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LOCAL LIPSCHITZ CONTINUITY OF THE STOP OPERATOR

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Abstract. On a closed convex set Z in $\mathbb{R}^N$ with sufficiently smooth ($\mathbf{W}^{2,\infty}$) boundary, the stop operator is locally Lipschitz continuous from $\mathbf{W}^{1,1}([0,T],\mathbb{R}^N)\times Z$ into $\mathbf{W}^{1,1}([0,T],\mathbb{R}^N)$. The smoothness of the boundary is essential: A counterexample shows that $\mathcal{C}^1$-smoothness is not sufficient.

Keywords: hysteresis, stop operator, differential inclusion, Lipschitz continuity

MSC 2000: 34A60, Secondary 49J40

1. Introduction and main result

Throughout the paper we will use the following notation: For $1\leqslant p<\infty$, an interval $[0,T]$, and a set $Z\subset\mathbb{R}^N$, the space $\mathbf{W}^{1,p}([0,T],Z)$ denotes the space of absolutely continuous functions $f\colon [0,T]\to Z$ whose derivative is in $\mathbf{L}^p$. We use the norm

$$\|f\|^p_{\mathbf{W}^{1,p}}=\int_0^T |f(t)|^p\,\mathrm{d}t+\int_0^T |f'(t)|^p\,\mathrm{d}t.$$

If $\Omega\subset\mathbb{R}^M$ is a domain, $\mathbf{W}^{k,\infty}(\Omega,Z)$ is the space of functions $f\colon\Omega\to Z$ whose partial derivatives up to order $k-1$ are Lipschitz continuous. By $B(x,r)$ we mean the closed ball with center x and radius r.

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Let $Z \subset \mathbb{R}^N$ be a closed convex set. Given $x_0 \in Z$ and a function $u \in \mathbf{W}^{1,1}([0,T],\mathbb{R}^N)$, we seek a function $x \in \mathbf{W}^{1,1}([0,T],\mathbb{R}^N)$ such that

- $x(0) = x_0$.
- $x(t) \in Z$ for all $t \in [0,T]$.
- For almost all t, $x'(t)$ is as close as possible to $u'(t)$.

Then x is characterized by the variational inequality

$$\begin{aligned} &x(0) = x_0, \\ &x(t) \in Z, \\ &(\forall y \in Z)\ \langle u'(t) - x'(t), y - x(t)\rangle \leqslant 0. \end{aligned} \tag{1.1}$$

We denote by ∂Z the boundary and by Z° the interior of Z. By $N_Z(x)$ we denote the normal cone of Z at the point x. We can rewrite the variational inequality as a differential inclusion

$$\begin{aligned} &x(0) = x_0, \\ &x(t) \in Z, \\ &u'(t) - x'(t) \in N_Z(x(t)). \end{aligned}$$

If Z is the closure of an open domain Z° with $\mathcal{C}^1$-boundary, so that for each point $x \in \partial Z$ the outward unit normal vector $n(x)$ is defined and depends continuously on x, then the differential inclusion is in fact a differential equation

$$x'(t) = \begin{cases} u'(t) \text{ if } x(t) \in Z^\circ, \\ u'(t) \text{ if } x(t) \in \partial Z \text{ and } \langle n(x(t)), u'(t)\rangle < 0, \\ u'(t) - \langle n(x(t)), u'(t)\rangle\ n(x(t)) \\ \qquad \text{if } x(t) \in \partial Z \text{ and } \langle n(x(t)), u'(t)\rangle \geqslant 0. \end{cases} \tag{1.2}$$

Given any closed convex set Z, it is shown in [6], that for any $x_0 \in Z$ and any $u \in \mathbf{W}^{1,1}([0,T],\mathbb{R}^N)$ there exists a unique function $x \in \mathbf{W}^{1,1}([0,T],Z)$ solving (1.1). (See also [7, Proposition 2.2], [8].) The operator

$$\mathcal{S}\colon \begin{cases} \mathbf{W}^{1,1}([0,T],\mathbb{R}^N) \times Z & \to \mathbf{W}^{1,1}([0,T],Z), \\ (u, x_0) & \mapsto x \end{cases}$$

is called the stop operator with characteristic Z. This operator plays a fundamental role in the theory of elastoplastic materials (see, e.g., the monographs [3], [6], [8], [11]).

According to [7, Proposition 3.1 and Corollary 3.4], the stop operator maps $\mathbf{W}^{1,p}([0,T],\mathbb{R}^N)\times Z$ continuously into $\mathbf{W}^{1,p}([0,T],\mathbb{R}^N)$ for $1\leqslant p<\infty$. Moreover, global Lipschitz continuity has been proved on $\mathbf{W}^{1,1}\times Z$ into $\mathbf{W}^{1,1}$, if $Z\subset\mathbb{R}$ is an interval [10], and, more generally, if $Z\subset\mathbb{R}^N$ is a (bounded or unbounded) polyhedron [4]. If $p>1$, the stop operator is not Lipschitz continuous from $\mathbf{W}^{1,p}\times Z$ into $\mathbf{W}^{1,p}$ [10]. The unit ball in $\mathbb{R}^2$ provides a counterexample to global Lipschitz continuity in $\mathbf{W}^{1,1}$ for general convex sets, however, if Z is a ball in $\mathbb{R}^N$, the stop operator satisfies a local Lipschitz condition

$$\begin{aligned}|x(t)-y(t)|+\int_0^T|x'(t)-y'(t)|\,\mathrm{d}t\\ \leqslant M(u)\biggl[|x_0-y_0|+\int_0^T|u'(t)-v'(t)|\,\mathrm{d}t\biggr]\end{aligned}$$

if $x=\mathcal{S}(u,x_0)$, $y=\mathcal{S}(v,y_0)$, and $M(u)$ is a Lipschitz constant depending on $\int_0^T|u'(t)|\,\mathrm{d}t$ [2, Corollary A.4 and Example A.6].

It is announced without proof in [6, Chapter 4, Theorem 20.1] that a similar local Lipschitz condition holds on domains with smooth boundaries. In this paper we give a proof for the local Lipschitz continuity of the stop operator if the domain Z is smooth enough so that there exists a unique outward unit normal vector $n(x)$ to ∂Z at every boundary point $x\in\partial Z$ and $n(x)$ depends Lipschitz continuously on x.

Hypothesis 1.1. *Let $Z\subset\mathbb{R}^N$ be a closed convex set with $\mathbf{W}^{2,\infty}$-boundary, i.e., for all $z\in\partial Z$ there exists an orthonormal system $(v_1,\dots,v_N)$, some $\varepsilon>0$ and a map $a\in\mathbf{W}^{2,\infty}([-\varepsilon,\varepsilon]^{N-1},\mathbb{R})$ such that $a(0,\dots,0)=0$ and for all $\xi_j\in[-\varepsilon,\varepsilon]$ the following holds:*

$$z+\sum_{j=1}^{N-1}\xi_jv_j+(a(\xi_1,\dots,\xi_{N-1})+\xi_N)v_N\in Z\quad\text{iff}\quad\xi_N\geqslant 0.$$

By $n(z)$ we will denote the outward unit normal vector at z:

$$n(z)=\frac{1}{\sqrt{1+\sum\limits_{j=1}^{N-1}\frac{\partial a}{\partial\xi_j}(0)^2}}\biggl(\sum_{j=1}^{N-1}\frac{\partial a}{\partial\xi_j}(0)v_j-v_N\biggr).$$

With this assumption we prove the following theorem:

Theorem 1.1. *Let $Z \subset \mathbb{R}^N$ satisfy Hypothesis* 1.1, *and let K be a compact subset of Z. Let $R > 0$ be fixed. Then there exists a constant $L > 0$ (depending on K and R) such that the following local Lipschitz estimate holds:*

If $x_0, y_0 \in K$ and $u, v \in \mathbf{W}^{1,1}([0,T], \mathbb{R}^N)$ for some $T > 0$ with

$$\int_0^T (|u'(t)| + |v'(t)|)\,\mathrm{d}t \leqslant R, \tag{1.3}$$

then $x = \mathcal{S}(u, x_0)$ and $y = \mathcal{S}(v, y_0)$ satisfy

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \leqslant L\biggl[|x_0 - y_0| + \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t\biggr]. \tag{1.4}$$

We will give the proof in Section 2. The smoothness assumption on ∂Z is essential: In Section 3 we present a cone in $\mathbb{R}^3$ as a counterexample to local Lipschitz continuity of the stop operator in general convex sets. Moreover, in Example 3.2 we show that Hölder continuous dependence of the normal vector $n(x)$ on x is not sufficient to imply that the stop operator is locally Lipschitz.

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2. Proof of the main result

For the proof of the main theorem, we will require some simple facts from differential geometry. Let V be a relatively compact subset of ∂Z. The tubular neighborhood of radius $\delta > 0$ around V is defined by

$$\operatorname{Tub}^\delta V = \{x + \lambda n(x) \mid \lambda \in (-\delta, \delta)\}.$$

If ∂Z is a $\mathcal{C}^2$-manifold, the implicit function theorem can be used to show that for sufficiently small $\delta > 0$, the map

$$\operatorname{can}\begin{cases} V \times (-\delta, \delta) & \to \operatorname{Tub}^\delta V \\ (x, \lambda) & \mapsto x + \lambda n(x) \end{cases}$$

is a $\mathcal{C}^1$-diffeomorphism. (This is, e.g., a special case of the situation treated in [1, Section 2.7].) Since we have required less smoothness than $\mathcal{C}^2$, the map can will in general not be contained in $\mathcal{C}^1$, and the standard versions of the implicit function theorem do not work. We will therefore relax the smoothness assumption a little and give a different proof:

Lemma 2.1. *Let Z be as in Hypothesis* 1.1, *and $z \in \partial Z$. For $x \in \mathbb{R}^N$ we define*

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial Z) & \text{if } x \in Z, \\ -\operatorname{dist}(x, \partial Z) & \text{if } x \notin Z. \end{cases}$$

For $\delta > 0$ let U_δ be the tubular neighborhood of radius δ around $\partial Z \cap B(z, \delta)$. Then $\delta > 0$ may be chosen sufficiently small, such that the following assertions hold:

(i) can: $[\partial Z \cap B(z, \delta)] \times (-\delta, \delta) \to U_\delta$ *is a Lipschitz continuous homeomorphism with a Lipschitz continuous inverse.*

(ii) *d is differentiable on U_δ, and its gradient $\nabla d(x)$ depends Lipschitz continuously on x. Namely, if $x = \operatorname{can}(y, \lambda)$, then $\nabla d(x) = -n(y)$.*

Proof. Let $z \in \partial Z$. We utilize the chart generated by $v_1, \ldots, v_N$, $\varepsilon > 0$, and the function a as in Hypothesis 1.1. Without loss of generality (by rotation of the coordinate system, if necessary) we may assume that $n(z) = -v_N$. We define

$$T\colon \begin{cases} (-\varepsilon, \varepsilon)^{N-1} \times (-\varepsilon, \varepsilon) & \to \mathbb{R}^N, \\ (\xi, \lambda) & \mapsto z + \sum\limits_{j=1}^{N-1} \xi_j v_j + (a(\xi) + \lambda) v_N, \end{cases}$$

and write the map can and the normal vector n in local coordinates:

$$\tilde{n}\colon \begin{cases} (-\varepsilon, \varepsilon)^{N-1} & \to \mathbb{R}^N, \\ \xi & \mapsto n\Big(z + \sum\limits_{j=1}^{N-1} \xi_j v_j + a(\xi_1, \ldots, \xi_{N-1}) v_N\Big), \end{cases}$$

$$\widetilde{\operatorname{can}}\colon \begin{cases} (-\varepsilon, \varepsilon)^{N-1} \times (-\varepsilon, \varepsilon) & \to \mathbb{R}^N, \\ (\xi, \lambda) & \mapsto z + \sum\limits_{j=1}^{N-1} \xi_j v_j + a(\xi) v_N + \lambda \widetilde{n}(\xi). \end{cases}$$

We have to prove that $\widetilde{\operatorname{can}}$ has a Lipschitz continuous inverse on a suitable sufficiently small neighborhood of z. It is easy to prove that T^{-1} exists and is Lipschitz continuous on a suitable neighborhood of z. Let M be a Lipschitz constant for T^{-1}. Notice that

$$(T - \widetilde{\operatorname{can}})(\xi, \lambda) = \lambda(v_N - \widetilde{n}(\xi)).$$

Therefore, if $\eta \in (0,\varepsilon)$ is sufficiently small, $(T - \widetilde{\text{can}})$ is Lipschitz on $(-\eta,\eta)^{N-1} \times (-\eta,\eta)$ with a Lipschitz constant $L < 1/(2M)$. From the contraction principle [5, 10.1.3] we infer that for y sufficiently close to z, there exists a unique solution to

$$(\xi,\lambda) = T^{-1}[y + T(\xi,\lambda) - \widetilde{\text{can}}(\xi,\lambda)],$$

which is equivalent to

$$y = \widetilde{\text{can}}(\xi,\lambda).$$

The proof of the contraction principle shows that this solution depends Lipschitz continuously on y. Therefore can possesses a Lipschitz continuous inverse on a sufficiently small neighborhood of W of z.

Now choose a neighborhood V of z and $\delta > 0$ sufficiently small, such that $U = \text{Tub}^\delta V \subset W$ and for any $x \in U$ the closest point $\Pi(x)$ to x on ∂Z is contained in W. For $x \in U$, elementary geometry shows that

$$\text{can}^{-1}(x) = (\Pi(x), -d(x)).$$

The proof above implies therefore that d is Lipschitz continuous. However, we can improve the result and obtain continuous differentiability of d. Let $x \in U$ and Δx be sufficiently small. We define

$$\begin{aligned} \Delta\Pi &= \Pi(x+\Delta x) - \Pi(x), \\ \Delta d &= d(x+\Delta x) - d(x), \\ \Delta n &= n(\Pi(x+\Delta x)) - n(\Pi(x)). \end{aligned}$$

Notice that by the Lipschitz continuity of n and can^{-1}, all of the following terms, $\Delta\Pi$, Δd, and Δn are of order $O(\Delta x)$. Thus

$$\begin{aligned} \Delta x &= \big[\Pi(x+\Delta x) - d(x+\Delta x)n(\Pi(x+\Delta x))\big] - \big[\Pi(x) - d(x)n(\Pi(x))\big] \\ &= \Pi(x) + \Delta\Pi - (d(x) + \Delta d)[n(\Pi(x)) + \Delta n] - \Pi(x) + d(x)n(\Pi(x)) \\ &= \Delta\Pi - d(x)\Delta n - (\Delta d)n(\Pi(x)) + o(\Delta x). \end{aligned}$$

Since n is normalized, we infer that $\langle n(\Pi(x)), \Delta n\rangle = o(\Delta x)$, and since n is orthogonal to ∂Z, we infer that $\langle n(\Pi(x)), \Delta\Pi\rangle = o(\Delta x)$. We obtain therefore

$$\langle n(\Pi(x)), \Delta x\rangle = -\Delta d + o(\Delta x).$$

This says that $\nabla d(x) = -n(\Pi(x))$. □

The following lemma is the core of the proof of Theorem 1.1.

Lemma 2.2. *Let Z be as in Hypothesis* 1.1, *and $z \in Z$. Then there exists a neighborhood V of z, a constant $R > 0$ and a constant $M > 0$ such that the stop operator satisfies the following local Lipschitz condition:*

If $T > 0$, $x_0, y_0 \in V$, $u, v \in \mathbf{W}^{1,1}([0,T], \mathbb{R}^N)$ with

$$\int_0^T (|u'(t)| + |v'(t)|)\,\mathrm{d}t \leqslant R,$$

and $x = \mathcal{S}(u, x_0)$, $y = \mathcal{S}(v, y_0)$, then

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \leqslant M\biggl[|x_0 - y_0| + \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t\biggr].$$

Proof. If $z \in Z^\circ$, then choose a neighborhood V and a constant $R > 0$ such that $V + B(0, R)$ is entirely contained in Z°. Since $|x'(t)| \leqslant |u'(t)|$, $|y'(t)| \leqslant |v'(t)|$ (e.g., [4, Proposition 1.2]), we infer that $x(t)$ and $y(t)$ remain in Z° for $t \leqslant T$, so that $x' = u'$ and $y' = v'$. In this case, the assertion is trivial.

Assume now that $z \in \partial Z$. According to Lemma 2.1 we choose a neighborhood $U = U_\delta$ of Z such that d is differentiable with Lipschitz continuous derivative on U. For shorthand we denote $n(x_0) = -\nabla d(x_0)$. This notation is consistent with the fact that $n(x_0)$ is the outward unit normal vector to Z at x_0, if $x_0 \in \partial Z$. Let L be a Lipschitz constant for n on U. Notice also that $|n(x_0)| \leqslant 1$ for any $x_0 \in U$, since n is the negative gradient of a distance. Again we choose a constant $R > 0$ and a neighborhood V of z such that $V + B(0, R) \subset U$, therefore $x(t)$ and $y(t)$ remain in U for $t \leqslant T$.

We keep track of the functions $|x'(t) - y'(t)|$, $|x(t) - y(t)|$ and $\beta(t) = |d(x(t)) - d(y(t))|$. Let t be a Lebesgue point of all of the following functions, x', y', $[d(x)]'$, $[d(y)]'$, and $|x(t) - y(t)|'$, and such that (1.2) holds. From [7, (2.6)] we infer easily that

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t) - y(t)| \leqslant |u'(t) - v'(t)|.$$

Thus

$$|x(t) - y(t)| \leqslant |x_0 - y_0| + \int_0^t |u'(s) - v'(s)|\,\mathrm{d}s. \tag{2.1}$$

To handle the other two functions, we will prove the inequality

$$\begin{aligned} &|x'(t) - y'(t)| + \beta'(t) \\ &\leqslant 2|u'(t) - v'(t)| + 2L(|u'(t)| + |v'(t)|)\,|x(t) - y(t)|. \end{aligned} \tag{2.2}$$

Once this equation is proved, we may integrate and obtain

$$\begin{aligned}
&\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \\
&\quad \leqslant \beta(0) - \beta(T) + 2\int_0^T |u'(t) - v'(t)|\,\mathrm{d}t \\
&\qquad + 2L\int_0^T (|u'(t)| + |v'(t)|)|x(t) - y(t)|\,\mathrm{d}t \\
&\quad \leqslant |x_0 - y_0| - 0 + 2\int_0^T |u'(t) - v'(t)|\,\mathrm{d}t \\
&\qquad + 2L\biggl(|x_0 - y_0| + \int_0^T |u'(s) - v'(s)|\,\mathrm{d}s\biggr)\int_0^T |u'(t) + v'(t)|\,\mathrm{d}t \\
&\quad \leqslant (2LR + 1)|x_0 - y_0| + (2LR + 2)\int_0^T |u'(t) - v'(t)|\,\mathrm{d}t.
\end{aligned}$$

Therefore, Lemma 2.2 is proved, if we can show (2.2). For this purpose we distinguish the following cases:

Case 1: $x(t) \in Z^\circ$, $y(t) \in Z^\circ$:

In this case, $x' = u'$ and $y' = v'$. For shorthand we will omit the argument (t) in the following computations. Thus

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\beta &\leqslant \Bigl|\frac{\mathrm{d}}{\mathrm{d}t}d(x) - \frac{\mathrm{d}}{\mathrm{d}t}d(y)\Bigr| = |-\langle n(x), u'\rangle + \langle n(y), v'\rangle| \\
&\leqslant |\langle n(x), u' - v'\rangle| + |\langle n(x) - n(y), v'\rangle| \leqslant |u' - v'| + L|x - y|\,|v'|.
\end{aligned}$$

Equation (2.2) follows easily.

Case 2: $x(t) \in \partial Z$ and $y(t) \in \partial Z$.

Since x is differentiable at the point t and $x(t) \in \partial Z$ while $x(s) \in Z$ for all s, the derivative $x'(t)$ is necessarily in the tangent space of Z at $x(t)$. This is only possible if $u'(t)$ does not point strictly inward, i.e. $\langle n(x), u'\rangle \geqslant 0$. The same argument holds for y'. We have therefore

$$x' = u' - \langle n(x), u'\rangle n(x), \quad y' = v' - \langle n(y), v'\rangle n(y).$$

We infer that

$$\begin{aligned}
|x' - y'| &= |u' - \langle n(x), u'\rangle n(x) - v' + \langle n(y), v'\rangle n(y)| \\
&\leqslant |u' - v' - \langle n(x), u' - v'\rangle n(x)| + |\langle n(x) - n(y), v'\rangle n(x)| \\
&\quad + |\langle n(y), v'\rangle (n(x) - n(y))| \\
&\leqslant |u' - v'| + 2L|x - y|\,|v'|.
\end{aligned}$$

Since x' and y' are tangential to ∂Z, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta \leqslant \left|\frac{\mathrm{d}}{\mathrm{d}t}d(x(t))\right| + \left|\frac{d}{\mathrm{d}t}d(y(t))\right| = 0.$$

Summing up these estimates, we infer again (2.2).

Case 3: $x(t) \in \partial Z$ and $y(t) \in Z^\circ$, or vice versa.

Again $\langle n(x), u'\rangle \geqslant 0$ and x' is tangential to ∂Z. Then

$$|x' - y'| = |u' - \langle n(x), u'\rangle n(x) - v'| \leqslant |u' - v'| + \langle n(x), u'\rangle.$$

Notice that in this case $d(x) = 0$, $d(y) > 0$, and again $\frac{d}{\mathrm{d}t}d(x) = 0$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\beta = \frac{d}{\mathrm{d}t}(d(y) - d(x)) &= \langle -n(y), v'\rangle - 0\\ &\leqslant |\langle n(y) - n(x), v'\rangle| + |\langle n(x), v' - u'\rangle| - \langle n(x), u'\rangle\\ &\leqslant L|x - y|\,|v'| + |u' - v'| - \langle n(x), u'\rangle.\end{aligned}$$

This implies again the estimate (2.2). □

P r o o f o f T h e o r e m 1.1. For each $z \in Z$, choose a neighborhood $V(z)$ and constants $M(z), R(z)$ according to Lemma 2.2. Let $W(z)$ be a neighborhood of z and let $\delta(z)$ be sufficiently small, such that $W(z) + B(0, \delta(z)) \subset V(z)$. We cover $K + B(0, R)$ by a finite union of neighborhoods $W(z_i)$ $(i = 1, \ldots, m)$. Put $M = \max\{M(z_i) \mid i = 1, \ldots, m\}$, $S = \min\{R, R(z_1), \ldots, R(z_m)\}$ and $\delta = \min\{\delta(z_i) \mid i = 1, \ldots, m\}$. We start proving Equation (1.4) with R replaced by S in (1.3), and with the assumption that

$$x_0, y_0 \in K + B(0, R) \text{ with } |x_0 - y_0| < \delta. \tag{2.3}$$

Choose i such that $x_0 \in W(z_i) \subset V(z_i)$. Assumption (2.3) implies $y_0 \in V(z_i)$. Therefore we may apply Lemma 2.2 on the set $V(z_i)$ and obtain exactly Equation (1.4) with $L = M$.

Next we remove the condition (2.3). Let $x_0, y_0 \in K + B(0, R)$ with $|x_0 - y_0| \leqslant k\delta$, and let $u, v \in \mathbf{W}^{1,1}([0, T], \mathbb{R}^N)$ satisfy (1.3) with S instead of R. For $j = 0, \ldots, k$ we define functions $z_j = \mathcal{S}(u_j, x_j)$ with $u_j = u + \frac{j}{k}(v - u)$ and $x_j = x_0 + \frac{j}{k}(y_0 - x_0)$. Notice that $x = z_0$ and $y = z_k$, and the initial data satisfy $|x_j - x_{j-1}| \leqslant \delta$. Therefore

(1.4) holds for each of the differences $z_j - z_{j-1}$ and we obtain

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \leqslant \sum_{j=1}^{k} \int_0^T |z'_{j-1}(t) - z'_j(t)|\,\mathrm{d}t$$
$$\leqslant M \sum_{j=1}^{k} \left[|x_{j-1} - x_j| + \int_0^T |u'_{j-1}(t) - u'_j(t)|\,\mathrm{d}t \right]$$
$$= M \left[|x_0 - y_0| + \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t \right].$$

Finally we ged rid of the assumption that R is replaced by S in (1.3). Assume that $R \leqslant kS$ with fixed k. Let $x_0, y_0 \in K$ and let $u, v \in \mathbf{W}^{1,1}([0,T], \mathbb{R}^N)$ satisfy (1.3). Since $|x'(t)| \leqslant |u'(t)|$, we infer that $x(t) \in K + B(0,R)$ for all $t \in [0,T]$. The same holds for $y(t)$. Choose $0 = t_0 < t_1 < \ldots < t_k = T$ such that

$$\int_{t_k}^{t_{k+1}} (|u'(t)| + |v'(t)|)\,\mathrm{d}t \leqslant S.$$

The estimate (1.4) holds on the intervals $[t_{j-1}, t_j]$. Utilizing Equation (2.1), we obtain

$$\int_{t_{j-1}}^{t_j} |x'(t) - y'(t)|\,\mathrm{d}t$$
$$\leqslant M \left[|x(t_{j-1}) - y(t_{j-1})| + \int_{t_{j-1}}^{t_j} |u'(t) - v'(t)|\,\mathrm{d}t \right]$$
$$\leqslant M \left[|x_0 - y_0| + \int_0^{t_{j-1}} |u'(t) - v'(t)|\,\mathrm{d}t + \int_{t_{j-1}}^{t_j} |u'(t) - v'(t)|\,\mathrm{d}t \right]$$
$$\leqslant M \left[|x_0 - y_0| + \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t \right].$$

Summing up all intervals we obtain

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \leqslant kM \left[|x_0 - y_0| + \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t \right].$$

Therefore (1.4) holds with $L = kM$. □

We show that the local Lipschitz condition proved in Theorem 1.1 for smooth domains is not valid in general convex sets. Our first counterexample is a cone of revolution in $\mathbb{R}^3$. For preparation we show that a local Lipschitz condition in a cone in fact implies a global condition.

Lemma 3.1. *Let $Z \subset \mathbb{R}^N$ be a closed convex cone with vertex 0. Suppose that there exist $R > 0$, $M > 0$ and $T > 0$ such that for all $u, v \in \mathbf{W}^{1,1}([0,T], \mathbb{R}^N)$ with*

$$\int_0^T (|u'(t)| + |v'(t)|)\,\mathrm{d}t \leqslant R,$$

the solutions $x = \mathcal{S}(u,0)$ and $y = \mathcal{S}(v,0)$ satisfy the estimate

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \leqslant M \int_0^T |u'(t) - v'(t)|\,\mathrm{d}t.$$

Then for all $x_0, y_0 \in Z$ and all $w \in \mathbf{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathbb{R}^N)$ the solutions $x = \mathcal{S}(w, x_0)$, $y = \mathcal{S}(w, y_0)$ satisfy

$$\int_0^\infty |x'(t) - y'(t)|\,\mathrm{d}t \leqslant M|x_0 - y_0|. \tag{3.1}$$

Proof. Let $w \in \mathbf{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathbb{R}^N)$, let $x_0, y_0 \in K$ and $x = \mathcal{S}(w, x_0)$, $y = \mathcal{S}(w, y_0)$. For $\eta > 0$ define x_η, y_η, u_η, v_η by $u_\eta(0) = v_\eta(0) = 0$ and

$$x_\eta(t) = \begin{cases} t x_0 & \text{if } t \in [0,\eta], \\ \eta x(\frac{t}{\eta} - 1) & \text{if } t \geqslant \eta, \end{cases} \qquad y_\eta(t) = \begin{cases} t y_0 & \text{if } t \in [0,\eta], \\ \eta y(\frac{t}{\eta} - 1) & \text{if } t \geqslant \eta, \end{cases}$$
$$u'_\eta(t) = \begin{cases} x_0 & \text{if } t \in [0,\eta], \\ w'(\frac{t}{\eta} - 1) & \text{if } t \geqslant \eta, \end{cases} \qquad v'_\eta(t) = \begin{cases} y_0 & \text{if } t \in [0,\eta], \\ w'(\frac{t}{\eta} - 1) & \text{if } t \geqslant \eta. \end{cases}$$

For $t \leqslant \eta$ we have $x'_\eta(t) = x_0 = u_\eta(t)$. For $t \geqslant \eta$ we obtain

$$u'_\eta(t) - x'_\eta(t) = w'\Big(\frac{t}{\eta} - 1\Big) - x'\Big(\frac{t}{\eta} - 1\Big) \in N_Z\Big(x\Big(\frac{t}{\eta} - 1\Big)\Big) = N_Z(x_\eta(t)).$$

Here we have used that Z is a cone. Thus $x_\eta = \mathcal{S}(u_\eta, 0)$. Similarly, $y_\eta = \mathcal{S}(v_\eta, 0)$.

Now we fix some $S > 0$. Notice that for any $\eta > 0$,

$$\begin{aligned}
&\int_0^{\eta(S+1)} (|u_\eta'(t)| + |v_\eta'(t)|)\,\mathrm{d}t \\
&= \int_0^{\eta} (|x_0| + |y_0|)\,\mathrm{d}t + 2\int_\eta^{\eta(S+1)} \Big|w'\Big(\frac{t}{\eta} - 1\Big)\Big|\,\mathrm{d}t \\
&= \eta(|x_0| + |y_0|) + 2\eta\int_0^S |w'(s)|\,\mathrm{d}s.
\end{aligned}$$

Therefore we can pick η sufficiently small such that $\eta(S+1) \leqslant T$ and

$$\int_0^{\eta(S+1)} (|u'(t)| + |v'(t)|)\,\mathrm{d}t < R.$$

Then by assumption we have

$$\begin{aligned}
&\int_0^S |x'(s) - y'(s)|\,\mathrm{d}t = \frac{1}{\eta}\int_\eta^{\eta(S+1)} |x_\eta'(t) - y_\eta'(t)|\,\mathrm{d}t \\
&\leqslant \frac{M}{\eta}\int_0^T |u_\eta'(t) - v_\eta'(t)|\,\mathrm{d}t = \frac{M}{\eta}\int_0^\eta |x_0 - y_0|\,\mathrm{d}t \\
&= M|x_0 - y_0|.
\end{aligned}$$

As $S \to \infty$, we obtain (3.1). □

Now we give our counterexample.

E x a m p l e 3.1. Consider the cone

$$Z = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{R}^3 \mid \xi_3 \geqslant \sqrt{\xi_1^2 + \xi_2^2} \right\}.$$

Then for any $R > 0$, $M > 0$, and any $T > 0$, there are functions $u, v \in \mathbf{W}^{1,1}([0,T],\mathbb{R}^3)$ and $x = \mathcal{S}(u,0)$, $y = \mathcal{S}(v,0)$, with

$$\int_0^T (|u'(t)| + |v'(t)|)\,\mathrm{d}t \leqslant R$$

and

$$\int_0^T |x'(t) - y'(t)|\,\mathrm{d}t > M\int_0^T |u'(t) - v'(t)|\,\mathrm{d}t.$$

P r o o f. Assume the contrary. Then the assumptions for Lemma 3.1 are satisfied. We construct w, x and y in order to arrive at a contradiction to (3.1). We put

$$x(t) = \begin{pmatrix} (t+1)^{-1}\cos(t) \\ (t+1)^{-1}\sin(t) \\ (t+1)^{-1} \end{pmatrix}, \quad y(t) = 0,$$
$$w'(t) = \begin{pmatrix} (1-(t+1)^{-2})\cos(t) - (t+1)^{-1}\sin(t) \\ (1-(t+1)^{-2})\sin(t) + (t+1)^{-1}\cos(t) \\ -1-(t+1)^{-2} \end{pmatrix}, \quad w(0) = 0.$$

Thus

$$x'(t) = \begin{pmatrix} -(t+1)^{-2}\cos(t) - (t+1)^{-1}\sin(t) \\ -(t+1)^{-2}\sin(t) + (t+1)^{-1}\cos(t) \\ -(t+1)^{-2} \end{pmatrix}.$$

The normal cone at zero is given by

$$N_Z(0) = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \mathbb{R}^3 \mid -\xi_3 \geqslant \sqrt{\xi_1^2 + \xi_2^2} \right\}.$$

A straightforward computation shows that $w'(t) \in N_Z(0)$ for all t, thus $\mathcal{S}(w, 0) = 0 = y$. At the other points of ∂Z, the normal cone is given by

$$N_Z\left(\begin{pmatrix} \gamma\cos(t) \\ \gamma\sin(t) \\ \gamma \end{pmatrix}\right) = \left\{ \lambda \begin{pmatrix} \cos t \\ \sin t \\ -1 \end{pmatrix} \mid \lambda \geqslant 0 \right\}.$$

Thus

$$w'(t) - x'(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ -1 \end{pmatrix} \in N_Z(x(t)).$$

Thus $x = \mathcal{S}(w, x(0))$. From (3.1) one infers

$$\int_0^\infty |x'(t)| \, \mathrm{d}t = \int_0^\infty |x'(t) - y'(t)| \, \mathrm{d}t \leqslant M|x(0)|.$$

However,

$$|x'(t)| = \sqrt{2(t+1)^{-4} + (t+1)^{-2}} \geqslant (t+1)^{-1},$$

so that x' is not integrable on $[0, \infty)$. □

R e m a r k 3.1. Although Example 3.1 shows an unbounded convex set, a careful analysis of the proof shows that also a truncated cone provides a counterexample.

The following example shows that the stop operator is not necessarily locally Lipschitz continuous if the characteristic is a domain of type $\mathcal{C}^1$, i.e., the normal vector $n(x)$ in each boundary point $x \in \partial Z$ is unique and depends continuously on x. In fact, the normal vector in the following counterexample depends Hölder continuously on x.

E x a m p l e 3.2. Let

$$Z = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{R}^2 \mid \xi_2 \geqslant \beta(|\xi_1|) \right\}$$

with

$$\beta(\xi) = \int_0^x \gamma(\tau)\,\mathrm{d}\tau, \quad \gamma(\tau) = \sqrt{\frac{\tau}{\tau+2}}.$$

Then for all $R > 0$ and $M > 0$ there exist $x_0, y_0 \in Z$, $T > 0$, $u \in \mathbf{W}^{1,1}([0.T], \mathbb{R}^2)$, $x = \mathcal{S}(x_0, u)$, $y = \mathcal{S}(y_0, u)$ with $|x_0| \leqslant R$, $|y_0| \leqslant R$,

$$\int_0^T |u'(t)|\,\mathrm{d}t \leqslant R \quad \text{and} \quad \int_0^T |x'(t) - y'(t)|\,\mathrm{d}t \geqslant M|x_0 - y_0|. \tag{3.2}$$

P r o o f. Notice that Hypothesis 1.1 holds everywhere except at the origin. To exploit the singularity at the origin we will construct a forcing function u and solutions

$$x(t) = \begin{pmatrix} \xi(t) \\ \beta(|\xi(t)|) \end{pmatrix} \in \partial Z, \quad y(t) = \begin{pmatrix} \eta(t) \\ \beta(\eta(t)) \end{pmatrix} \in \partial Z,$$

such that $\xi \leqslant 0$ and $\eta \geqslant 0$ oscillate in a neighborhood of the origin. More precisely, we construct sequences $0 = t_0 < t_1 < t_2 \ldots$ and $q_0 > q_1 > q_2 > \ldots > 0$ with

$$\xi(t_i) = \begin{cases} -q_i & \text{for even } i, \\ 0 & \text{for odd } i, \end{cases} \quad \text{and } \eta(t_i) = \begin{cases} 0 & \text{for even } i, \\ q_i & \text{for odd } i, \end{cases} \tag{3.3}$$

$$q_i \geqslant \frac{q_0}{1 + iq_0}, \tag{3.4}$$

$$\int_{t_{i-1}}^{t_i} |u'(t)|\,\mathrm{d}t \leqslant q_{i-1}\sqrt{2} \leqslant q_0\sqrt{2}, \tag{3.5}$$

$$\int_{t_{i-1}}^{t_i} |x'(t) - y'(t)|\,\mathrm{d}t \geqslant \frac{\sqrt{2}}{3\sqrt{3}}\, q_{i-1}^{3/2}. \tag{3.6}$$

We will show later that this construction ensures that the solutions satisfy (3.2).

With

$$K = \frac{2\sqrt{2}}{3\sqrt{3}} \left(1 - \frac{1}{\sqrt{1 + R/\sqrt{8}}} \right)$$

we choose $q_0 > 0$ sufficiently small such that

$$q_0 < \min\Big(\Big\{1, \frac{K^2}{4M^2}, \frac{R}{\sqrt{8}}\Big\}\Big) \text{ and } \sqrt{q_0^2 + \beta(q_0)^2} < 2q_0.$$

We put $t_0 = 0$, $x_0 = (-q_0, \beta(q_0))^T$, $y_0 = (0,0)^T$ and proceed by induction. Suppose sequences t_i and q_i and a forcing function $u \in \mathbf{W}^{1,1}([0,t_n], \mathbb{R}^N)$ have been established such that the conditions (3.3), (3.4), (3.5) and (3.6) are satisfied up to t_n. Without loss of generality we assume that n is even. The other case is treated similarly with the roles of x and y interchanged. We put $t_{n+1} = t_n + q_n$ and continue the forcing function u on the interval $[t_n, t_{n+1}]$ by

$$u'(t) = \begin{pmatrix} 1 \\ -\gamma(q_n - t_n + t) \end{pmatrix}.$$

Put $\xi(t) = -q_n + t - t_n$. Obviously $x = (\xi(t), \beta(|\xi(t)|)^T$ satisfies $x' = u'$, so that $x = \mathcal{S}(x_0, u)$. In particular $\xi(t_{n+1}) = 0$. We obtain $y(t)$ by

$$y'(t) = \alpha(t) \begin{pmatrix} 1 \\ \gamma(\eta(t)) \end{pmatrix}$$

with

$$\alpha(t) = \frac{1 - \gamma(|\xi(t)|)\gamma(\eta(t))}{1 + \gamma^2(\eta(t))}.$$

Consider the outward unit normal vector $n(y(t))$ to ∂Z given by

$$n(y(t)) = \frac{1}{\sqrt{1 + \gamma^2(\eta(t))}} \begin{pmatrix} \gamma(\eta(t)) \\ -1 \end{pmatrix}$$

and let

$$\lambda(t) = \frac{\gamma(|\xi(t)|) + \gamma(\eta(t))}{\sqrt{1 + \gamma^2(\eta(t))}} \geqslant 0.$$

A straightforward computation shows that $y'(t) + \lambda(t)n(y(t)) = u'(t)$ so that $y = \mathcal{S}(y_0, u)$.

Since $0 \leqslant \alpha(t) \leqslant 1$ we infer that $\eta(t) \leqslant q_n$ for $t \in [t_n, t_{n+1}]$. A more careful estimate shows now that

$$\alpha(t) \geqslant \frac{1 - \gamma^2(q_n)}{1 + \gamma^2(q_n)} = \frac{1 - \frac{q_n}{q_n+2}}{1 + \frac{q_n}{q_n+2}} = \frac{1}{q_n + 1}.$$

We put $q_{n+1} = \eta(t_n)$ and obtain

$$q_{n+1} \geqslant (t_{n+1} - t_n) \min_{t\in[t_n,t_{n+1}]} (\alpha(t)) \geqslant \frac{q_n}{q_n + 1} \geqslant \frac{q_0}{1 + (n+1)q_0}.$$

Using the inequalities $q_0 \leqslant 1$ and $\gamma(\tau) \leqslant 1$ we obtain

$$\int_{t_n}^{t_{n+1}} |u'(t)| \,\mathrm{d}t = \int_{t_n}^{t_{n+1}} \sqrt{1 + \gamma^2(|\xi(t)|)} \,\mathrm{d}t \leqslant q_n\sqrt{2} \leqslant q_0\sqrt{2},$$

and

$$\begin{aligned}\int_{t_n}^{t_{n+1}} |x'(t) - y'(t)| \,\mathrm{d}t &= \int_{t_n}^{t_{n+1}} \lambda(t) \,\mathrm{d}t \geqslant \int_{t_n}^{t_{n+1}} \frac{\gamma(|\xi(t)|)}{\sqrt{2}} \,\mathrm{d}t \\ &= \frac{1}{\sqrt{2}} \int_{t_n}^{t_{n+1}} \sqrt{\frac{t_{n+1} - t}{t_{n+1} - t + 2}} = \frac{1}{\sqrt{2}} \int_0^{q_n} \sqrt{\frac{s}{s+2}} \,\mathrm{d}s \\ &\geqslant \frac{1}{\sqrt{6}} \int_0^{q_n} \sqrt{s} \,\mathrm{d}s = \frac{\sqrt{2}}{3\sqrt{3}} q_n^{3/2}.\end{aligned}$$

At this point the inductive construction is complete.

We choose now an integer n such that $nq_0\sqrt{2} \leqslant R < (n+1)q_0\sqrt{2}$. Since $q_0 \leqslant R/\sqrt{8}$ this implies $R/\sqrt{8} \leqslant nq_0 \leqslant R/\sqrt{2}$. From (3.5) we infer immediately

$$\int_0^{t_n} |u'(t)| \,\mathrm{d}t \leqslant R.$$

From (3.4) and (3.6) we infer now

$$\begin{aligned}\int_0^{t_n} |x'(t) - y'(t)| \,\mathrm{d}t &\geqslant \frac{\sqrt{2}}{3\sqrt{3}} \sum_{i=0}^{n-1} \Big(\frac{q_0}{1 + iq_0}\Big)^{3/2} \\ &\geqslant \frac{\sqrt{2}}{3\sqrt{3}} \int_0^n \Big(\frac{q_0}{1 + sq_0}\Big)^{3/2} \,\mathrm{d}s = q_0^{1/2} \frac{2\sqrt{2}}{3\sqrt{3}} \big(1 - (1 + nq_0)^{-1/2}\big) \\ &= q_0^{1/2} \frac{2\sqrt{2}}{3\sqrt{3}} \left(1 - \frac{1}{\sqrt{1 + R/\sqrt{8}}}\right) = Kq_0^{1/2} \\ &\geqslant 2Mq_0 \geqslant M|x_0 - y_0|.\end{aligned}$$

$\square$

R e m a r k 3.2. Again the domain in Example 3.2 can be modified to be bounded.

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