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STEADY VORTEX RINGS WITH SWIRL IN AN IDEAL FLUID: ASYMPTOTICS FOR SOME SOLUTIONS IN EXTERIOR DOMAINS

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Abstract. In this paper, the axisymmetric flow in an ideal fluid outside the infinite cylinder ($r \leqslant d$) where (r,θ,z) denotes the cylindrical co-ordinates in $\mathbb{R}^3$ is considered. The motion is with swirl (i.e. the θ-component of the velocity of the flow is non constant). The (non-dimensional) equation governing the phenomenon is (Pd) displayed below. It is known from e.g. [9] that for the problem without swirl ($f_q = 0$ in (f)) in the whole space, as the flux constant k tends to ∞,

1) $\operatorname{dist}(0z, \partial A) = O(k^{1/2})$; $\operatorname{diam} A = O(\exp(-c_0 k^{3/2}))$;

2) $(k^{1/2}\Psi)_{k\in\mathbb{N}}$ converges to a vortex cylinder U_m (see (1.2)).

We show that for the problem with swirl, as $k \nearrow \infty$, 1) holds; if $m \leqslant q+2$ then 2) holds and if $m > q+2$ it holds with U_{q+2} instead of U_m. Moreover, these results are independent of f_0, f_q and $d > 0$.

Keywords: vortex rings, potential theory, elliptic equations

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1. Introduction

Let (r,θ,z) denote the cylindrical coordinates in $\mathbb{R}^3$. We consider an axisymmetric (w.r.t. Oz) flow in an ideal fluid occupying the exterior domain

$$\Omega_d := \{(r,\theta,z) \mid r > d,\ \theta = 0,\ z \in \mathbb{R}\}, \quad d > 0.$$

The problem is then posed in the half plane $\Pi_d := \{x = (r,z) \mid r > d, z \in \mathbb{R}\}$.

It si known (see e.g. [1]) that if $\mathbf{q} = (u,S,v)$ denotes the velocity of the flow and $\omega = (w_1,\omega,w_2) = \operatorname{curl}\mathbf{q}$ its vorticity, then $w_1 = -S_z$; $\omega = u_z - v_r$; $w_2 = -(1/r)\{rS\}_r$.

The mass conservation ($\mathrm{div}\mathbf{q} = 0$) implies that there is a stream function Ψ such that $u = -\Psi_z/r$, $v = \Psi_r/r$ whence $\omega = \{\Psi_{rr} - \Psi_r/r + \Psi_{zz}\}/r$.

From Bernoulli's equation $|\mathbf{q}|^2/2 + p/\varrho = H(\Psi)$ where p and ϱ denote the pressure and the density and H a scalar function, the dynamical equation $\mathbf{q}\times\omega - \mathbf{q}_t = \mathrm{grad}\, H$, rS and ω/r are constant on each stream line and for $rS = C(\Psi)$ we have $\omega/r \equiv L\Psi := r^{-2}\{\Psi_{rr} - \Psi_r/r + \Psi_z\} = r^{-2}C(\Psi)C'(\Psi) - H'(\Psi)$.

So, as seen in [1], the non-dimensional equations (see [9]) governing the flow are

$$(Pd)\begin{cases} L\psi := \dfrac{1}{r^2}\Big\{\partial_r^2 - \dfrac{1}{r}\partial_r + \partial_z^2\Big\}\psi = -\lambda f(r,\Psi) \quad \text{in } \Pi_d \\ \psi|_{r=0} = 0; \quad \psi \text{ and } |\nabla\psi| \searrow 0 \quad \text{as } |x| = \sqrt{r^2+z^2} \nearrow \infty \text{ in } \Pi_d \end{cases}$$

where the stream functions are related by $\psi(x) := \Psi(x) + r^2/2 + k$, the vorticity function f is here defined for some $m, q \geqslant 0$ and $f_0, f_q \geqslant 0$ by

$$\text{(f)} \qquad f(r,t) := f_q\,\frac{\{t_+\}^q}{r^2} + f_0\,\{t_+\}^m$$

where $t_+ := \max\{t, 0\}$ and $f_q = 0$ *for the problem without swirl.* The parameter $\lambda > 0$ is a Lagrangian multiplier, determined a posteriori. The parameter $k > 0$ denotes the flux constant, measuring the flux of the fluid between the boundary $r = d$ and the boundary of the ring ∂A where

$$\text{(A)} \qquad A := \{x \in \Pi_d \mid \Psi(x) > 0\}$$

denotes the cross-section of the ring. The problem is then to find solutions $\psi \in C^1(\overline{\Pi_d})$ and the corresponding A for (Pd).

We are concerned with the variational solutions of the type found in [3], i.e. local maximizers of the functional

$$\text{(Z)} \qquad Z(u) := \int_{\Pi_d} F(r,U)\,\mathrm{d}\tau;\ \mathrm{d}\tau := r\,\mathrm{d}r\,\mathrm{d}z; \quad F(r,T) := \int_0^T f(r,s)\,\mathrm{d}s$$

on the sphere $S_1(\Pi_d) := \{u \in H_d := H(\Pi_d) \mid \|u\|^2 = 1\}$ where $U(x) := u(x) - r^2/2 - k$ and H_d denotes the completion of $C_0^\infty(\Pi_d)$ in the norm

$$\text{(1.1a)} \qquad \|u\| \equiv \|u\|_{H_d} := \left(\int_{\Pi_d} \frac{u_r^2 + u_z^2}{r^2}\,\mathrm{d}\tau\right)^{1/2}.$$

Note that for $u \in C_0^\infty(\Pi_d)$, $\|u\| = \left(-\int_{\Pi_d} uLu\,\mathrm{d}\tau\right)^{1/2}$ and H_d is a Hilbert space with the scalar product

$$\text{(1.1b)} \qquad \langle u, v\rangle_{H_d} := \int_{\Pi_d} \frac{1}{r^2}\{u_r v_r + u_z v_z\}\,\mathrm{d}\tau.$$

The problem without swirl for which Z is replaced by

$$J_m(u) := f_0 \int_\Pi \frac{\{U_+\}^{m+1}}{m+1}\, \mathrm{d}\tau \tag{J}$$

in $\Pi := \Pi_0$ has variational solutions ψ such that ([3])

a) $\psi \in C^2(\overline{\Pi})$ if $m > 0$ and $\psi \in C^2(\overline{\Pi} \setminus \overline{\partial A}) \bigcap C^1(\overline{\Pi})$ for $m = 0$;

b) ψ is an even function of z and $\psi_z < 0$ if $z > 0$;

c) for $k > 0$, A has a finite number of simply connected components ([9]) and is simply connected if $m \geqslant 1$.

For the asymptotics of these solutions ([9]), for large values of k,

d) $|a^2 - (2/3)k| = O(k^{-1/2} \log k) a := (r_1 + r_2)/2$ where $r_1 := \inf\{r > 0 \mid (r, 0) \in A\}$ $(r_2 := \sup\{r > 0 \mid (r, 0) \in A\}$;

e) for $\varepsilon > 0$ such that $\operatorname{diam} A = 2a\varepsilon$ and $c_0 = 8\pi(2/3)^{3/2}$, $\varepsilon \leqslant C \exp\{-c_0 k^{3/2}\}$, $\lambda \leqslant C k^{(m-2)/2} \exp\{2c_0 k^{3/2}\}$ and $|\Psi|_{C(\overline{A})} = O(k^{-1/2})$; let

$$\hat{\Pi} := \{\zeta = (\xi, \eta) \mid \xi > -1/\varepsilon, \quad \eta \in \mathbb{R}\} \tag{1.Π}$$

denote the image of Π in the transformation

$$r = a(1 + \varepsilon\xi), \quad z = a\varepsilon\eta \tag{1.ζ}$$

and for u defined in Π let $\hat{u}(\zeta) := u(a(1+\varepsilon\xi), a\varepsilon\eta)$; when $k \nearrow \infty$ the functions $k^{1/2}\Psi$ converge in $C^1(\Pi)$ to a function U_m such that $\hat{U}_m$ is radial; namely

$$\hat{U}_m(\sigma) = \frac{\sqrt{6}}{4\pi \varrho_m^2} Q_m(\varrho_m \sigma) \tag{1.2}$$

where Q_m is the unique solution of

$$Q'' + Q'/\sigma = -Q_+^m, \quad \sigma > 0; \quad Q(0) = 1; \quad Q'(0) = 0 \tag{1.Q}$$

and ϱ_m is its unique positive zero ([8]). In this context the function U will be called a *vortex cylinder*.

In the sequel, for any function φ, $\Phi(x) := \{\varphi(x) - r^2/2 - k\}$ and diverse constants C will denote generic constants.

By the maximum principle all solutions ψ are positive in their respective domains. The main results that we obtain are:

1) The variational solutions of (Pd) satisfy a)–e) where for i) in a), Π_d *replaces* Π*,* $mq > 0$ *and* $mq = 0$ *replace respectively* $m > 0$ *and* $m = 0$*;*

ii) the estimates in d)–e) are independent of d*.*

2) Independently of d, f_0 and f_q, the functions $k^{1/2}\hat{\Psi}_d$ converge to $\hat{U}_m$ if $m \leqslant q+2$ and to $\hat{U}_{q+2}$ if $m > q+2$.

3) For large k we deduce variational solutions of the problem in Π from those of $\{(P_d)\}_{d\in(0,1]}$, and they have the same estimates.

2. Existence of solutions

2.1. Preliminaries. For $b > 0$, let $D \equiv D_b$ denote a regular convex domain ($\partial D_b \in C^l$; $l \geqslant 2$) in Π_d such that the rectangle $(d, d+b) \times (-2b, 2b)$ is contained in $\overline{D}$. Define the spaces $L^p(D) := \{u \mid |u|_{p;D} := \left(\int_D |u|^p \,\mathrm{d}\tau\right)^{1/p} < \infty\}$ and denote by $H(D)$ the completion of $C_0^\infty(D)$ in the norm $\|u\|_D := \left(\int_D \{(u_r^2 + u_z^2)/r^2\} \,\mathrm{d}\tau\right)^{1/2}$. We have the imbeddings ([3])

$$H(D) \subset W_2^1(D) \subset L^p(D); \quad p \geqslant 1 \tag{2.1}$$

where the second imbedding is compact. In fact, if $u \in H(D)$ has its support in $R := (r_0 - \alpha, r_0) \times (-2\beta, 2\beta)$ then

$$|u|_{p,R} \leqslant C_p r_0^{(2+p)/2p} (2\alpha\beta)^{1/p} \|u\|_R; \quad p \geqslant 1. \tag{2.2}$$

From the Sobolev inequality ([2])

$$\forall u \in H(\Pi) \quad \forall p \geqslant 2, \quad \int_\Pi \frac{|u|^p}{r^{2+p/2}} \,\mathrm{d}r \,\mathrm{d}z \leqslant (A_p \|u\|_\Pi)^p \tag{2.3}$$

where A_p depends only on p, we have the following lemma:

Lemma 2.1. *Let $u \in H(\Pi)$ be such that $A(u) := \{x \in \Pi \mid U(x) := u(x) - k - r^2/2 > 0\}$ has a non void interior. Then $\forall p \geqslant 1$ and $l > 0$ with $\mu := 8p - 6$,*

$$\forall u \in H(\Pi) \quad \int_\Pi \Big(\frac{U_+^l}{r^2}\Big)^p \,\mathrm{d}\tau \leqslant k^{\mu/2} A_\mu^\mu (|U|_{2pl})^{pl} \|u\|^{\mu/2}; \tag{2.4a}$$

$$\forall u \in H(Db) \quad \int_{Db} \Big(\frac{U^l}{r^2}\Big)^p \leqslant (C_{2pl})^{pl} b^{(1+pl)/2} (\operatorname{diam} D_b)^{pl} \|u\|_{Db}^{pl}. \tag{2.4b}$$

Also for $p \geqslant 2$,

$$\int_{A(u)} \frac{1}{r^{2p}} \,\mathrm{d}\tau \leqslant k^{3-2p} \big(A_{4p-6}\|u\|\big)^{4p-6}, \tag{2.4c}$$

where A_μ and C_l are from (2.3) and $|.|_l := |.|_{l;\Pi}$.

P r o o f. As $u > k$ on $A(u)$, by the Hölder inequality we have

$$\int_{A(u)} r^{-2p} U^{pl}\,\mathrm{d}\tau \leqslant (|U|_{2pl})^{pl} k^{\mu/2} \Big(\int_{A(u)} u^{\mu}/r^{2+\mu/2}\,\mathrm{d}r\,\mathrm{d}z\Big)^{1/2}$$

and (2.4a) follows. The other assertions follow from (2.2) and (2.4a). □

Maps between $H(\Pi)$ and the space V_5: Π becomes a meridional half-plane in $\mathbb{R}^N$, $N \geqslant 3$, if we define $z = x_N$; $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_{N-1}^2}$.

Let V_N denote the completion of

$$C_{0,c}^{\infty}(\mathbb{R}^N) := \{\varphi \in C_0^{\infty}(\mathbb{R}^N) \mid u \text{ depends only on } (r, z)\}$$

in the norm $\|\varphi\|_{V_N} := \big(\int_{\Pi} (\varphi_r^2 + \varphi_z^2) r^{N-2}\,\mathrm{d}r\,\mathrm{d}z\big)^{1/2}$.

From [2], the map $\varphi \mapsto \overline{\varphi}$; $\overline{\varphi}(x) := r^{-(N-1)/2}\varphi(x)$ is a homeomorphism from $H(\Pi)$ to V_N with

$$\|\overline{\varphi}\|_{V_N}^2 = \|\varphi\|_{H(\Pi)}^2 + \frac{(N-1)(N-5)}{4} \int_{\Pi} |\varphi|^2 r^{-3}\,\mathrm{d}r\,\mathrm{d}z. \tag{2.5}$$

Thus for $N = 5$ the map is an isometry between $H(\Pi)$ and V_5.

2.2. Solutions in bounded $Db \subset \Pi_d$. As $d > 0$, L is uniformly elliptic in Π_d and $\forall u \in H_d$ with $Z_{\alpha}(u) := (1/\alpha^2) J_q(u) + J_m(u)$, we have

$$Z_{2b}(u) \leqslant Z(u) \leqslant Z_d(u). \tag{2.6}$$

Theorem 2.2. *$\forall k$, $b > 0$, the problem*

$$L\psi = -\lambda f(r, \Psi) \quad \text{in } Db; \quad \psi|_{\partial Db} = 0 \tag{Db}$$

has a solution ψ which is a maximizer of Z on $S_1(Db)$. For some $\nu \in (0, 1]$, if $mq > 0$ there is

$$\overline{\psi} \in C^{2,\nu}(\overline{Db}) \quad (\in C^{1,\nu}(\overline{Db}) \bigcap C^{2,\nu}(\overline{Db} \setminus \overline{\partial A}) \quad \text{if } mq = 0) \tag{2.7}$$

such that $\psi(x) = r^2\overline{\psi}(x)$. Moreover, $\overline{\psi}$ is an even function in z with $\overline{\psi}_z < 0$ for $z > 0$ in Db.

P r o o f. From (2.4) and (2.6), as J_m is in $H(D)$ (see [3]), Z is bounded on $S_1(Db)$ and continuous w.r.t. the weak convergences of $H(Db)$ (hence w.r.t. the strong convergences in $L^p(Db)$, $p \geqslant 1$). Thus there is $\psi \equiv \psi_b \in S_1(Db)$ such that

i) $Z(\psi) = \max\limits_{S_1(Db)} Z(u)$;

ii) Z has a Frechet derivative Z' defined by

$$\langle Z'(u), \varphi \rangle_{H(Db)} := \int_{Db} \varphi f(r, U)\, \mathrm{d}\tau \ \forall \varphi \in H(Db);$$

iii) ψ is a critical point of Z whence there is $\lambda > 0$ such that

$$\forall \varphi \in H(Db) \ \ \langle \psi, \varphi \rangle_{H(Db)} = \lambda \int_{Db} \varphi f(r, \Psi)\, \mathrm{d}\tau$$

and ψ is a weak solution of (Db) with

$$\lambda \equiv \lambda_b = \left(\int_{Db} \psi_b f(r, \Psi_b)\, \mathrm{d}\tau \right)^{-1} \leqslant \{Z(u)\}^{-1} \qquad \forall u \in H(D_b). \tag{2.8}$$

By taking large p in (2.4), the elliptic theory implies that $\psi_b \in C^{1,\nu}(\overline{Db})$ for any $\nu \in (0,1]$. Let $\varphi := \overline{\psi_b}$, the image of ψ_b in the isometry in (2.5); then

$$\triangle_5 \varphi := \varphi_{rr} + \frac{3\varphi_r}{r} + \varphi_{zz} = -\lambda f(r, \Psi_b) \quad in \quad Db_5; \quad \varphi|_{\partial Db_5} = 0 \tag{D_5}$$

and φ satisfies (2.7). The proof is completed by the fact that the equation in (D_5) is even in z (see [4]). □

2.3. Solutions in Π_d for a fixed $k > 0$. For a $b > 0$ and $b_i := ib$, $i \in \mathbb{N}$, let $Di := Db_i$ and let (ψ_i, λ_i) be the corresponding solutions of (Di) where ψ_i is extended by 0 outside Di. From (2.8),

$$\forall i > 1, \qquad \lambda_i \leqslant \{Z(\psi_1)\}^{-1}. \tag{2.9}$$

Lemma 2.3. *There is a bounded domain* $\Omega_k \subset \Pi_d$ *such that*

$$A_i := A(\psi_i) \subset \Omega_k \qquad \forall i \in \mathbb{N}. \tag{2.10}$$

Consequently, Z is uniformly bounded and continuous on $S_1(\Pi_d)$.

P r o o f. Let D be any of the Di, (ψ, λ) the corresponding solution and $A := A(\psi)$. Let $r_2 := \sup\{r > 0 \mid (r, 0) \in \overline{A}\}$. The Green function G of L in D satisfies for $S^2 := (r - r_2)^2 + z^2$ with $x = (r, z)$, $x_2 = (r_2, 0)$ and $\sigma := 4\sqrt{rr_2}/s$ the inequality $(16/\sqrt{rr_2})\, G(x_2, x) \leqslant \sinh^{-1}(1/\sigma) = \log\{1/\sigma + \sqrt{1 + 1/\sigma^2}\}$ (see [6]) whence for a small $\alpha > 0$ and $A\alpha := \{x \in A;\ s < \alpha r_2\}$ we have $r_2^2/2 + k = \psi(x_2) \leqslant (r_2/2\pi)\lambda \int_{A\alpha} \log 1/s\, f(r, \Psi)\, \mathrm{d}\tau + (r_2\lambda/2\pi k) \log(24 r_2/\alpha) \int_A \psi f(r, \Psi)\, \mathrm{d}\tau$ whence

$$k + r_2^2/2 \leqslant C\, r_2 \frac{r_2}{2\pi k}\{\log(24 r_2/\alpha)\} + \lambda\alpha r_2^{m'} \tag{2.11}$$

for some $m' := m'(q, m)$. From (2.9), λ is bounded and so is r_2; in fact if we suppose that r_2 is very big, then for $\alpha > 0$ such that $1/r_2 < \lambda\alpha r_2^{m'} < 1$, (2.11) implies that $r_2 \leqslant \{\log r_2 + \log \lambda\}$. From [3], it is known that if $(r, z) \in A$ then $|z| < k^{-1}$. The existence of Ω_k is obtained.

Let $\varphi \in S_1(\Pi_d) \bigcap C_0^\infty(\Pi_d)$; there is $l \in \mathbb{N}$ such that $\varphi \in S_1(Dl)$ whence from (2.1), (2.4) and (2.9),

$$Z(\varphi) \leqslant Z(\psi_l) \leqslant C(\Omega_k). \tag{2.11'}$$

The uniform continuity then follows as in the case of J_m in [3]. □

Theorem 2.4. *(Pd) has a solution ψ which is a maximizer of Z on $S_1(\Pi_d)$ such that*

1) *there is $\varphi \in C^{2,\nu}(\overline{\Pi_d})$ if $mq > 0$ and $\varphi \in C^{1,\nu}(\overline{\Pi_d}) \bigcap C^{2,\nu}(\overline{\Pi_d \setminus \partial A(\varphi)})$ if $mq = 0$ such that $\psi(x) = r^2\varphi(x)$ in Π_d; ψ is an even function of z and $\psi_z < 0$ if $z > 0$;*

2) *the cross-section A is simply connected if $m, q \geqslant 1$ and has a finite number of components otherwise.*

P r o o f. As $\forall i \quad H(Di) \subset H(D_{i+1}) \subset H(\Pi_d)$,

$$\lim_{i \nearrow \infty} Z(\psi_i) = \max_{S_1(\Pi_d)} Z(u) := \sigma_d; \tag{2.12}$$

so the uniform continuity and boundedness of Z on S_1 implies that there is a subsequence (ψ_i') which converges weakly to a $\psi \in H(\Pi_d)$ with $\|\psi\|_{H(\Pi_d)} \leqslant 1$. If we suppose that $\psi \notin S_1$ then $u := \psi/\|\psi\| \in S_1$ with $Z(u) > \sigma_d$ which is absurd. As (ψ_l') converges strongly in $L^p(\Omega_k) \quad \forall p \geqslant 1$, (λ_l') converges to a $\lambda > 0$ and ψ is a weak solution of (Pd) with $\lambda := \left(\int_{\Pi_d} \psi f(r, \Psi)\, \mathrm{d}\tau\right)^{-1}$. The proof is completed by similar arguments as for the last theorem.

2) As the domain is away from $r = 0$, if $m, q \geqslant 1$ then a slight extension of the results in [3] and [5] shows that A is simply connected.

Assume only that $m, q \geqslant 0$.

If A has an infinite number of disjoint connected components then for some $\theta > 0$, $A^\theta := A \bigcap \{z = -\theta\}$ and $A^0 := A \bigcap \{z = 0\}$ have an infinite number of components (t_i, t_{i+1}) and (r_i, r_{i+1}) with $t_i < r_i < r_{i+1} < t_{i+1}\ \forall i \in \mathbb{N}$. As Ω_k is bounded, the sequences (t_i) and (r_i) converge to the same limit. We then have a contradiction as $\forall i$, $\Psi(r_i, 0) = 0$ and $\Psi(t_i, 0) = -\theta$ ([9]). □

2.4. Estimates for ψ in Π_d for large $k > 0$. Let $(\psi, \lambda) \equiv (\psi_k, \lambda_k)$ denote the solution in Π_d corresponding to $k > 0$.

Lemma 2.5. *For any $d > 0$ with $c_0 := 8\pi(2/3)^{3/2}$, as $k \nearrow \infty$, we have*

$$\lambda \leqslant Ck^{(m-2)/2} \exp\{2c_0 k^{3/2}\}. \tag{2.13}$$

Proof. Let $v(x) := \psi_0(r-d, z)$ where ψ_0 denotes the solution of the problem without swirl with $m = 0$ (see [9]). We have $v \in S_1(\Pi_d)$ and for large k, $J_m(v) \geqslant Ck^{(2-m)/2)} \exp(-2c_0 k^{3/2})$ (see [9]).

The estimate then derives from the fact that $\lambda \leqslant \{Z(v)\}^{-1} \leqslant \{J_m(v)\}^{-1}$. □

Define $r_1 := \inf\{r > 0;\ (r, 0) \in A\}$ $(r_2 := \sup\{r > 0;\ (r, 0) \in A\})$.

Theorem 2.6. *For any $d > 0$, as $k \nearrow \infty$,*

$$\left|r_i^2 - \frac{2}{3}k\right| = O(k^{-1/2} \log k); \tag{2.14}$$

$$\operatorname{diam} A \leqslant Ck^{1/2} \exp\{-c_0 k^{3/2}\}; \tag{2.15}$$

$$|\psi|_{C(\bar{A})} = O(k); \tag{2.16}$$

$$\lambda k |f(.\,, \Psi)|_{C(A)} |A|_\tau = \lambda k |f(.\,, \Psi)|_{C(A)} \int_A \mathrm{d}\tau = O(1); \tag{2.17}$$

$$|\Psi|_{C(A)} \leqslant Ck^{-1/2}. \tag{2.18}$$

Proof. As the Green function P of L in Π_d has the same estimates as that in D, (2.11) and (2.13) imply that for large k, $r_i^2/2 + k \leqslant Cr_i\{k^{1/2} + \log r_2\}$ after taking a suitable value for α (note that (2.11) shows also that λ_k cannot be bounded as $k \nearrow \infty$). The last estimate implies that $r_i = O(k^{1/2})$ and (2.16) is similarly obtained as the estimate holds for any $x = (r, z) \in A$. The capacity theory ([7]) shows that for large k, as $(r_2 - r_1)/r_1$ is bounded and A is moving away from $z = 0$ as $k \nearrow \infty$, if $\operatorname{diam} A = 2\varepsilon r_0$ where $r_0 := (r_1 + r_2)/2$, then we have the estimate

$$\varepsilon \leqslant C \exp\{-c_0 k^{3/2}\}.$$

In fact, the capacity of a closed subset E of Π relative to the operator L is defined as the quantity

$$\begin{aligned} \operatorname{Cap}_L(E) &:= \inf\left\{-\int_{\Pi \setminus E} uLu \,\mathrm{d}\tau \mid u \in C_0^\infty(\Pi),\ u|_E \geqslant 1\right\} \\ &= \inf\{\|u\|^2 \mid u \in H(\Pi);\ u|_E \geqslant 1\}. \quad ([7]) \end{aligned}$$

For $E := [a(1-\varepsilon),\, a(1+\varepsilon)] \times \{z=0\}$, if $\varepsilon > 0$ is small enough, then

$$\mathrm{Cap}_L(E) = 2\pi\big(\log(16/e^2\varepsilon)\big)^{-1}\{1 + O(\varepsilon \log 1/\varepsilon)\}.$$

(Theorem 3 of [7].)

Thus we have $2\pi\big(r_1 \log(16e^{-2}/\varepsilon)\big)^{-1} = \mathrm{Cap}_L A \leqslant (r_1^2/2 + k)^{-2}$ whence $\varepsilon \leqslant 16e^{-2}\exp\{-2\pi k^{3/2}s^{-1/2}(s^2/2+1)^2\}$ with $r_1^2 \simeq sk$ for large k. $y(s) := 2\pi s^{-1/2}(s^2/2+1)^2$ has its minimum $c_0 = 8\pi(2/3)^{3/2}$ at $s_0 = 2/3$. As $y''(s_0) > 0$, if $r_1^2 \simeq (s_0 + \tau^2)k$ for large k, (2.13) and (2.17) imply that $k^{m'}\exp(-\tau^2 k^{3/2}) = O(1)$ and this leads to (2.14). (2.17) follows from the fact that $\lambda k \int_A f(r,\Psi)\,\mathrm{d}\tau = O(1)$ for large k.

As $\psi \in S^1(\Pi)$ and for large k we have $|A(\psi)|_\tau := \int_{A(\psi)} \mathrm{d}\tau \simeq |A(\psi_0)|_\tau$ where ψ_0 denotes the solution for the problem without swirl with $m = 0$, for large k we obtain $(|A(\psi)|_\tau)^{-1}J_0(\psi) \leqslant (|A(\psi_0)|_\tau)^{-1}J_0(\psi_0) \leqslant |\Psi_0|_{C(A(\psi_0))} = Ck^{-1/2}$ and (2.18) follows. □

Theorem 2.7. *Suppose that for large k, $|\Psi|_{C(A)} = O(k^{-\alpha})$ for some $\alpha > 0$ and define*

$$g(\Psi) := \begin{cases} f_0\Psi_+^m & \text{if} \quad m < q+2; \\ f_q'\Psi_+^{q+1/\alpha} := (2f_q/3)\Psi_+^{q+1/\alpha} & \text{if } m > q+2; \\ f_{q0}\Psi_+^m := ((2f_q/3) + f_0)\Psi_+^m & \text{if } m = q+2. \end{cases} \tag{2.19}$$

Then as $k \nearrow \infty$, ψ becomes the solution for the problem without swirl

$$L\psi = -\lambda g(\Psi) \quad \text{in } \Pi_d. \tag{2.20}$$

Proof. From (2.14), for large k, in A we have $f(r,\Psi) \simeq (2f_q/3)\Psi^{q+1/\alpha} + f_0\Psi^m \simeq k^{-(1+\alpha q)}\{2f_q/3 + f_0 k^{\alpha(q-m)+1}\}$, hence $f(r,\Psi) \simeq g(\Psi)$ in A for large k. As $\|\psi\|^2_{H(\Pi_d)} = \lambda \int_A \psi f(r,\Psi)\,\mathrm{d}\tau$, we then have $\lim\limits_{k\nearrow\infty}\big\{\langle\psi,\psi\rangle_{H(\Pi_d)} - \lambda\int_A \psi g(\Psi)\,\mathrm{d}\tau\big\} = 0$ and (2.20) follows. □

3. Estimates in the stretched plane $\hat{\Pi}_d$

Define $\forall k > 0$ $a \equiv a(k) > 0$ such that

$$\nabla\Psi(a,0) = 0 \quad \text{and} \quad \Psi(a,0) = \max_{x\in A}\Psi(x). \tag{3.1}$$

Let $\hat{\Pi}_d$ be the image of Π_d in the transformation $(1.\zeta)$ where ε satisfies $\operatorname{diam} A = 2a\varepsilon$. $\hat{D}$ will denote the image of any $D \subset \Pi_d$. Define $u(\zeta) := u(a(1+\varepsilon\xi), a\varepsilon\eta)$ for u defined in Π_d and $\hat{f}(U) := f(a(1+\varepsilon\xi), U(\zeta))$. For large k,

$$\hat{A} \subset B^1 := \{\zeta \mid |\zeta|^2 = \xi^2 + \eta^2 < 1\} \text{ and } \operatorname{diam}\hat{A} = 2. \tag{3.2a}$$

For $x, x_0 \in \Pi$ with $x = (r,z)$ and $x_0 = (r_0, z_0)$, the Green function of L in Π is

$$P(x,x_0) = \frac{rr_0}{2\pi}\int_0^{\pi}\frac{\cos\theta\,\mathrm{d}\theta}{\{r^2 + r_0^2 - 2rr_0\cos\theta + (z-z_0)^2\}^{1/2}}$$

([3], [6]). So, provided that $\varepsilon|\zeta_0|, \varepsilon|\zeta| \in (0,1)$, the Green function P of L in $\hat{\Pi}$ satisfies for $P(\zeta,\zeta_0) := P((a(1+\varepsilon\xi), a\varepsilon\eta), (a(1+\varepsilon\xi_0), a\varepsilon\eta_0))$ ([6], [9])

$$P(\zeta,\zeta_0) = \frac{a}{2\pi}\Big\{\log\frac{8e^{-2}}{\varepsilon|\zeta-\zeta_0|} + R_1(\zeta,\zeta_0)\log\frac{8}{\varepsilon|\zeta-\zeta_0|} + R_2(\zeta,\zeta_0)\Big\} \tag{3.2b}$$

where for $|\alpha| \in \mathbb{N}$, $|D^\alpha R_i| = O(\varepsilon)$. Under those conditions we have the following estimates for large k:

$$P(\zeta_0,\zeta) = \frac{a}{2\pi}\log\frac{8e^{-2}}{\varepsilon|\zeta-\zeta_0|} + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big) \tag{3.3}$$

$$= Ck^2 + \frac{a}{2\pi}\log\frac{1}{|\zeta-\zeta_0|} + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big). \tag{3.4a}$$

If $\zeta' \in \hat{A}$, we get for large k and $|\nabla_{\zeta'}P(\zeta,\zeta')| := \sqrt{P_{\xi'}^2 + P_{\eta'}^2}$

$$|\nabla_{\zeta'}P(\zeta,\zeta')| \leqslant \text{const } a\Big\{\varepsilon\log\frac{1}{\varepsilon} + \frac{1}{|\zeta-\zeta'|} + \varepsilon\log\frac{e}{|\zeta-\zeta'|}\Big\}. \tag{3.4b}$$

Lemma 3.1. *For large k,*

$$\Psi(\zeta_0) = \frac{\lambda a^4\varepsilon^2}{2\pi}\int_{\hat{A}}\hat{f}(\Psi)\log\frac{1}{|\zeta-\zeta_0|}\,\mathrm{d}\xi\,\mathrm{d}\eta + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big) \tag{3.5a}$$

$$= O(k^{-1/2}) \quad \textit{in } \hat{A}, \tag{3.5b}$$

$$\textit{and } \big||\psi|_{C(A)} - 4k/3\big| = O(k^{-1/2}). \tag{3.6}$$

P r o o f. From (3.4), for large k, with $\delta := \varepsilon \log 1/\varepsilon$ and $\zeta_0 \in \hat{A}$, we have $\psi(\zeta_0) = Ck + (\lambda a^4 \varepsilon^2/2\pi) \int_{\hat{A}} \log(1/|\zeta - \zeta_0|)\, \hat{f}(\Psi)(1 + \varepsilon\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta + O(\delta)$. From (2.17), for large k, we have $\lambda a^5 \varepsilon^2 |\hat{f}(\Psi)|_{\hat{A}} = O(1)$ hence the integral term above is $O(k^{-1/2})$. The fact that $|\psi|_{C(A)} = O(k)$ then leads to (3.5a). The formula (3.6) follows from (2.14) and (3.5b). □

For any $k > 0$, define $u_k := k^{1/2}\Psi$. In $\hat{\Pi}_d$

$$\text{(3.7)} \qquad \forall k > 0 \quad \nabla u_k(0) = 0 \quad \text{and} \quad |u_k|_{C(\hat{A})} = O(1) \quad \text{for large } k.$$

As $\forall k > 0$ $\lambda \int_A \psi f(r, \Psi)\, \mathrm{d}\tau = 1$, by (2.19), (3.5b) and (3.6) each of the quantities

$$\text{(3.8)} \qquad \begin{cases} \dfrac{4}{3}\lambda a^2 \varepsilon^2 k^{(2-m)/2} f_0 u_k(0)^m |\hat{A}| & \text{if } m < q + 2, \\ \dfrac{4}{3}\lambda a^3 \varepsilon^2 k^{-q/2} f'_q u_k(0)^{q+2} |\hat{A}| & \text{if } m > q + 2, \\ \text{and } \dfrac{4}{3} f_{q0} \lambda a^3 \varepsilon^2 k^{(2-m)/2} u_k(0)^m |\hat{A}| & \text{if } m = q + 2 \end{cases}$$

converges to 1 as $k \nearrow \infty$.

Theorem 3.2. *Let $\mu \in (0, 1]$; then $(u_k)_{k\in\mathbb{N}}$ converges to u, such that*
1) $u \in C^{2,\mu}(\hat{\Pi}_d)$ if $mq > 0$ and $u \in C^{1,\mu}(\hat{\Pi}_d) \bigcap C^{2,\mu}(\hat{\Pi} \setminus \partial\hat{A}(u))$ if $mq = 0$;
2) u is radial and independent of d, f_0 and f_q. In fact, for $\sigma := |\zeta|$ we have

$$\text{(3.9)} \qquad u(\sigma) = \begin{cases} \dfrac{\sqrt{6}}{4\pi\varrho_m^2} Q_m(\varrho_m \sigma) & \text{if } m \leqslant q + 2 \\ \dfrac{\sqrt{6}}{4\pi\varrho_{q+2}^2} Q_{q+2}(\varrho_{q+2}\sigma) & \text{if } m > q + 2 \end{cases}$$

where Q_l and ϱ_l are defined in (1.Q).

P r o o f. Let B be a (bounded) ball centered at the origin in $\hat{\Pi}_d$ and let k be large. From the equation $L\Psi = -\lambda f(r, \Psi)$, with $A_k := A(\psi)$ we obtain $(\partial_\xi^2 - \varepsilon\partial_\xi/(1 + \varepsilon\xi) + \partial_\eta^2) u_k = -\lambda a^4 \varepsilon^2 k^{1/2} (1 + \varepsilon\xi)^2 \hat{f}(\Psi)$ in $\hat{A}_k$.

From (2.17) and (3.8), the second member of this equation is bounded uniformly on $\hat{A}_k$ and the elliptic theory implies that $\|u_k\|_{W_p^2(B)}$ is uniformly bounded as easy calculations show that $|u_k(\zeta)| \leqslant C|\zeta|$ for $\zeta \notin \hat{A}_k$.

In fact, from (3.4b), as $\varepsilon^2 a^5 \lambda |\hat{f}(\Psi)|_{C(\hat{A})} = O(1)$, we obtain

$$\begin{aligned} |\nabla\psi(\zeta)| &\leqslant \mathrm{const}\, \varepsilon^2 a^4 \lambda |\hat{f}(\Psi)|_{C(\hat{A})} \\ &\quad \times \left\{ \varepsilon \log(1/\varepsilon) + \int_{\hat{A}} (1/|\zeta - \zeta'|)\, \mathrm{d}\xi'\, \mathrm{d}\eta' + \varepsilon \int_{\hat{A}} \log(e/|\zeta - \zeta'|)\, \mathrm{d}\xi'\, \mathrm{d}\eta' \right\} \\ &\leqslant (\mathrm{const}/a)\{\varepsilon \log(1/\varepsilon) + O(1 + \varepsilon)\} = O(k^{-1/2}). \end{aligned}$$

For $\zeta'' \in \partial\hat{A}$ satisfying $\mathrm{dist}(\zeta, \partial\hat{A}) = |\zeta - \zeta''|$, as $u_k(\zeta'') = 0$, we conclude $|u_k(\zeta)| = |u_k(\zeta) - u_k(\zeta'')| \leqslant |\nabla u_k||\zeta - \zeta''| \leqslant |\nabla u_k||\zeta| \leqslant \mathrm{const}|\zeta|$ as $|\nabla u_k| \leqslant \mathrm{const}k^{1/2}|\nabla\psi| + O(k^{3/2}\varepsilon)$. The existence of $u \in C^{1,\nu}(B)$ as the limit of a subsequence of (u_k) follows from the Sobolev imbedding theorems. The regularity of u follows from the elliptic theory.

Let $m < q + 2$. From (2.19) and (3.5), for large k and $\delta := \varepsilon \log(1/\varepsilon)$, we have $u_k(\zeta) \simeq (1/2\pi)(\sqrt{2/3})\lambda a^3\varepsilon^2 k^{(2-m)/2} f_0 \int_{\hat{A}} u_k^m \log(1/(|\zeta' - \zeta|)\, \mathrm{d}\xi'\, \mathrm{d}\eta' + O(\delta)$. So, as from (3.8) $\lim_{k\nearrow\infty} (\sqrt{2/3})\lambda a^3\varepsilon^2 k^{(2-m)/2} f_0 = \sqrt{6}/(4|\hat{A}(u)|u(0)^m) := \nu_m$, u is a fixed point of N where

$$\forall\zeta \in \hat{\Pi}_d \qquad N\varphi(\zeta) = \frac{\nu_m}{2\pi}\int_{\hat{A}(u)} \varphi_+^m \log\frac{1}{|\zeta' - \zeta|}\, \mathrm{d}\xi'\, \mathrm{d}\eta'. \tag{3.10}$$

Then from [8], u is radial and $\nu_m = \sqrt{6}/(4\pi u(0)^m)$. For $\sigma := |\zeta|$, $u'' + u'/\sigma = -\nu_m u_+^m$, $\sigma > 0$; $u'(0) = 0$, $u(0) := u_0$ for some $u_0 > 0$.

With $t := \{\nu_m u_0^{m-1}\}^{1/2}\sigma$ and $W(t) =: u(\sigma)/u_0$, we have $W'' + W'/t = -W_+^m$; $W(0) = 1$, $W'(0) = 0$ whence $u(\sigma) = u_0 Q_m(\{\nu_m u_0^{m-1}\}^{1/2}\sigma)$. $u(1) = 0 \Longrightarrow \sqrt{6}/(4\pi u_0) = \varrho_m^2$ and

$$u(0) = \frac{\sqrt{6}}{4\pi\varrho_m^2}. \tag{3.11}$$

Because $u(0)$ is independent of the choice of the subsequence, (u_k) converges to u. The cases when $m \geqslant q + 2$ follow by the same arguments. □

4. Existence of variational solutions in Π for large k and their estimates

Lemma 4.1. *Let $\mu \in (0, 1]$; there is $K > 0$ such that for $k > K$*

$$L\psi = -\lambda f(r, \Psi) \quad \text{in } \Pi \tag{4.1}$$

with $\psi \in H(\Pi)$ has a solution $\psi \in C^{2,\mu}(\Pi)$ if $mq > 0$ ($\in C^{1,\mu}(\Pi) \bigcap C^{2,\mu}(\Pi \setminus \partial A(\psi))$ if $mq = 0$) which is a maximizer of Z on $S_1(\Pi)$.

P r o o f. Theorem 2.6 implies that there is $K > 0$ such that $\forall d \in (0, 1]$ and $k > K$ we have $r_1 := \inf\{r > 0 \mid (r, 0) \in A(\psi_d)\} > 1$.

Consider a decreasing sequence $(d_i)_{i\in\mathbb{N}}$ in $(0, 1]$ such that $d_i \searrow 0$ and a fixed $k > K$. The proof is similar to that of Theorem 2.4 as $\forall i \in \mathbb{N}$, $S_1(\Pi_{d_i}) \subset S_1(\Pi_{d_{i+1}}) \subset S_1(\Pi)$. □

Theorem 4.2. *Theorems* 2.6, 2.7 *and* 3.2 *hold for the variational solutions* $\{\psi_k\}_{k>K}$ *of the problems in* Π.

P r o o f. This follows from the facts that the Green function of L in Π has the same upper bounds as that in Π_d and the estimates (3.3)–(3.4) hold for it as well. It has to be noted that $\hat{\Pi}_d \simeq \hat{\Pi}$ for large k. □

To my uncles G. Tatuam, P. Wabo and JB. Topa Chatue, in memoriam.

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