.3 are satisfied with $0 \leqslant m < \infty$ (except for the orientation of $\partial\Omega$). Attention: Convention 1.12 is taken into account.)

3.5. Theorem (Green's theorem for a domain consisting of elementary domains). *Let a bounded domain Ω satisfy the condition*

$$\overline{\Omega} = \bigcup_{k=1}^{n} \overline{\Omega}_k, \quad \Omega_i \cap \Omega_j = \emptyset \quad (i, j = 1, \ldots, n), \tag{3.1}$$

where $\Omega_1, \ldots, \Omega_n$ are elementary domains with respect to a given Cartesian coordinate system x, y. Let functions P, Q and their derivatives $\partial P/\partial y$, $\partial Q/\partial x$ be continuous on $\overline{\Omega}$. Let the boundary $\partial\Omega$ of the domain Ω be piecewise smooth and positively oriented with respect to the domain Ω. Then

$$\iint_{\overline{\Omega}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega} P \, \mathrm{d}x + Q \, \mathrm{d}y. \tag{3.2}$$

This theorem is proved in every basic course of mathematical analysis.

3.6. Example of a domain consisting of elementary domains. Let $\overline{\Omega}$ be a closed domain bounded by two circles with the same center at the origin and with radii R_1, R_2 ($0 < R_1 < R_2$). The axes x, y divide the domain $\overline{\Omega}$ into four closed domains $\overline{\Omega}_1, \ldots, \overline{\Omega}_4$ with disjoint interiors $\Omega_1, \ldots, \Omega_4$: $\Omega_i \cap \Omega_j = \emptyset$ ($i \neq j$; $i, j = 1, \ldots, 4$). The domains $\overline{\Omega}_1, \ldots, \overline{\Omega}_4$ are elementary with respect to the Cartesian coordinate system x, y and each two adjacent domains $\overline{\Omega}_i$, $\overline{\Omega}_j$ have a common segment $\overline{\Omega}_i \cap \overline{\Omega}_j$ which lies either on the x-axis, or on the y-axis.

Theorem 3.4 is not sufficiently general for applications. The verification of its assumptions is in many simple cases quite impossible. To divide, for example, "a slice of cheese with many bubbles"

$$\overline{\Omega} = \overline{K}_0 - \bigcup_{j=1}^{N} K_j \quad (\overline{K}_j \subset K_0,\ \overline{K}_i \cap \overline{K}_j = \emptyset,\ i,j = 1,\dots,N),$$

where $\overline{K}_0, \overline{K}_1, \dots, \overline{K}_N$ are circles, into a union of a finite number of elementary domains with mutually disjoint interiors is a very difficult and often impossible task.

Thus our aim is to substitute in the Green's theorem the assumption concerning the elementary subdomains by assumptions concerning the properties of the boundary $\partial\Omega$ of a domain Ω. We start in the next section with some auxiliary theorems.

4. Auxiliary theorems

4.1. Definition (of a polygon and a polyhedron). a) By a polygon we understand every nonempty, bounded and closed domain in $\mathbb{R}^2$ the boundary of which can be expressed as a union of a finite number of segments.

b) By a polyhedron we understand every nonempty, bounded and closed domain in $\mathbb{R}^3$ the boundary of which can be expressed as a union of a finite number of polygons with mutually disjoint interiors.

4.2. Lemma (on a decomposition of a polygon and a polyhedron into convex components). a) *For every polygon $\overline{\Omega}$ there exists a finite number of convex polygons with mutually disjoint interiors the union of which is $\overline{\Omega}$.*

b) *For every polyhedron $\overline{\Omega}$ there exists a finite number of convex polyhedrons with mutually disjoint interiors the union of which is $\overline{\Omega}$.*

P r o o f. The proof is part of the proof of a more general theorem (see [3]). However, because of the importance of the lemma we reproduce the corresponding part of Křížek's proof in a slight extended form.

The proof is presented in the three-dimensional case; this part of Lemma 4.2 will play a fundamental role in the proof of the Gauss-Ostrogradskij theorem. In the two-dimensional case the proof of Lemma 4.2 is analogous but simpler.

Let $\overline{\Omega}$ be an arbitrary polyhedron and let $\pi^1, \dots, \pi^m$ be polygons the union of which is the boundary $\partial\Omega$. Let $\varrho^1, \dots, \varrho^m$ be such planes that $\pi^i \subset \varrho^i$, $i = 1, \dots, m$. It may happen that some of these planes will coincide. Without loss of generality let us assume that $\varrho^1, \dots, \varrho^k$ $(k \leqslant m)$ are mutually different planes and each ϱ^i $(k < i \leqslant m)$ belongs to the set $\{\varrho^1, \dots, \varrho^k\}$. Let $\Omega_1, \dots, \Omega_r \subset \mathbb{R}^3$ be all components of the set $\overline{\Omega} - \bigcup_{i=1}^{k} \varrho^i$ (i.e. the components which arise after "cutting up" the polyhedron $\overline{\Omega}$ by the planes ϱ^i). The number of these components is finite (at most 2^k). We assert that $\overline{\Omega}_j$ $(j = 1, \dots, r)$ are the sought convex polyhedrons.

First we show that Ω_j are open sets. As $\partial\Omega \subset \bigcup_{i=1}^{k} \varrho^i$ we have

$$\overline{\Omega} - \bigcup_{i=1}^{k} \varrho^i = \Omega - \bigcup_{i=1}^{k} \varrho^i.$$

This set is open because Ω is an open set and $\bigcup_{i=1}^{k} \varrho^i$ is a closed set, and components of an open set are open.

Further we prove the convexity of $\overline{\Omega}_j$. Let $j \in \{1, \dots, r\}$ be an arbitrary fixed integer. Each plane ϱ^i $(i = 1, \dots, k)$ divides the space $\mathbb{R}^3$ into two half-spaces. Let us denote by Q^i the closed half-space which is bounded by the plane ϱ^i and which contains $\overline{\Omega}_j$, and let us denote $M := \bigcap_{i=1}^{k} Q^i$. Then we have $\overline{\Omega}_j \subset M$. The opposite inclusion will be proved by contradiction. Let us assume that there exists a point $P \in M - \overline{\Omega}_j$. As $\overline{\Omega}_j$ is a closed set we have $R = \operatorname{dist}(P, \overline{\Omega}_j) > 0$; this means that

$$M - \overline{\Omega}_j \supset M \cap \mathcal{S}_R(P) \neq \emptyset,$$

where $\mathcal{S}_R(P)$ is an open ball of a radius R and with the center at P, which lies in the convex set M. Let $X \in M \cap \mathcal{S}_R(P)$ be such a point that does not belong to any plane $\varrho^1, \dots, \varrho^k$ and let Y be an arbitrary interior point of $\overline{\Omega}_j$ (such a point certainly exists because Ω_j is a domain). Then inside the segment $\overline{XY}$ there exists such a point Z that $Z \in \partial\Omega_j$ (because $X \notin \overline{\Omega}_j$). As Z is a boundary point of $\overline{\Omega}_j$ there exists a plane ϱ^s $(1 \leqslant s \leqslant k)$ such that $Z \in \varrho^s$ and this plane separates the points X a Y because $X \notin \varrho^s$, $Y \notin \varrho^s$. This implies that $X \notin Q^s$, which contradicts the fact that $X \in M \subset Q^s$. Hence

$$\overline{\Omega}_j = \bigcap_{i=1}^{k} Q^i$$

and this intersection is evidently bounded and has at least one interior point. In other words, $\overline{\Omega}_j$ is a convex polyhedron.

Further, the definition of components Ω_j $(j = 1, \dots, r)$, i.e., the relation

$$\overline{\Omega} - \bigcup_{i=1}^{k} \varrho^i = \bigcup_{j=1}^{r} \Omega_j,$$

immediately implies that $\overline{\Omega} = \bigcup_{j=1}^{r} \overline{\Omega}_j$. $\square$

4.3. Definition (of cusp-points (turning points)). a) Let Γ be a simple piecewise smooth curve. A point $B \in \Gamma$ is called a *cusp-point* if the curve Γ is not smooth at this point, if B is the end point of two smooth parts of Γ (let us denote them Γ_1 and Γ_2) and if Γ_1 and Γ_2 have at B a common nonoriented tangent.

b) A cusp-point is called *one-sided* if there exists a neighbourhood of this point such that in it both smooth parts lie in the same half-plane which is determined by the common tangent. In the opposite case a cusp-point is called *two-sided*.

4.4. Theorem (on extension of functions with keeping their class). *Let the boundary $\partial\Omega$ of a bounded domain $\overline{\Omega} \subset \mathbb{R}^2$ be piecewise smooth and let it have no cusp-points. Let a function $f(x, y)$ and its derivatives $\partial f/\partial x$, $\partial f/\partial y$ be continuous on the closed domain $\overline{\Omega}$ and let*

$$\max_{\overline{\Omega}} |f(x, y)| \leqslant M, \quad \max_{\overline{\Omega}} \left|\frac{\partial f}{\partial x}(x, y)\right| \leqslant M, \quad \max_{\overline{\Omega}} \left|\frac{\partial f}{\partial y}(x, y)\right| \leqslant M.$$

Then there exist a circle $K_R \supset \overline{\Omega}$ (the magnitude of its radius R depends on the function $f(x, y)$) and a function $\widetilde{f}(x, y)$ which is continuous and continuously differentiable in the whole plane $\mathbb{R}^2$, outside the circle K_R is equal to zero and satisfies

$$\widetilde{f}(x, y) = f(x, y) \quad \forall [x, y] \in \overline{\Omega},$$
$$\max_{\overline{K_R}} \bigl|\widetilde{f}(x, y)\bigr| \leqslant C, \quad \max_{\overline{K_R}} \left|\frac{\partial \widetilde{f}}{\partial x}(x, y)\right| \leqslant C, \quad \max_{\overline{K_R}} \left|\frac{\partial \widetilde{f}}{\partial y}(x, y)\right| \leqslant C.$$

The proof of this theorem is given, for example, in [1, pp. 674–687] (see also the German translation [2, pp. 550–562]). It is noted in [1], [2] that the origins of the proof belong to H. Whitney and M.R. Hestenes. A generalization of these considerations leads to Nikolskij-Babič extension theorem (see [5, pp. 75–77], or in a more general form in [7, pp. 18–24]).

Domains whose boundaries satisfy the assumptions contained in Theorem 4.4 form a subset of domains with Lipschitz-continuous boundaries. Therefore, we introduce

4.5. Definition. We say that a bounded domain $\Omega \subset \mathbb{R}^2$ has an S-Lipschitz continuous boundary $\partial\Omega$ if $\partial\Omega$ is piecewise smooth and has no cusp-points.

Domains with S-Lipschitz continuous boundaries cover the set of domains with Lipschitz continuous boundaries used in applications.

5.1. Theorem (Green). *Let a* (*simply or multiply connected*) *domain* Ω *have an* S*-Lipschitz continuous boundary. Let the boundary* $\partial\Omega$ *of the domain* Ω *be oriented positively with respect to the domain* Ω*. Let finally* P, $Q \in C^1(\overline{\Omega})$*. Then relation* (3.2) *holds.*

Proof. We will present two different proofs of Theorem 5.1. The first will be generalized in [8] in the proof of the Gauss-Ostrogradskij theorem, the second is based on special properties of $\mathbb{R}^2$.

1) Let us write the expression for the boundary $\partial\Omega$ of the domain Ω in the form

$$\partial\Omega = \bigcup_{k=0}^{m} \Gamma_k, \tag{5.1}$$

where $\Gamma_1, \ldots, \Gamma_m$ are interior curves, which form also the boundaries of the "holes", and Γ_0 is the external curve.

Let us choose $\varepsilon > 0$ and let us approximate the bounded closed domain $\overline{\Omega}$ by a convenient polygon $\overline{\Omega}_\varepsilon$ such that

$$\sum_{k=1}^{N(\varepsilon)} \text{meas}_2 \Delta_k < \varepsilon, \tag{5.2}$$

where $\Delta_1, \ldots, \Delta_{N(\varepsilon)}$ are crescent-shaped domains created by segments which form $\partial\Omega_\varepsilon$, and smooth parts of $\partial\Omega$, which are approximated by these segments.

Theorem 3.5 and Lemma 4.2 imply

$$\iint_{\overline{\Omega}_\varepsilon} \left(\frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{P}}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega_\varepsilon} \widetilde{P} \, \mathrm{d}x + \widetilde{Q} \, \mathrm{d}y, \tag{5.3}$$

where $\widetilde{P}$ and $\widetilde{Q}$ are the extensions of functions P and Q, respectively, according to Theorem 4.4. Relation (5.3) can be written in the form

(5.4)
$$\iint_{\overline{\Omega}} \left(\frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{P}}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y + \iint_{\overline{\Omega}_\varepsilon} \left(\frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{P}}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y - \iint_{\overline{\Omega}} \left(\frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{P}}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y$$
$$= \int_{\partial\Omega} \widetilde{P} \, \mathrm{d}x + \widetilde{Q} \, \mathrm{d}y + \int_{\partial\Omega_\varepsilon} \widetilde{P} \, \mathrm{d}x + \widetilde{Q} \, \mathrm{d}y - \int_{\partial\Omega} \widetilde{P} \, \mathrm{d}x + \widetilde{Q} \, \mathrm{d}y.$$

The assumptions of Theorem 5.1 and relation (5.2) give

$$\begin{aligned}(5.5)\qquad &\left|\iint_{\overline{\Omega}_\varepsilon}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y-\iint_{\overline{\Omega}}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right|\\ &\leqslant\sum_{k=1}^{N(\varepsilon)}\left|\iint_{\Delta_k}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right|\leqslant K\sum_{k=1}^{N(\varepsilon)}\mathrm{meas}_2\Delta_k\leqslant K\varepsilon.\end{aligned}$$

As to the difference of the last two integrals on the right-hand side of (5.4) we have (because $\partial\Omega_\varepsilon$ is oriented positively with respect to Ω_ε and $\partial\Omega$ positively with respect to Ω)

$$\int_{\partial\Omega_\varepsilon}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y-\int_{\partial\Omega}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y=\sum_{k=1}^{N(\varepsilon)}\delta_k\int_{\partial\Delta_k}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y,$$

where $\delta_k=1$ if the segment forming part of the boundary $\partial\Delta_k$ lies outside the domain Ω, and $\delta_k=-1$ if this segment lies in $\overline{\Omega}$. As each Δ_k is a convex domain, we further have

$$\begin{aligned}(5.6)\qquad &\left|\int_{\partial\Omega_\varepsilon}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y-\int_{\partial\Omega}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y\right|\\ &\leqslant\sum_{k=1}^{N(\varepsilon)}\left|\int_{\partial\Delta_k}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y\right|=\sum_{k=1}^{N(\varepsilon)}\left|\iint_{\Delta_k}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\right|<K\varepsilon.\end{aligned}$$

As $\varepsilon>0$ is arbitrary, relations (5.4)–(5.6) imply the assertion of Theorem 5.1.

2) In the second proof let us approximate the domain Ω by a domain Ω_h with a polygonal boundary $\partial\Omega_h$ the vertices of which lie on $\partial\Omega$. This approximation need not be close; however, it must have the property that the set bounded by the curves $\partial\Omega$ and $\partial\Omega_h$ is a union of convex domains $G_1,\dots,G_n$ which have the crescent-shaped form.

We have, according to Theorem 3.5 and Lemma 4.2,

$$(5.7)\qquad \iint_{\overline{\Omega}_h}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\int_{\partial\Omega_h}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y,$$

where $\widetilde{P}$ and $\widetilde{Q}$ are extensions of functions P and Q, respectively, according to Theorem 4.4. Relation (5.7) can be written in the form

(5.8)

$$\iint_{\overline{\Omega}_h}\left(\frac{\partial\widetilde{Q}}{\partial x}-\frac{\partial\widetilde{P}}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\int_{\partial\Omega_h}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y-\int_{\partial\Omega}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y+\int_{\partial\Omega}\widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y.$$

The crescent-shaped domains lying outside Ω_h will be denoted by $G^+_{k_i}$, the crescent-shaped domains lying in the interior of Ω_h will be denoted by $G^-_{k_j}$. We have

(5.9)
$$\int_{\partial\Omega_h} \widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y-\int_{\partial\Omega} \widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y = -\sum_i \int_{\partial G^+_{k_i}} \widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y+\sum_j \int_{\partial G^-_{k_j}} \widetilde{P}\,\mathrm{d}x+\widetilde{Q}\,\mathrm{d}y;$$

the boundary of each crescent-shaped domain G_k is oriented positively, hence the minus sign at the first sum on the right-hand side of (5.9) follows. Using Green's theorem for a convex domain we obtain

$$\int_{\partial G_k} \widetilde{P}\,\mathrm{d}x + \widetilde{Q}\,\mathrm{d}y = \iint_{G_k} \left(\frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{P}}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y. \tag{5.10}$$

The last relation which we need has the form

$$\iint_{\Omega_h} \widetilde{F}\,\mathrm{d}x\,\mathrm{d}y + \sum_i \iint_{G^+_{k_i}} \widetilde{F}\,\mathrm{d}x\,\mathrm{d}y - \sum_j \iint_{G^-_{k_j}} \widetilde{F}\,\mathrm{d}x\,\mathrm{d}y = \iint_{\Omega} \widetilde{F}\,\mathrm{d}x\,\mathrm{d}y, \tag{5.11}$$

where $\widetilde{F} := \partial\widetilde{Q}/\partial x - \partial\widetilde{P}/\partial y$. Combining (5.8)–(5.11) we obtain (3.2).

The second proof surprises by the fact that we did not need to use the limit passage from Ω_h to Ω. This is only possible in the two-dimensional case. □

6. The divergence form of Green's theorem

Every point $[\varphi(t), \psi(t)] \in \Gamma$ at which relation (1.3) is satisfied, is called *an ordinary point* of the curve Γ.

6.1. Lemma (on a tangent of a smooth curve). *At every ordinary point of a smooth curve Γ there exists a tangent, the direction cosines of which satisfy*

$$\cos\alpha = \beta \frac{\dot\varphi(t)}{\sqrt{[\dot\varphi(t)]^2 + [\dot\psi(t)]^2}}, \quad \sin\alpha = \beta \frac{\dot\psi(t)}{\sqrt{[\dot\varphi(t)]^2 + [\dot\psi(t)]^2}} \tag{6.1}$$

provided the orientation of the tangent is defined as a continuation of the orientation of the curve Γ. The symbol β has the same meaning as in Definition 2.1b.

For the proof see [1] or [2]. The following lemma is an immediate consequence of Definition 2.1 and Lemma 6.1.

6.2. Lemma. *Let Γ be a smooth oriented curve and let the orientation of the tangent to Γ be at every point of Γ a continuation of the orientation of Γ. Let $\alpha(x, y)$*

be the angle between the oriented tangent at the point $[x, y] \in \Gamma$ *and the positive direction of the* x*-axis. Then*

$$\int_\Gamma P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \int_\Gamma [P(x,y)\cos\alpha(x,y) + Q(x,y)\sin\alpha(x,y)]\,\mathrm{d}s. \tag{6.2}$$

6.3. Theorem (divergence form of Green's theorem). *Let a (simply or multiply connected) domain* Ω *have an* S*-Lipschitz continuous boundary. Let the boundary* $\partial\Omega$ *of the domain* Ω *be oriented positively with respect to the domain* Ω*. Then for all functions* $P_1, P_2 \in C^1(\overline{\Omega})$ *we have*

$$\iint_\Omega \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} (P_1\cos\omega + P_2\sin\omega)\,\mathrm{d}s \equiv \int_{\partial\Omega} (P_1 n_1 + P_2 n_2)\,\mathrm{d}s, \tag{6.3}$$

where ω *is the angle between the unit vector* (n_1, n_2) *of the outer normal to the boundary* $\partial\Omega$ *and the positive direction of the* x*-axis.*

P r o o f. Let us denote $P = -P_2$, $Q = P_1$. According to the assumptions of Theorem 6.3, we can use Theorem 5.1 and write

$$\iint_{\overline{\Omega}} \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y = \int_{\partial\Omega} (-P_2\,\mathrm{d}x + P_1\,\mathrm{d}y). \tag{6.4}$$

According to Lemma 6.2 the right-hand side satisfies

$$\int_{\partial\Omega} (-P_2\,\mathrm{d}x + P_1\,\mathrm{d}y) = \int_{\partial\Omega} (-P_2\cos\alpha + P_1\sin\alpha)\,\mathrm{d}s. \tag{6.5}$$

As the boundary $\partial\Omega$ is oriented positively with respect to the domain Ω, the angles α a ω satisfy the relation $\omega = \alpha - \pi/2$. Hence

$$\cos\omega = \sin\alpha, \quad \sin\omega = -\cos\alpha. \tag{6.6}$$

Substituting (6.6) into the right-hand side of (6.5) and using (6.4) we obtain (6.3). □

6.4. R e m a r k. Let $u, v \in C^1(\overline{\Omega})$. Setting $P_i = uv$, $P_j \equiv 0$ in (6.3) we obtain another form of Green's formula:

$$\iint_\Omega \frac{\partial u}{\partial x_i} v\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\partial\Omega} uvn_i\,\mathrm{d}s - \iint_\Omega u\frac{\partial v}{\partial x_i}\,\mathrm{d}x_1\,\mathrm{d}x_2 \tag{6.7}$$

where we set $x_1 := x$, $x_2 := y$.

7.1. Lemma (on the invariance of the line integral). *Let (x, y) and (ξ, η) be two Cartesian coordinate systems related by the transformation*

$$x = x_0 + a_1\xi + a_2\eta, \quad y = y_0 + b_1\xi + b_2\eta, \tag{7.1}$$

where $[x_0, y_0]$ is the origin of the system (ξ, η) in the system (x, y). Let the (nonoriented) smooth curve Γ be expressed in the system (x, y) by

$$y = f(x), \quad x \in \langle a, b\rangle, \quad a < b, \tag{7.2}$$

where $f \in C^1\langle a, b\rangle$, and in the system (ξ, η) by

$$\eta = \varphi(\xi), \quad \xi \in \langle \alpha, \beta\rangle, \quad \alpha < \beta, \tag{7.3}$$

where $\varphi \in C^1\langle \alpha, \beta\rangle$. Let

$$a = x_0 + a_1\alpha + a_2\varphi(\alpha), \quad b = x_0 + a_1\beta + a_2\varphi(\beta) \tag{7.4}$$

or

$$a = x_0 + a_1\beta + a_2\varphi(\beta), \quad b = x_0 + a_1\alpha + a_2\varphi(\alpha). \tag{7.4*}$$

Let $a_1 + a_2\varphi'(\xi)$ not change its sign on $\langle \alpha, \beta\rangle$. If $F\colon\ \Gamma \to \mathbb{R}^1$ is a continuous function with values $F(x, y)$ for $[x, y] \in \Gamma$ then

$$\int_a^b [F(x, f(x))]^2 \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x = \int_\alpha^\beta [\Phi(\xi, \varphi(\xi))]^2 \sqrt{1 + [\varphi'(\xi)]^2}\, \mathrm{d}\xi, \tag{7.5}$$

where

$$\Phi(\xi, \eta) = F(x_0 + a_1\xi + a_2\eta, y_0 + b_1\xi + b_2\eta). \tag{7.6}$$

P r o o f. Let $P \in \Gamma$ be arbitrary but fixed. By (7.2) we can write $P = [x, f(x)]$ and by (7.3), $P = [\xi, \varphi(\xi)]$. According to (7.1), these coordinates satisfy the relations

$$x = x_0 + a_1\xi + a_2\varphi(\xi), \tag{7.7}$$

$$f(x) = y_0 + b_1\xi + b_2\varphi(\xi). \tag{7.8}$$

Using (7.4), (7.6)–(7.8) and the theorem on transformation of an integral we obtain

$$
\begin{aligned}
(7.9)\qquad & \int_a^b [F(x, f(x))]^2 \sqrt{1+[f'(x)]^2}\,\mathrm{d}x \\
&= \pm \int_\alpha^\beta \operatorname{sign}(a_1 + a_2\varphi'(\xi))[\Phi(\xi, \varphi(\xi))]^2 \\
&\quad\times \sqrt{1+[f'_x(x_0 + a_1\xi + a_2\varphi(\xi))]^2}\,|a_1 + a_2\varphi'(\xi)|\,\mathrm{d}\xi
\end{aligned}
$$

where the sign plus occurs in the case (7.4) and the sign minus in the case (7.4*). As the left-hand side of (7.9) is positive, we have

$$
(7.10)\qquad \pm \operatorname{sign}(a_1 + a_2\varphi'(\xi)) = 1
$$

with the possible exception of a finite number of points, because $\alpha < \beta$ and the remaining terms on the right-hand side of (7.9) are nonnegative.

Using the theorem on differentiation of a composite function and relation (7.8) we find

$$
(7.11)\qquad
\begin{aligned}
f'_x(x_0 + a_1\xi + a_2\varphi(\xi))(a_1 + a_2\varphi'(\xi)) &= \tfrac{\mathrm{d}}{\mathrm{d}\xi}[f(x)] \\
= \tfrac{\mathrm{d}}{\mathrm{d}\xi}(y_0 + b_1\xi + b_2\varphi(\xi)) &= b_1 + b_2\varphi'(\xi).
\end{aligned}
$$

Relation (7.11) implies

$$
(7.12)\qquad
\begin{aligned}
&\sqrt{1+[f'_x(x_0 + a_1\xi + a_2\varphi(\xi))]^2}\,|a_1 + a_2\varphi'(\xi)| \\
&= \sqrt{[a_1 + a_2\varphi'(\xi)]^2 + [b_1 + b_2\varphi'(\xi)]^2}.
\end{aligned}
$$

As the transformation between the coordinate systems (x, y) and (ξ, η) is orthogonal the column vectors of the matrix

$$
\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}
$$

form a system of orthonormal vectors. Hence,

$$
(7.13)\qquad \sqrt{[a_1 + a_2\varphi'(\xi)]^2 + [b_1 + b_2\varphi'(\xi)]^2} = \sqrt{1+[\varphi'(\xi)]^2}.
$$

Relations (7.9)–(7.13) imply (7.5). □

7.2. Remark. In the same way we can prove the following assertions:

A. *Let (x, y) and (ξ, η) be two Cartesian coordinate systems related by transformation* (7.1). *Let a smooth curve Γ be expressed in the system (x, y) by* (7.2) *and in the system (ξ, η) by*

$$\xi = \psi(\eta), \quad \eta \in \langle \alpha, \beta \rangle, \quad \alpha < \beta, \tag{7.14}$$

where $\psi \in C^1\langle \alpha, \beta \rangle$ and

$$a = x_0 + a_1\psi(\alpha) + a_2\alpha, \quad b = x_0 + a_1\psi(\beta) + a_2\beta \tag{7.15}$$

or

$$a = x_0 + a_1\psi(\beta) + a_2\beta, \quad b = x_0 + a_1\psi(\alpha) + a_2\alpha. \tag{7.15*}$$

Let $a_1\psi'(\eta) + a_2$ not change its sign on $\langle \alpha, \beta \rangle$. If $F\colon \Gamma \to \mathbb{R}^1$ is a continuous function with values $F(x, y)$ for $[x, y] \in \Gamma$ then

$$\int_a^b [F(x, f(x))]^2 \sqrt{1 + [f'(x)]^2}\, \mathrm{d}x = \int_\alpha^\beta [\varPhi(\psi(\eta), \eta)]^2 \sqrt{1 + [\psi'(\eta)]^2}\, \mathrm{d}\eta, \tag{7.16}$$

where the function $\varPhi(\xi, \eta)$ is given by (7.6).

B. *Let (x, y) and (ξ, η) be two Cartesian coordinate systems related by transformation* (7.1). *Let a smooth curve Γ be expressed in the system (x, y) by*

$$x = g(y), \quad y \in \langle a, b \rangle, \quad a < b, \tag{7.17}$$

where $g \in C^1\langle a, b \rangle$, and in the system (ξ, η) by (7.14) *where*

$$a = y_0 + b_1\psi(\alpha) + b_2\alpha, \quad b = y_0 + b_1\psi(\beta) + b_2\beta \tag{7.18}$$

or

$$a = y_0 + b_1\psi(\beta) + b_2\beta, \quad b = y_0 + b_1\psi(\alpha) + b_2\alpha. \tag{7.18*}$$

Let $b_1\psi'(\eta) + b_2$ not change its sign on $\langle \alpha, \beta \rangle$. If $F\colon \Gamma \to \mathbb{R}^1$ is a continuous function with values $F(x, y)$ for $[x, y] \in \Gamma$ then

$$\int_a^b [F(g(y), y)]^2 \sqrt{1 + [g'(y)]^2}\, \mathrm{d}y = \int_\alpha^\beta [\varPhi(\psi(\eta), \eta)]^2 \sqrt{1 + [\psi'(\eta)]^2}\, \mathrm{d}\eta, \tag{7.19}$$

where the function $\varPhi(\xi, \eta)$ is given by (7.6).

C. *Let (x,y) and (ξ,η) be two Cartesian coordinate systems related by transformation* (7.1). *Let a smooth curve Γ be expressed in the system (x,y) by* (7.17) *and in the system (ξ,η) by* (7.3) *where*

$$a = y_0 + b_1\alpha + b_2\varphi(\alpha), \quad b = y_0 + b_1\beta + b_2\varphi(\beta) \tag{7.20}$$

or

$$a = y_0 + b_1\alpha + b_2\varphi(\alpha), \quad b = y_0 + b_1\beta + b_2\varphi(\beta). \tag{7.20*}$$

Let $a_1\psi'(\eta)+a_2$ not change its sign on $\langle\alpha,\beta\rangle$. If $F\colon\ \Gamma \to \mathbb{R}^1$ is a continuous function with values $F(x,y)$ for $[x,y]\in\Gamma$ then

$$\int_a^b [F(g(y),y)]^2\sqrt{1+[g'(y)]^2}\,\mathrm{d}y = \int_\alpha^\beta [\varPhi(\xi,\varphi(\xi))]^2\sqrt{1+[\varphi'(\xi)]^2}\,\mathrm{d}\xi, \tag{7.21}$$

where the function $\varPhi(\xi,\eta)$ is given by (7.6).

7.3. Example. Let us consider a circle

$$x^2+y^2=R^2 \quad (R>0)$$

in the Cartesian coordinate system (x,y). Let us denote

$$A_1=[-\tfrac{\sqrt2}{2}R,-\tfrac{\sqrt2}{2}R],\ A_2=[\tfrac{\sqrt2}{2}R,-\tfrac{\sqrt2}{2}R],\ A_3=[\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R],\ A_4=[-\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R],$$
$$B_1=[0,-R],\ B_2=[R,0],\ B_3=[0,R],\ B_4=[-R,0].$$

The curves A_1A_2 and A_4A_3 have the expressions

$$y=-\sqrt{R^2-x^2} \quad\text{and}\quad y=\sqrt{R^2-x^2}, \quad x\in\big\langle-\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R\big\rangle$$

and the curves A_2A_3 and A_1A_4 have the expressions

$$x=\sqrt{R^2-y^2} \quad\text{and}\quad x=-\sqrt{R^2-y^2}, \quad y\in\big\langle-\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R\big\rangle.$$

Let (ξ,η) be the Cartesian coordinate system which is connected with the system (x,y) by the transformation

$$x=\tfrac{\sqrt2}{2}\xi-\tfrac{\sqrt2}{2}\eta, \quad y=\tfrac{\sqrt2}{2}\xi+\tfrac{\sqrt2}{2}\eta.$$

The curves B_1B_2 and B_4B_3 have the expressions

$$\eta=-\sqrt{R^2-\xi^2} \quad\text{and}\quad \eta=\sqrt{R^2-\xi^2}, \quad \xi\in\big\langle-\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R\big\rangle$$

and the curves B_2B_3 and B_1B_4 have the expressions

$$\xi=\sqrt{R^2-\eta^2} \quad\text{and}\quad \xi=-\sqrt{R^2-\eta^2}, \quad \eta\in\big\langle-\tfrac{\sqrt2}{2}R,\tfrac{\sqrt2}{2}R\big\rangle.$$

Thus each of the eight curves A_iB_j satisfies either Lemma 7.1 or one of the assertions **A**, **B**, **C** from Remark 7.2.

Remark 2.5, Theorem 2.6, Lemma 7.1 and Remark 7.2 imply

7.4. Theorem. *Let (x, y) and (ξ, η) be two Cartesian coordinate systems which are connected by transformation (7.1). Let $u(x, y) \in L_2(\partial\Omega)$ where $\partial\Omega$ is an S-Lipschitz continuous boundary of a domain Ω. Then the function*

$$U(\xi, \eta) = u(x_0 + a_1\xi + a_2\eta, y_0 + b_1\xi + b_2\eta)$$

belongs to $L_2(\partial\Omega)$ and we have

$$\int_{\partial\Omega} [u(x, y)]^2 \,\mathrm{d}s = \int_{\partial\Omega} [U(\xi, \eta)]^2 \,\mathrm{d}s,$$

where the line integrals are defined without use of the partition of unity.

7.5. Lemma (trace theorem). *Let Ω be a domain with an S-Lipschitz continuous boundary. Then there exists a unique bounded linear mapping $\gamma\colon H^1(\Omega) \equiv W_2^1(\Omega) \to L_2(\partial\Omega)$ such that we have $(\gamma v)(x, y) = v(x, y)$ for all $v \in C^\infty(\overline{\Omega})$ and $[x, y] \in \partial\Omega$. The boundedness of the mapping is expressed by the inequality*

$$\|v\|^2_{0,\partial\Omega} \equiv \int_{\partial\Omega} [v(x, y)]^2 \,\mathrm{d}s \leqslant C\|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega), \tag{7.22}$$

where the line integral is defined without use of the partition of unity.

For the proof see [5, pp. 15–16].

7.6. Lemma (density theorem). *Let Ω be a domain with an S-Lipschitz continuous boundary. Then $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.*

For the proof see [4, pp. 271–272].

7.7. Theorem (Green's formula). *Let a domain Ω have an S-Lipschitz continuous boundary. Then for all functions $u, v \in H^1(\Omega)$ we have*

$$\iint_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \int_{\partial\Omega} uvn_i \,\mathrm{d}s - \iint_\Omega u\frac{\partial v}{\partial x_i} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \tag{7.23}$$

where (n_1, n_2) is the outer unit normal vector and where we write simply u, v instead of γu, γv.

P r o o f. Let $\{u_k\} \subset C^\infty(\overline{\Omega})$, $\{v_k\} \subset C^\infty(\overline{\Omega})$ be such sequences that

$$u_k \to u, \quad v_k \to v \quad \text{in} \quad H^1(\Omega). \tag{7.24}$$

Then, according to Remark 6.4,

$$\iint_\Omega \frac{\partial u_k}{\partial x_i} v_k \,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\partial\Omega} u_k v_k n_i \,\mathrm{d}s - \iint_\Omega u_k \frac{\partial v_k}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{7.25}$$

We have

$$\begin{aligned} \int_{\partial\Omega} u_k v_k n_i \,\mathrm{d}s &= \int_{\partial\Omega} (u_k - u + u)(v_k - v + v) n_i \,\mathrm{d}s \\ &= \int_{\partial\Omega} (u_k - u)(v_k - v) n_i \,\mathrm{d}s + \int_{\partial\Omega} u(v_k - v) n_i \,\mathrm{d}s \\ &\quad + \int_{\partial\Omega} (u_k - u) v n_i \,\mathrm{d}s + \int_{\partial\Omega} u v n_i \,\mathrm{d}s. \end{aligned} \tag{7.26}$$

As

$$\|u_k - u\|_{L_2(\partial\Omega)} \leqslant C\|u_k - u\|_{H^1(\Omega)} \to 0,$$
$$\|v_k - v\|_{L_2(\partial\Omega)} \leqslant C\|v_k - v\|_{H^1(\Omega)} \to 0,$$

the first three terms on the right-hand side of (7.26) tend to zero with $k \to \infty$. Hence

$$\int_{\partial\Omega} u_k v_k n_i \,\mathrm{d}s \to \int_{\partial\Omega} u v n_i \,\mathrm{d}s \quad \text{for} \quad k \to \infty. \tag{7.27}$$

Similarly we find

$$\iint_\Omega \frac{\partial u_k}{\partial x_i} v_k \,\mathrm{d}x_1\,\mathrm{d}x_2 \to \iint_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{7.28}$$

$$\iint_\Omega u_k \frac{\partial v_k}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2 \to \iint_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{7.29}$$

Passing to the limit in (7.25) with $k \to \infty$ we obtain, according to (7.27)–(7.29), relation (7.23). □

7.8. Theorem (divergence form of Green's theorem). *Let a domain Ω have an S-Lipschitz continuous boundary. Then for all functions $P_1, P_2 \in H^1(\Omega)$ we have*

$$\begin{aligned} \iint_\Omega \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y &= \int_{\partial\Omega} (P_1 \cos\omega + P_2 \sin\omega) \,\mathrm{d}s \\ &\equiv \int_{\partial\Omega} (P_1 n_1 + P_2 n_2) \,\mathrm{d}s, \end{aligned} \tag{7.30}$$

where ω is the angle between the unit vector (n_1, n_2) of the outer normal to the boundary $\partial\Omega$ and the positive direction of the x-axis.

P r o o f. Let us set $u := P_i$, $v \equiv 1$ in (7.23). Summing up the result from $i = 1$ to $i = 2$ we obtain relation (7.30). □

Our next aim is to generalize Theorem 7.7 to the case when $u \in W^{1,p}(\Omega)$ and $v \in W^{1,q}(\Omega)$.

7.9. Lemma (density theorem). *Let Ω be a domain with an S-Lipschitz continuous boundary. Then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ with $p \geqslant 1$.*

For the proof see [4, pp. 271–272].

7.10. Lemma (trace theorem). *Let a domain $\Omega \subset \mathbb{R}^2$ have an S-Lipschitz continuous boundary. Let $1 \leqslant p < 2$, $q = p/(2-p)$. Then there exists a uniquely determined continuous linear mapping $\gamma\colon W^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

P r o o f. Let $u \in C^\infty(\overline{\Omega})$. Let the norm $\|\cdot\|_{L_q(\partial\Omega)}$ be defined by the line integral *without use* of partition of unity. (Using an analogue of Theorem 7.4 we see that this norm is well defined.) The proof of [4, Th. 6.4.1] must be modified. Combining it with the approach of [5, pp. 15–16] we find

$$\int_{\partial\Omega} |\gamma u(x,y)|^q \,\mathrm{d}s \leqslant c\|u\|^q_{W^{1,p}(\Omega)} + \iint_\Omega |u(x,y)|^q \,\mathrm{d}x\,\mathrm{d}y \quad \forall u \in C^\infty(\overline{\Omega}),$$

where $q = 2/(2-p)$. Hence

$$\|\gamma u\|_{L_q(\partial\Omega)} \leqslant C_1\|u\|_{W^{1,p}(\Omega)} + C_2\|u\|_{L_q(\Omega)} \quad \forall u \in C^\infty(\overline{\Omega}). \tag{7.31}$$

Using [4, Th. 5.7.7(i)] with $N = 2$, $k = 1$, we obtain

$$W^{1,p}(\Omega) \subset L_q(\Omega) \quad \text{algebraically and topologically.} \tag{7.32}$$

Combining (7.31), (7.32) and using Lemma 7.9 we find that

$$\|\gamma u\|_{L_q(\partial\Omega)} \leqslant C\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \tag{7.33}$$

which proves the theorem. □

7.11. Theorem (trace theorem). *Let a domain Ω have an S-Lipschitz continuous boundary. Let $p \geqslant 2$. Then for any $q \geqslant 1$ there exists a unique continuous linear mapping $\gamma\colon W^{1,p}(\Omega) \to L_q(\partial\Omega)$ such that $\gamma u = u\big|_{\partial\Omega}$ for all $u \in C^\infty(\overline{\Omega})$.*

P r o o f. Because of its shortness we reproduce the proof of [4, Th. 6.4.2] and correct simultaneously a misprint appearing in this proof.

Let $q \geqslant 1$ be an arbitrary fixed number. Since the function $\nu(t) = t/(2-t)$ increases from 1 to ∞ on the interval $\langle 1, 2)$, there exists a $\overline{p} \in \langle 1, 2)$ such that $q = \nu(\overline{p})$. According to Lemma 7.10 we have a uniquely defined linear mapping $\mathcal{M}\colon W^{1,\overline{p}}(\Omega) \to L_q(\partial\Omega)$, $\mathcal{M}u = u\big|_{\partial\Omega}$ if $u \in C^\infty(\overline{\Omega})$. Composing it with the identity mapping I from $W^{1,p}(\Omega)$ into $W^{1,\overline{p}}(\Omega)$, we can see that $\gamma = \mathcal{M} \circ I$. □

7.12. Theorem (Green's formula). *Let a domain Ω have an S-Lipschitz continuous boundary. Let $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} \leqslant \frac{3}{2}$ if $1 \leqslant p < 2$, $1 \leqslant q < 2$, or with $q > 1$ if $p \geqslant 2$, or with $p > 1$ if $q \geqslant 2$. Then*

$$\iint_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \int_{\partial\Omega} u v n_i \,\mathrm{d}s - \iint_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1 \,\mathrm{d}x_2, \tag{7.34}$$

where (n_1, n_2) is the unit outer normal vector and the line integral is defined without use of the partition of unity.

P r o o f. According to Remark 6.4 we can write

$$\iint_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \int_{\partial\Omega} u v n_i \,\mathrm{d}s - \iint_\Omega u \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1 \,\mathrm{d}x_2 \quad \forall u, v \in C^\infty(\overline{\Omega}). \tag{7.35}$$

Using Lemmas 7.9, 7.10, Theorem 7.11 and imbedding theorems [4, Thms. 5.7.7 and 5.7.8], we can proceed similarly as in the proof of Theorem 7.7; for more details see the part of the proof of [5, Th. 3.1.1] which follows relation (1.9) on page 122, or the proof of [8, Th. 14.3] where the considerations are presented in a broader way. □

7.13. Theorem (divergence form of Green's theorem). *Let a domain Ω have an S-Lipschitz continuous boundary. Then for all functions $P_1, P_2 \in W^{1,p}(\Omega)$ $(p \geqslant 1)$ we have*

$$\begin{aligned} \iint_\Omega \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_2}{\partial y} \right) \mathrm{d}x \,\mathrm{d}y &= \int_{\partial\Omega} (P_1 \cos\omega + P_2 \sin\omega) \,\mathrm{d}s \\ &\equiv \int_{\partial\Omega} (P_1 n_1 + P_2 n_2) \,\mathrm{d}s, \end{aligned} \tag{7.36}$$

where ω is the angle between the unit vector (n_1, n_2) of the outer normal to the boundary $\partial\Omega$ and the positive direction of the x-axis.

P r o o f. Let us set $u := P_i$, $v \equiv 1$ in (7.34). Summing up the result from $i = 1$ to $i = 2$, we obtain relation (7.36). □

In this case we restrict ourselves to the Riemann integral because the assumption of trace theorems concerning the boundary is not satisfied. Also the Extension Theorem 4.4 does not hold; thus we must assume the functions P, Q to be defined in a domain $\widetilde{\Omega} \supset \overline{\Omega}$. However, such situations are met in some real-life physical problems; for example, a two-dimensional body with non-Lipschitz continuous boundary (but with a piecewise smooth boundary) immersed into a two-dimensional physical field.

It should be noted that some domains with cusp-points are in certain coordinate systems elementary domains (as, for example, the domain from Remark 3.2c).

Let us note that every domain with an S-Lipschitz continuous boundary is a domain with a piecewise smooth boundary; however, a domain with piecewise smooth boundary can have cusp-points.

8.1. Remark. Besides one-sided and two-sided cusp points we shall also distinguish *internal* and *external* cusp points. We explain these two notions by an example. Besides the curve σ (with end-points A_1, A_3) from Remark 3.2c let us consider the curve ϱ defined by the relations

$$x = -1 + \cos t, \quad y = 1 + \sin t, \quad \left\langle \tfrac{3}{2}\pi, 2\pi \right\rangle.$$

Let $A_4 = [-1, 0]$, $B_1 = [1, -1]$, $B_2 = [-1, -1]$, $B_3 = [1, 2]$, $B_4 = [-1, 2]$ and let ω_1 be the bounded domain whose boundary consists of the curves ϱ, σ and segments A_4B_2, B_2B_1, B_1A_3. Further, let ω_2 be the bounded domain whose boundary consists of the curves ϱ, σ and segments A_3B_3, B_3B_4, B_4A_4. Then in the case of the domain ω_1 the point A_1 is an external cusp point and in the case of the domain ω_2 the same point is an internal cusp point.

The most general theorem which can be proved in this case reads as follows:

8.2. Theorem (Green). *Let a domain Ω be bounded and let it have a piecewise smooth boundary. Let functions $P(x, y)$, $Q(x, y)$ be continuous and bounded together with their derivatives $\partial P/\partial y$, $\partial Q/\partial x$ in a simply connected domain $\widetilde{\Omega} \supset \overline{\Omega}$. Let the boundary $\partial\Omega$ of the domain Ω be oriented positively with respect to the domain Ω. Then relation* (3.2) *holds.*

Proof. A) First we consider the case of a simply connected domain Ω. For greater simplicity, let us consider a bounded domain Ω with one cusp point only. (A generalization to the case of a domain with more cusp points is straightforward.) If this point is an internal cusp point then we can use Theorem 3.5 (and do not need the functions P, Q to be defined outside $\overline{\Omega}$).

In the case of an external two-sided cusp point we can proceed in the same way as in the first proof of Theorem 5.1. If the external cusp point is one-sided it suffices to cut off (in thought) the "beak" and to complete it into a convex domain. The beak-shaped domain separated in thoughts should have a sufficiently small measure; only in this case the corresponding crescent-shaped domain has small measure.

B) Let now Ω be a bounded multiply connected domain. This means that

$$\overline{\Omega} = \overline{\Omega}_0 - \bigcup_{k=1}^{n} \Omega_k \quad (\overline{\Omega}_k \subset \Omega_0;\ \overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset,\ i, j, k = 1, \dots, n). \tag{8.1}$$

We have by (8.1)

$$\begin{aligned} \iint_{\overline{\Omega}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \iint_{\overline{\Omega}_0} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y \\ - \sum_{k=1}^{n} \iint_{\overline{\Omega}_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y. \end{aligned} \tag{8.2}$$

We have proved in part A that

$$\iint_{\overline{\Omega}_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_{\partial \Omega_i} P \, \mathrm{d}x + Q \, \mathrm{d}y \quad (i = 0, 1, \dots, n). \tag{8.3}$$

Inserting (8.3) into the right-hand side of (8.2) we obtain (3.2). □

We close this section with an exceptional case.

8.3. Theorem (Green). *Let a domain Ω be bounded and let it have a piecewise smooth boundary $\partial\Omega$, which has only two-sided interior cusp points. Let the boundary $\partial\Omega$ be oriented positively with respect to the domain Ω. Let $P, Q \in C^1(\overline{\Omega})$. Then relation* (3.2) *holds.*

P r o o f. Let us approximate the domain Ω by a domain Ω_h with a polygonal boundary $\partial\Omega_h$ the vertices of which lie on $\partial\Omega$. Let this approximation have the property that the set bounded by the curves $\partial\Omega$ and $\partial\Omega_h$ is a union of convex domains $G_1, \dots, G_n$ which have the crescent-shaped form.

Let B be a cusp point. Then there exists a neighbourhood $U(\varepsilon, B)$ of B such that we have

$$G_i \subset U(\varepsilon, B) \quad \Rightarrow \quad G_i \subset \overline{\Omega}. \tag{8.4}$$

This means that we need not extend the functions P, Q in the neighbourhoods of the cusp points. As Theorem 4.4 can be generalized in such a way that it allows us to extend the functions P, Q locally (see [1] or [2]), the rest of the proof follows the same lines as the second proof of Theorem 5.1. □

8.4. R e m a r k . a) Another proof of Theorem 8.3 follows from the fact that the domain Ω can be divided into a finite number of closed domains with S-Lipschitz continuous boundaries and mutually non-overlapping interiors.

b) In neighbourhoods of the cusp points the condition of Theorem 2.6 mentioned in Remark 2.2b is violated. However, for the proof of Theorem 2.6 property (8.4) is sufficient.

c) The results of Section 7 can be generalized to the case of domains Ω from Theorem 8.3.

Acknowledgement. The author is indebted to the referee for his comments which helped him to elaborate the final form of the paper.

References

[1] *G.M. Fichtengolc*: Differential and Integral Calculus I. Gostechizdat, Moscow, 1951. (In Russian.)
[2] *G.M. Fichtenholz*: Differential- und Integralrechnung I. VEB Deutscher Verlag der Wissenschaften, Berlin, 1968.
[3] *M. Křížek*: An equilibrium finite element method in three-dimensional elasticity. Apl. Mat. *27* (1982), 46–75.
[4] *A. Kufner, O. John, S. Fučík*: Function Spaces. Academia, Prague, 1977.
[5] *J. Nečas*: Les Méthodes Directes en Théorie des Equations Elliptiques. Academia, Prague, 1967.
[6] *M. Novotný*: Integral Calculus. SPN, Prague, 1958. (In Czech.)
[7] *L.A. Oganesjan, L.A. Ruchovec*: Variational-Difference Methods for the Solution of Elliptic Problems. Izdat. Akad. Nauk ArSSR, Jerevan, 1979. (In Russian.)
[8] *A. Ženíšek*: Gauss-Ostrogradskij theorem from the point of view of applications. to appear.

Author's address: *Alexander Ženíšek*, Department of Mathematics, Technical University Brno, Technická 2, 616 69 Brno, Czech Republic, e-mail: `zenisek@mat.fme.vutbr.cz`.