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HOW THE DESIGN OF AN EXPERIMENT INFLUENCES THE NONSENSITIVENESS REGIONS IN MODELS WITH VARIANCE COMPONENTS

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Abstract. Nonsensitiveness regions for estimators of linear functions, for confidence ellipsoids, for the level of a test of a linear hypothesis on parameters and for the value of the power function are investigated in a linear model with variance components.

The influence of the design of an experiment on the nonsensitiveness regions mentioned is numerically demonstrated and discussed on an example.

Keywords: mixed linear model, model with variance components, nonsensitiveness region

MSC 2000: 62J05

Introduction

Experiments in natural science, i.e. physics, chemistry, biology, etc., are provided by measurement equipments characterized, among other, by parameters of the accuracy. It is well known that the efficiency of estimators of useful parameters (they represent the aim of the experiment) heavily depends on values of the accuracy parameters. If their true values are not used in calculation, then estimators of the useful parameters are worse in comparison with estimators using true values; this fact can produce the greater loss in the investment on experiments the greater is the investment itself.

Thus it seems to be important to investigate boundaries of regions which cover such values of deviations of the accuracy parameters from the true values which

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cannot destroy essentially the quality of estimators and other statistical inferences connected with the useful parameters.

The aim of the paper is to give some rules for determining such boundaries in different situations and to present some experience obtained in an application of these rules. Papers [1], [3], [4], [5] and [6] represent the starting point of the contribution.

1. Notation and definitions

Let Y be an n-dimensional random vector and $\mathscr{F} = \{F(\cdot,\beta,\vartheta)\colon\ \beta \in \mathbb{R}^k, \vartheta \in \underline{\vartheta}\}$ a class of distribution functions assigned to Y. Here β is a k-dimensional useful parameter (the first order parameter; the aim of the experiment is to determine it), $\mathbb{R}^k$ is a k-dimensional real vector space, ϑ is a p-dimensional accuracy parameter (its components $\vartheta_1,\dots,\vartheta_p$ are variance components—the second order parameters) and $\underline{\vartheta}$ is an open set in the Euclidean topology of $\mathbb{R}^p$.

The class $\mathscr{F}$ satisfies two conditions:

$$\forall\{\beta \in \mathbb{R}^k, \vartheta \in \underline{\vartheta}\} \int_{\mathbb{R}^n} u \,\mathrm{d}F(u,\beta,\vartheta) = X\beta = \mathrm{E}(Y)$$

(here X is a known $n \times k$ matrix) and

$$\forall\{\beta \in \mathbb{R}^k, \vartheta \in \underline{\vartheta}\} \int_{\mathbb{R}^n} (u - X\beta)(u - X\beta)' \,\mathrm{d}F(u,\beta,\vartheta) = \sum_{i=1}^{p} \vartheta_i V_i = \Sigma(\vartheta)$$

(here $V_1,\dots,V_p$, are known $n \times n$ symmetric matrices).

The model described is written either in the form

$$\Big(Y, X\beta, \Sigma(\vartheta)\Big),\ \beta \in \mathbb{R}^k,\ \vartheta \in \underline{\vartheta},$$

or in the form

$$Y \sim \Big(X\beta, \Sigma(\vartheta)\Big),\ \beta \in \mathbb{R}^k,\ \vartheta \in \underline{\vartheta}.$$

For the sake of simplicity in the following the regular version (cf. [2]) of this model will be considered, i.e. $r(X) = k < n\ \forall\{\vartheta \in \underline{\vartheta}\}\ \Sigma(\vartheta)$ is positive definite ($r(X)$ denotes the rank of the matrix X).

The ϑ-LBLUE (locally best linear unbiased estimator) of β is

$$\hat{\beta}(Y,\vartheta) = [X'\Sigma^{-1}(\vartheta)X]^{-1}X'\Sigma^{-1}(\vartheta)Y$$

(cf. [7]). The quality of this estimator is naturally influenced by the chosen design of the experiment. To investigate its affect is the aim of the numerical example 2.8.

Definition 1.1. The random variable

$$\partial\hat{\beta}_i(Y,\vartheta)/\partial\vartheta_j,\ i=1,\ldots,k;\ j=1,\ldots,p,$$

is called the *sensitivity of the ith component $\hat{\beta}_i(Y,\vartheta)$ of the estimator $\hat{\beta}(Y,\vartheta)$ on the jth component ϑ_j of ϑ.*

Let $g(\beta)=g'\beta$, $g\in\mathbb{R}^k$, $\beta\in\mathbb{R}^k$, be a given linear function of β and let ϑ^* be the true value of ϑ. The $(\vartheta^*+\delta\vartheta)$-LBLUE of $g(\cdot)$ may be approximated in the following way:

$$g'\hat{\beta}(Y,\vartheta^*+\delta\vartheta)\doteq g'\hat{\beta}(Y,\vartheta^*)+g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta.$$

Thus the ϑ^*-LBLUE of $g(\cdot)$

$$g'\hat{\beta}(Y,\vartheta^*)=g'[X'\Sigma^{-1}(\vartheta^*)X]^{-1}X'\Sigma^{-1}(\vartheta^*)Y,$$

with the well known property

$$\forall\{\vartheta\neq\vartheta^*\}\ \mathrm{Var}[g'\hat{\beta}(Y,\vartheta^*)|\vartheta^*]\leqslant\mathrm{Var}[g'\hat{\beta}(Y,\vartheta)|\vartheta^*],$$

is depreciated by the random variable

$$g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta. \tag{1.1}$$

It can be proved (cf. Statement 2.1 (iii)) that the random variables

$$g'\hat{\beta}(Y,\vartheta^*)\quad\text{and}\quad g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta$$

are uncorrelated and therefore

$$\mathrm{Var}[g'\hat{\beta}(Y,\vartheta^*+\delta\vartheta)|\vartheta^*]\doteq\mathrm{Var}[g'\hat{\beta}(Y,\vartheta^*)|\vartheta^*]+\mathrm{Var}\left[g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta|\vartheta^*\right]; \tag{1.2}$$

in the following we denote

$$\sigma_g^2=\mathrm{Var}[g'\hat{\beta}(Y,\vartheta^*)|\vartheta^*]=g'[X'\Sigma^{-1}(\vartheta^*)X]^{-1}g.$$

Definition 1.2. The set

$$\left\{\delta\vartheta\colon\ \sqrt{\mathrm{Var}\left[g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta\Big|\vartheta^*\right]}\Big/\sigma_g\leqslant\varepsilon_g\right\},$$

where ε_g is a positive number chosen by the user, is called *the relative (with respect to σ_g) region of nonsensitiveness of the estimator of $g(\cdot)$ with respect to the second order parameters.*

The choice of the number ε_g depends on two factors: on the user's requirement expressed by the σ_g concerning the expected accuracy of the result, and on the actual accuracy of the experiment expressed by

$$\operatorname{Var}\left(g'\hat{\beta}(Y,\vartheta^*) + \frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta \middle| \vartheta^*\right).$$

Let, e.g., σ_g be much smaller number than the user requires, then the number ε_g in the relation

$$\sqrt{\sigma_g^2 + \operatorname{Var}\left(\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta \middle| \vartheta^*\right)} \leqslant \sigma_g\sqrt{1+\varepsilon_g^2}$$

may be large. If, on the other hand, σ_g is the maximum value permitted by the user, then ε_g must be zero.

Remark 1.3. The region of nonsensitiveness can be defined in another way, e.g. as the set

$$\left\{\delta\vartheta\colon \sqrt{\operatorname{Var}\left[g'\frac{\partial\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta \middle| \vartheta^*\right]} \leqslant \kappa_g\right\}$$

(an absolute region of nonsensitiveness), etc.

Regions of nonsensitiveness can be defined for another statistical inference, e.g. for a confidence ellipsoid and for a test of a linear hypothesis on β.

2. Estimator of the function $g(\cdot)$

In the following the relative regions of nonsensitiveness (see Definition 1.2) are taken into account only. Let us consider the regular model $Y \sim \big(X\beta, \Sigma(\vartheta)\big)$, $\beta \in \mathbb{R}^k$, $\vartheta \in \underline{\vartheta}$, and a known function $g(\beta) = g'\beta$, $\beta \in \mathbb{R}^k$, of β. Let ϑ^* be the true value of ϑ.

The statements of this section are proved in [5].

Statement 2.1. *Let*

$$v(Y,\vartheta^*) = Y - X\hat{\beta}(Y,\vartheta^*)$$

and

$$L_g' = g'[X'\Sigma^{-1}(\vartheta^*)X]^{-1}X'\Sigma^{-1}(\vartheta^*);$$

then

(i)

$$\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta = -L_g'\sum_{i=1}^{p} V_i\delta\vartheta_i\Sigma^{-1}(\vartheta^*)v(Y,\vartheta^*),$$

(ii)

$$\forall\{\beta\in\mathbb{R}^k,\ \vartheta\in\underline{\vartheta}\}\ \mathrm{E}\left[\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta\middle|\beta,\vartheta\right] = 0,$$

(iii)

$$\forall\{\beta\in\mathbb{R}^k\}\ \mathrm{cov}\left[\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta, \hat{\beta}(Y,\vartheta^*)\middle|\vartheta^*\right] = 0.$$

Statement 2.2.

$$\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta} \sim (0, W_g),$$

where

$$\{W_g\}_{i,j} = L_g'V_i[M_X\Sigma(\vartheta^*)M_X]^+V_jL_g,$$

$$[M_X\Sigma(\vartheta^*)M_X]^+ = \Sigma^{-1}(\vartheta^*) - \Sigma^{-1}(\vartheta^*)X[X'\Sigma^{-1}(\vartheta^*)X]^{-1}X'\Sigma^{-1}(\vartheta^*)$$

and $M_X = I - XX^+$.

Corollary 2.3.

$$\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta \sim (0, \delta\vartheta'W_g\delta\vartheta).$$

Statement 2.4. *On the basis of Definition* 1.2, *Statements* 2.1, 2.2 *and Corollary* 2.3, *the nonsensitiveness region of the estimator of the function* $g(\cdot)$ *with respect to the second order parameters can be written in the form*

$$\{\delta\vartheta\colon\ \delta\vartheta'W_g\delta\vartheta \leqslant \varepsilon_g^2\sigma_g^2\}. \tag{2.1}$$

If $\delta\vartheta$ *lies in this region, then*

$$\sqrt{\mathrm{Var}\left[\frac{\partial g'\hat{\beta}(Y,\vartheta^*)}{\partial\vartheta'}\delta\vartheta\middle|\vartheta^*\right]} \leqslant \varepsilon_g\sigma_g.$$

Corollary 2.5. *Let* $W_g = \sum_{i=1}^{s}\lambda_i f_i f_i'$ *be a spectral decomposition of the matrix* W_g, *and let* $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$. *Then the region*

$$\left\{\delta\vartheta\colon\ \|\delta\vartheta\| \leqslant \varepsilon_g\frac{\sigma_g}{\sqrt{\lambda_1}}\right\} \tag{2.2}$$

is included in the region (2.1).

The region (2.2) seems to be more suitable for practical purposes.

Corollary 2.6. *The critical direction of the change of the parameter ϑ on the estimator of the function $g'\beta$ is determined by the eigenvector f_1 belonging to the maximum eigenvalue of the matrix W_g. In this direction the change of the dispersion of the estimator of the function $g'\beta$ of the first order parameter β caused by a change $\delta\vartheta$ of the second order parameter is maximum. Thus $\delta\vartheta_{\text{crit}} = cf_1$. If $\delta\sigma_g$ means the maximum torelable value of the change of the standard deviation in the estimator of $g'\beta$, the constant c is a solution of the definition relation of $\delta\sigma_g$, i.e. of the equation*

$$\delta\sigma_g = \sqrt{\operatorname{Var}[g'\hat{\beta}_1(Y, \vartheta^* + cf_1)|\vartheta^*]} - \sqrt{\operatorname{Var}[g'\hat{\beta}_1(Y, \vartheta^*)|\vartheta^*]}.$$

Thus for c we obtain the formula

$$c = \sqrt{\left[\left(\delta\sigma_g + \sqrt{g'C^{-1}g}\right)^2 - g'C^{-1}g\right]/\lambda_1}; \tag{2.3}$$

it suffices to realize that

$$\begin{aligned}\operatorname{Var}[g'\hat{\beta}_1(Y, \vartheta^*)|\vartheta^* + cf_1] &= \operatorname{Var}[g'\beta_1(Y, \vartheta^*)|\vartheta^*] + cf_1'W_g cf_1 \\ &= g'C^{-1}g + cf_1'\left(\lambda_1 f_1 f_1' + \sum_{i=2}^{s} \lambda_i f_i f_i'\right)cf_1 \\ &= g'C^{-1}g + c^2\lambda_1.\end{aligned}$$

Corollary 2.7. *The tolerable shift $\delta\sigma_i$ in the standard deviation $\sigma_i = \sqrt{\vartheta_i}$, $i = 1, \ldots, p$ in dependence on the torelable shift $\delta\vartheta_i$ is*

$$\delta\sigma_i = \sqrt{\vartheta^* + \delta\vartheta_i} - \sigma_i^*; \tag{2.4}$$

vice versa, $\delta\vartheta_i = 2\sigma_i^*\delta\sigma_i + (\delta\sigma_i)^2$, $i = 1, \ldots, p$, thus $\delta\vartheta$ is tolerable iff $\delta\sigma = (\delta\sigma_1, \ldots, \delta\sigma_p)'$ is tolerable.

E x a m p l e 2.8. Let us consider a linear relation $y = \beta_1 + \beta_2 x$. In order to determine the best estimators of the parameters β_1 and β_2 two devices with dispersions $\vartheta_1^* = \sigma_1^2 = 0.001^2$ and $\vartheta_2^* = \sigma_2^2 = 0.004^2$, respectively, are at our disposal. The measurement of y may be performed at different points x of the set of observation points $S = \{x_1, \ldots, x_n\}$, when simultaneously each of the devices may be used just l

times (the result of a measurement is a $2l$-dimensional realization of the random vector Y that models the measurement). In the situation described there exist $\binom{n}{l}\binom{n-l}{l}$ different designs. The aim of the example is to illustrate how the choice of the design of the experiment influences the regions of nonsensitivity of the parameters β_1 and β_2.

In the following the abbreviate notation will be used for recording S: Let A be the minimum value and $B = A + (n-1)\Delta$ the maximum value of the observation point, Δ the difference between two sequential points. Then $S\{A(\Delta)B\}$ denotes the observation set $S = \{A, A+\Delta, A+2\Delta, \dots, B\}$ (which contains n points).

Let $l = 2$ (each of the devices may be used just twice), let the first device ($\sigma_1^2 = 0.001^2$) be used in first two experiments, which means that

$$\Sigma(\vartheta^*) = 10^{-6} \times \begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 16, & 0 \\ 0, & 0, & 0, & 16 \end{pmatrix},$$

and let $S = \{-10(5)10\}$, which means that $n = 5$ and therefore there exist 30 different designs.

In the following the dependence on the chosen design will be recorded by the left upper index k. This dependence occurs in the observation vector Y, the design matrix X, the eigenvalue λ and in the standard deviation σ_g. $\Sigma(\vartheta^*)$ does not depend on it if the experiment is performed in such a way that two first measurements are realized by the device whose accuracy is characterized by $\sigma_1^2 = 0.001^2$ and the second two by the device with $\sigma_2^2 = 0.004^2$.

The study is performed in two steps.

The first step consists in studying the relative and absolute regions of nonsensitiveness of parameters ϑ and σ of the parameter β_1 (i.e. for $g = e_1 = (1,0)'$) estimated on the basis of two extremum designs $k = 1$ and $k = 2$ (by $k = 1$ the design with the maximum eigenvalue ${}^1\lambda_1$ and by $k = 2$ the design with the minimum eigenvalue ${}^2\lambda_1$ is denoted.

Table 2.1a contains results concerning the relative nonsensitiveness regions of the parameter β_1.

	$\sigma_1^2 = 0.001^2$		$\sigma_2^2 = 0.004^2$				
k	x_1	x_2	x_3	x_4	λ_1	$\sigma_{e_1}^2$	$\sigma_{e_1}/\sqrt{\lambda_1}$
1	5	10	-5	0	584 946	2.3779×10^{-6}	2.016×10^{-6}
2	-5	5	0	10	22 486	0.4721×10^{-6}	4.582×10^{-6}

Table 2.1a: The relative nonsensitiveness regions for $g(\beta) = e_1'\beta = \beta_1$

In accordance with (2.2) the ratio $4.582/2.016 = 2.273$ shows that the design $k = 2$ with the minimum eigenvalue of the matrix W_g tolerates in the relative sense (with respect to σ_{e_1}) 2.273 times larger change in the parameter ϑ than the design $k = 1$ with the maximum eigenvalue. For the sake of completeness let us note that ${}^2\sigma_{e_1}/{}^1\sigma_{e_1} = 0.446$ and the ratio ${}^2\lambda_1/{}^1\lambda_1 = 0.038$. In other words, the change $\delta\vartheta$ of the parameter ϑ^* within the design $k = 1$ permits the change $2.237\delta\vartheta$ in the parameter ϑ^* within the design $k = 2$, which leads to the same value of the ratio

$$\sqrt{\operatorname{Var}\left[\frac{\partial e_1'\hat{\beta}({}^kY, \vartheta^*)}{\partial \vartheta'}\delta\vartheta\bigg|\vartheta^*\right]} \bigg/ {}^k\sigma_{e_1}$$

in the both designs mentioned.

The results concerning the absolute tolerable changes in ϑ and σ for β_1 under the condition that $\delta\sigma_{e_1\,\text{toler}} = {}^2\sigma_{e_1} = \sqrt{0.4721} \times 10^{-3} = 0.687 \times 10^{-3}$ (see Table 2.1a) are given in Table 2.1b.

The eigenvector f_1 belonging to the eigenvalues ${}^1\lambda_1$ and ${}^2\lambda_1$ is $f_1 = (16/\sqrt{257}, -1/\sqrt{257})'$ and it is the same for both the designs. The eigenvalues ${}^1\lambda_2$ and ${}^2\lambda_2$ are zero.

k	λ_1	$\sigma_{e_1} \times 10^3$	$c \times 10^6$	$\delta\vartheta_{\text{crit}} \times 10^6$	$\delta\sigma_{\text{crit}} \times 10^3$
1	584 946	1.542	2.104	$(2.100, -0.131)'$	$(0.761, -0.016)'$
2	22 486	0.687	7.935	$(7.920, -0.495)'$	$(1.987, 0.000)'$

Table 2.1b: The absolute nonsensitiveness regions for $g(\beta) = e_1'\beta = \beta_1$

The relations (2.3) and (2.4) were applied in Table 2.1b.

The ratio ${}^2c/{}^1c = 7.935/2.104 = 3.771$ expresses that the design $k = 2$ permits 3.771 times larger change in the parameter ϑ than the design $k = 1$ for saving the same quality of the estimator of the parameter β_1. Analogous conclusions concern the change in σ.

Table 2.2a for the function $g(\beta) = e_2'\beta = \beta_2$ is analogous to Table 2.1a. By $k = 3$ and $k = 4$ the design with the maximum and minimum eigenvalue, respectively, is denoted.

	$\sigma_1^2 = 0.001^2$		$\sigma_2^2 = 0.004^2$		λ_1	$\sigma_{e_2}^2$	$\sigma_{e_2}/\sqrt{\lambda_1}$
k	x_1	x_2	x_3	x_4			
3	0	5	−5	10	11 842	0.0512×10^{-6}	2.079×10^{-6}
4	−10	10	0	5	35.14908	0.0050×10^{-6}	11.882×10^{-6}

Table 2.2a: The relative nonsensitiveness regions for $g(\beta) = e_2'\beta = \beta_2$

The conclusions are analogous to the previous case.

As far as Table 2.2b is concerned, the eigenvector belonging to the eigenvalues ${}^3\lambda_1$ and ${}^4\lambda_1$ is $(16/\sqrt{257}, -1/\sqrt{257})'$ and the tolerable shift is $\delta\sigma_{e_2\ toler} = {}^4\sigma_{e_2} = 0.070 \times 10^{-3}$.

k	λ_1	$\sigma_{e_2} \times 10^3$	$c \times 10^6$	$\delta\vartheta_{\text{crit}} \times 10^6$	$\delta\sigma_{\text{crit}} \times 10^3$
3	11 842	0.226	1.760	$(1.757, -0.110)'$	$(0.660, -0.014)'$
4	35.149	0.070	20.450	$(20.410, -1.276)'$	$(3.627, -0.163)'$

Table 2.2b: The absolute nonsensitiveness regions for $g(\beta) = e_2'\beta = \beta_2$

Here the ratio is ${}^2c/{}^1c = 11.617$.

Since the second design is much better for determining β_1 than the first one, the shift $\delta\vartheta$ is different in different designs if we require the equality

$$\delta^1\sigma_g = \delta^2\sigma_g,$$

i.e. the equality

$$\sqrt{\text{Var}[\hat{\beta}_1({}^1Y, \vartheta^* + \delta^1\vartheta)|\vartheta^*]} - \sqrt{\text{Var}[\hat{\beta}_1({}^1Y, \vartheta^*)|\vartheta^*]}$$
$$= \sqrt{\text{Var}[\hat{\beta}_1({}^2Y, \vartheta^* + \delta^2\vartheta)|\vartheta^*]} - \sqrt{\text{Var}[\hat{\beta}_1({}^2Y, \vartheta^*)|\vartheta^*]}$$

to be fulfilled.

In both designs the tolerable inaccuracies in the parameters σ_1 and σ_2 are essentially larger at the more precise device.

Also here essential differences, regarding the values σ_g^2 and λ_1, among designs can be recognized.

In order to document in more detail the above mentioned facts, a survey of the values ${}^k\sigma_g/\sqrt{{}^k\lambda_1}$ for different E and different designs is given.

(i) The case of four measurements:

$$S = \{-10(1)10\}$$

number of designs 35,910

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	68.86×10^{-6}	1.996×10^{-6}
$g(\beta) = \beta_2$	75.86×10^{-6}	1.996×10^{-6}

$$S = \{1, 4, 6, 9\} \text{ or } S = \{2, 8, 12, 18\}$$

number of designs 6

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	6.541×10^{-6}	2.016×10^{-6}
$g(\beta) = \beta_2$	16.031×10^{-6}	1.996×10^{-6}

$$S = \{2,5,7,10\}$$

number of designs 6

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	7.221×10^{-6}	2.023×10^{-6}
$g(\beta) = \beta_2$	16.031×10^{-6}	1.996×10^{-6}

$$S = \{0,3,5,8\} \text{ or } S = \{0,6,10,16\}$$

number of designs 6

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	6.135×10^{-6}	2.010×10^{-6}
$g(\beta) = \beta_2$	16.031×10^{-6}	1.996×10^{-6}

$$S = \{-7,-1,3,9\}$$

number of designs 6

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	4.274×10^{-6}	2.074×10^{-6}
$g(\beta) = \beta_2$	16.031×10^{-6}	1.996×10^{-6}

$$S = \{1(1)20\}$$

number of designs 29,070

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	90.579×10^{-6}	1.996×10^{-6}
$g(\beta) = \beta_2$	75.865×10^{-6}	1.996×10^{-6}

$$S = \{10(1)29\}$$

number of designs 29,070

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$g(\beta) = \beta_1$	73.716×10^{-6}	1.996×10^{-6}
$g(\beta) = \beta_2$	75.865×10^{-6}	1.996×10^{-6}

(ii) The case of six measurements: here the measurement may be performed at different points x when simultaneously each of the devices may be used just three times (the result of a measurement is a 6-dimensional realization of the random vector Y which models the measurement). The number of different designs is $\binom{n}{3}\binom{n-3}{3}$.

$$S = \{1(1)7\}$$

number of designs 140

	$(\sigma_g/\sqrt{\lambda_1})_{\max}$	$(\sigma_g/\sqrt{\lambda_1})_{\min}$
$h(\beta) = \beta_1$	14.877×10^{-6}	1.998×10^{-6}
$h(\beta) = \beta_2$	10.390×10^{-6}	2.031×10^{-6}

Example 2.8 shows quite clearly that the preparation of a measurement when only the approximate values of variance components are at our disposal requires a thorough preliminary analysis.

3. Confidence ellipsoid of a vector function of β

The problem is to determine for a given confidence level $1-\alpha$ and for a chosen probability ε (this expresses the maximum acceptable reduction of the confidence level $1-\alpha$ reflecting the fact that the actual value ϑ^* of the second order parameter for estimating the parameter β is not known precisely) the region $\mathscr{K}_\varepsilon$ of those changes of the parameter ϑ that do not cause larger decrease of the confidence level than ε. Within an example the dependence of the region $\mathscr{K}_\varepsilon$ on the chosen design of the experiment is shown.

Let $Y \sim N_n(X\beta, \Sigma(\vartheta))$ and let G be a given $s\times k$ matrix, such that $r(G)=s<k$. Denote by $k_G(Y,\vartheta)$ the random variable

$$k_G(Y,\vartheta) = [\beta^* - \hat{\beta}(Y,\vartheta)]'G'\Big(G[X'\Sigma^{-1}(\vartheta)X]^{-1}G'\Big)^{-1}G[\beta^* - \hat{\beta}(Y,\vartheta)],$$

where $\vartheta = \vartheta^* + \delta\vartheta$ and β^*, ϑ^* are the actual values of the parameters β and ϑ, respectively. Obviously $k_G(Y,\vartheta^*) \sim \chi^2_s(0)$ (a random variable possessing the central chi-square distribution with s degrees of freedom). Let $\chi^2_s(0;1-\alpha)$ be the $(1-\alpha)$-quantile of this distribution.

The set

$$\Big\{u\colon\ [u - G\hat{\beta}(Y,\vartheta^*)]'\Big(G[X'\Sigma^{-1}(\vartheta^*)X]^{-1}G'\Big)^{-1} \times [u - G\hat{\beta}(Y,\vartheta^*)] \leqslant \chi^2_s(0;1-\alpha)\Big\} \tag{3.1}$$

represents the $(1-\alpha)$-confidence ellipsoid of $G\beta$, $\beta\in\mathbb{R}^k$.

The statements given in the following are proved in [6].

Statement 3.1. *The change*

$$\delta k_G = \delta\vartheta'\partial k_G(Y,\vartheta^*)/\partial\vartheta$$

of k_G caused by the change $\delta\vartheta$ of the second order parameter is

$$\begin{aligned}\delta k_G = &-2[\hat{\beta}(Y,\vartheta^*) - \beta^*]'X'U_G\Sigma(\delta\vartheta)\Sigma^{-1}(\vartheta^*)v\\ &-[\hat{\beta}(Y,\vartheta^*) - \beta^*]'X'U_G\Sigma(\delta\vartheta)U_GX[\hat{\beta}(Y,\vartheta^*) - \beta^*],\end{aligned}$$

where

$$U_G = \Sigma^{-1}(\vartheta^*)XC^{-1}G'(GC^{-1}G')^{-1}GC^{-1}X'\Sigma^{-1}(\vartheta^*).$$

Its mean value and variance are

$$E(\delta k_G|\beta^*,\vartheta^*) = -\operatorname{Tr}[U_G\Sigma(\delta\vartheta)] = -\delta\vartheta'[\operatorname{Tr}(U_GV_1),\dots,\operatorname{Tr}(U_GV_p)]' \tag{3.2}$$

and

$$\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*) = \delta\vartheta'(2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+})\delta\vartheta, \tag{3.3}$$

respectively; here

$$\{S_{U_G}\}_{i,j} = \operatorname{Tr}(U_GV_iU_GV_j),\ i,j=1,\dots,p,$$

and

$$\{C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}\}_{i,j} = \operatorname{Tr}\Big(U_GV_i[M_X\Sigma(\vartheta^*)M_X]^+V_j\Big),\ i,j=1,\dots,p.$$

The basis for defining the nonsensitiveness region of the confidence ellipsoid is the Chebyshev inequality ([7], p. 75)

$$P\Big\{\big|\delta k_G - \mathrm{E}(\delta k_G|\beta^*,\vartheta^*)\big| \geqslant t\sqrt{\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*)}\Big\} \leqslant \frac{1}{t^2},\ t>0.$$

According to it, for a sufficiently large t, we can write

$$P\Big\{\big|\delta k_G - \mathrm{E}(\delta k_G|\beta^*,\vartheta^*)\big| \geqslant t\sqrt{\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*)}\Big\} \approx 0.$$

This implies the inequality

$$\delta k_G \leqslant \mathrm{E}(\delta k_G|\beta^*,\vartheta^*) + t\sqrt{\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*)}$$

with sufficiently high probability. For δk_G limited in this way we get

$$\begin{aligned}&P\{\chi_s^2(0) + \delta k_G \geqslant \chi_s^2(0;1-\alpha)\\ &\leqslant P\{\chi_s^2(0) + \mathrm{E}(\delta k_G|\beta^*,\vartheta^*) + t\sqrt{\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*)} \geqslant \chi_s^2(0;1-\alpha)\}.\end{aligned}$$

For a chosen probability ε only those δk_G caused by the change $\delta\vartheta$ of the parameter ϑ^* are acceptable that satisfy the inequality

$$\mathrm{E}(\delta k_G|\beta^*,\vartheta^*) + t\sqrt{\operatorname{Var}(\delta k_G|\beta^*,\vartheta^*)} < \delta_\varepsilon,$$

where

$$\delta_\varepsilon = \chi^2_s(0; 1-\alpha) - \chi^2_s(0; 1-\alpha-\varepsilon),$$

since only those satisfy the inequality

$$P\{\chi^2_s(0) + \delta k_G \geqslant \chi^2_s(0; 1-\alpha)\} \leqslant \alpha + \varepsilon.$$

Thus according to (3.2) and (3.3) the inequality

$$-[\mathrm{Tr}(U_G V_1), \ldots, \mathrm{Tr}(U_G V_p)]\delta\vartheta + t\sqrt{\delta\vartheta'(2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+})\delta\vartheta} \leqslant \delta_\varepsilon$$

determines those changes of the second order parameter that ensure the tolerable change of the confidence region given by ε.

Definition 3.2. The set

$$\mathscr{K}_\varepsilon = \{\delta\vartheta\colon\ -\delta\vartheta' a + t\sqrt{\delta\vartheta' A \delta\vartheta} \leqslant \delta_\varepsilon,\ \delta\vartheta \in \mathbb{R}^p,\ t > 0\} \tag{3.4}$$

is said to be the *nonsensitiveness region of the confidence ellipsoid* (3.1). Here $a = [\mathrm{Tr}(U_G V_1), \ldots, \mathrm{Tr}(U_G V_p)]'$, $A = 2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$ and t has to fulfil the relation

$$P\Big\{|\delta k_G - \mathrm{E}(\delta k_G|\beta^*, \vartheta^*)| \geqslant t\sqrt{\mathrm{Var}(\delta k_G|\beta^*, \vartheta^*)}\Big\} \approx 0 \tag{3.5}$$

and δ_ε is given by the relationship

$$P\{\chi^2_s(0) \geqslant \chi^2_s(0; 1-\alpha) - \delta_\varepsilon\} = \alpha + \varepsilon. \tag{3.6}$$

Remark 3.3. With respect to the Chebyshev inequality

$$\text{if}\quad t = 5,\quad \text{then}\quad P\{|\delta k_G - \mathrm{E}(\delta k_G|\beta^*, \vartheta^*)| \geqslant t\sqrt{\mathrm{Var}(\delta k_G|\beta^*, \vartheta^*)}\} \leqslant 0.04.$$

If δk_G is approximately normally distributed, then

$$\text{if}\quad t = 3,\quad \text{then}\quad P\{|\delta k_G - \mathrm{E}(\delta k_G|\beta^*, \vartheta^*)| \geqslant t\sqrt{\mathrm{Var}(\delta k_G|\beta^*, \vartheta^*)}\} \approx 0.003.$$

Thus it seems that the proper value of t lies in the interval $[3, 5]$.

This is the reason that enables us to apply the almost practical certainty in the Chebyshev inequality (see the preceding case).

Remark 3.4. Let $A = S_{U_G} + C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$. Then $\mathscr{M}(S_{U_G}) \subset \mathscr{M}(A)$. If $a = [\mathrm{Tr}(U_G V_1), \ldots, \mathrm{Tr}(U_G V_p)]'$, then it can be proved (cf. [6]) that $a \in \mathscr{M}(S_{U_G}) \subset \mathscr{M}(A)$ and the equation $(t^2 A - aa')x_0 = a\delta_\varepsilon$ (with respect to x_0) is consistent.

Statement 3.5. *Let*

$$A = 2S_{U_G} + 4C_{U_G,[M_X\Sigma(\vartheta^*)M_X]^+}$$

and

$$a = [\mathrm{Tr}(U_G V_1), \ldots, \mathrm{Tr}(U_G V_p)]'.$$

Then the boundary of the domain $\mathscr{K}_\varepsilon$*, see* (3.4)*, is*

$$\overline{\mathscr{K}}_\varepsilon = \left\{u\colon\ u \in \mathbb{R}^k, (u-u_0)'(t^2A - aa')(u-u_0) = \delta_\varepsilon^2 \frac{t^2}{t^2 - a'A^-a}\right\}, \tag{3.7}$$

where $u_0 = \delta_\varepsilon A^- a/(t^2 - a'A^-a)$*, and for* t *see* (3.5)*.*

Statement 3.6. *Let* β^* *and* ϑ^* *be the actual values of* β *and* ϑ*, respectively. Let* G *be an* $s \times k$ *matrix such that* $r(G) = s \leqslant k$*. Then*

$$\delta\vartheta \in \mathscr{K}_\varepsilon \quad \Rightarrow$$
$$P\Big\{\beta^* \in \Big\{u\colon\ [u - G\hat\beta(Y,\vartheta^*+\delta\vartheta)]'\Big(G[X'\Sigma^{-1}(\vartheta^*+\delta\vartheta)X]^{-1}G'\Big)^{-1}$$
$$\times[u - G\hat\beta(Y,\vartheta^*+\delta\vartheta)] \leqslant \chi_s^2(0;1-\alpha)\Big\}\Big\} \geqslant 1-\alpha-\varepsilon.$$

Example 3.7 (continuation of Example 2.8). The problem is to construct and to compare the nonsensitiveness regions $\overline{\mathscr{K}}_\varepsilon$ for designs described in Tables 2.1a and 2.2a. Figs 3.1–3.6 give graphic description of the nonsensitiveness regions $\overline{\mathscr{K}}_\varepsilon$ defined by (3.7) for the confidence level $1-\alpha = 0.95$, $\varepsilon = 0.05$ especially for β_1, where the designs characterized by the design matrices $X_1 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ 5, & 10, & -5, & 0 \end{pmatrix}'$ and $X_2 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ -5, & 5, & 0, & 10 \end{pmatrix}'$ described in Table 2.1a were considered and β_2, where the designs characterized by the design matrices $X_3 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ 0, & 5, & -5, & 10 \end{pmatrix}'$ and $X_4 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ -10, & 10, & 0, & 5 \end{pmatrix}'$ (see Table 2.2a) were considered. The solid lines in the first column correspond to the design $k = 1$, the dot-and-dash lines in this column correspond to the design $k = 2$. Analogously in the second column of figures, the solid line corresponds to the design $k = 3$ and the dot-and-dash line to the design $k = 4$. Here the dimension of the estimated function is $s = 1$ and $\delta_\varepsilon = 1.135$ ($G = e_1' = (1,0)$ or $G = e_2' = (0,1)$).

Nonsensitiveness regions $\overline{\mathscr{K}}_\varepsilon$

$G = e_1' = (1, 0)$ $\quad$ $\alpha = 0.05$ $\quad$ $G = e_2' = (0, 1)$

$\beta_1 \quad (i = 1)$ $\quad$ $\varepsilon = 0.05$ $\quad$ $\beta_2 \quad (i = 2)$

$t = 3$

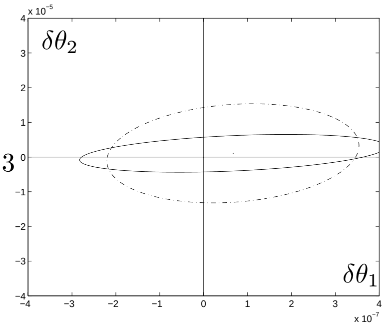

Fig. 3.1

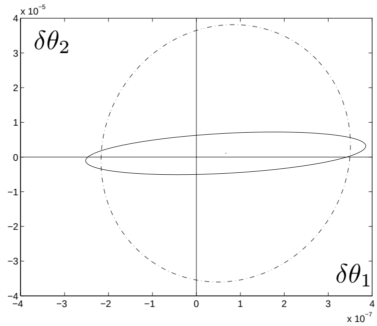

Fig. 3.4

$t = 4$

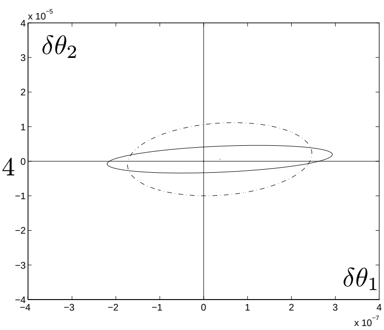

Fig. 3.2

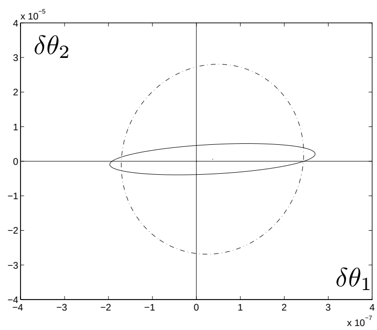

Fig. 3.5

$t = 5$

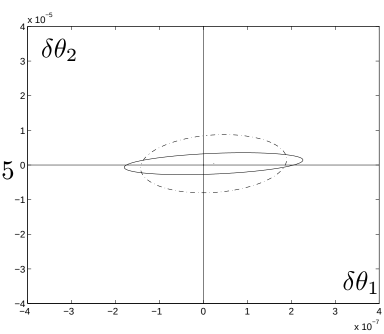

Fig. 3.3

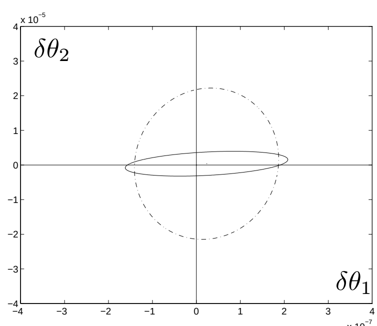

Fig. 3.6

As far as Fig. 3.7 is concerned, it shows for $t = 3$ (the solid line), $t = 4$ (the dashed line) and $t = 5$ (the dot-and-dash line) the nonsensitivenes region $\overline{\mathscr{K}}_\varepsilon$ of the whole

Nonsensitiveness regions $\overline{\mathscr{K}}_{\varepsilon}$

$G = I$
X_3

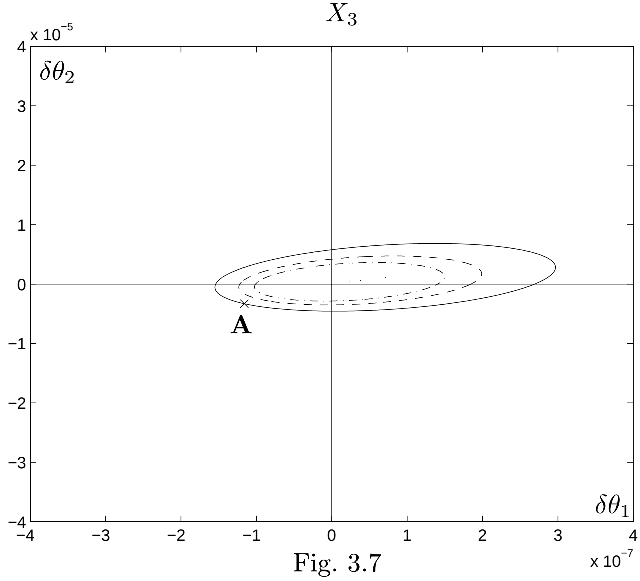

Fig. 3.7

Confidence ellipses

$G = I$
X_3
$t = 3$

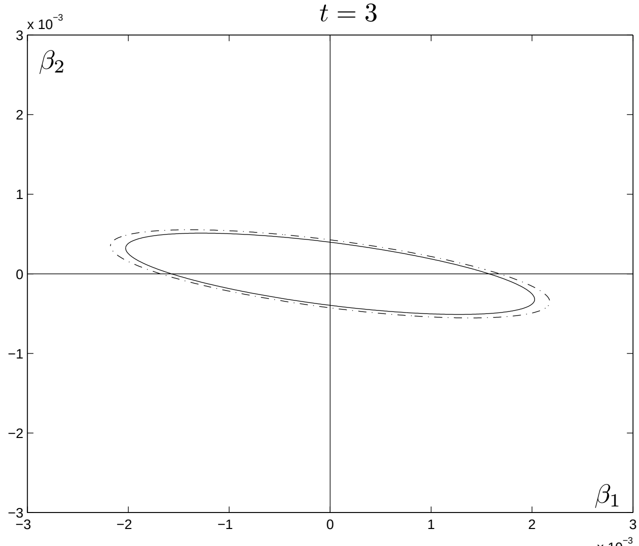

Fig. 3.8

vector $(\beta_1, \beta_2)'$ for the design $k = 3$. Here the dimension of the estimated function is $s = 2$ and $\delta_\varepsilon = 1.386$ $(G = I)$.

Fig. 3.8 demonstrates the change of the confidence ellipsoid of the vector $(\beta_1, \beta_2)'$ caused by the change of the second order parameter. The value $\delta\vartheta = [-1.16 \times 10^{-7}, -3.3 \times 10^{-6}]$ corresponds to the point A lying on the boundary of the nonsensitiveness region given on Fig. 3.7. The dot-and-dash line corresponds to the changed value $\vartheta^* + \delta\vartheta$ and the solid line to the actual value ϑ^*.

4. Linear hypothesis on β

The problem is to find such a neighbourhood of the parameter ϑ^* in which the substitution of ϑ for ϑ^* does not cause an essential change in the risk α of the test of the null hypothesis $H_0\colon\ H\beta^* + h = 0$ against an alternative $H_a\colon\ H\beta^* + h = \xi \neq 0$. Another problem is to find an analogous neighbourhood which does not cause an essential change in the value of the power function at the point ξ. To say it more precisely:

Let a test with the given risk α of the null hypothesis $H_0\colon\ H\beta^* + h = 0$ against the alternative $H_a\colon\ H\beta^* + h = \xi \neq 0$ be considered. Here two problems occur:

(i) to determine the region $\mathscr{R}_\varepsilon$ with the property

$$\delta\vartheta \in \mathscr{R}_\varepsilon \Rightarrow P\{T_H(Y, \vartheta^* + \delta\vartheta) \geqslant \chi_q^2(0; 1-\alpha) | H_0\} \leqslant \alpha + \varepsilon$$

and

(ii) for a given ξ to determine the region $\mathscr{H}_{\varepsilon,\xi}$ with the property

$$\begin{aligned} \delta\vartheta \in \mathscr{H}_{\varepsilon,\xi} \Rightarrow P\{T_H(Y, \vartheta^* + \delta\vartheta) &\geqslant \chi_q^2(0; 1-\alpha) | H_a\} \\ &\geqslant P\{T_H(Y, \vartheta^*) \geqslant \chi_q^2(0; 1-\alpha) | H_a\} - \varepsilon \end{aligned}$$

(for T_H see Statement 4.2). Both these problems can be solved on the basis of the following.

Let $Y \sim N_n(X\beta^*, \sum_{i=1}^{p} \vartheta_i^* V_i)$. Let the null hypothesis concerning β^* be $H_0\colon\ H\beta^* + h = 0$, where H is a $q \times k$ matrix such that $r(H) = q$; the alternative hypothesis is of the form $H_a\colon\ H\beta^* + h \neq 0$. The well known lemma is valid.

Lemma 4.1. *If $H_\beta^* + h = 0$, the statistic*

$$T_H(Y, \vartheta^*) = [H\hat{\beta}(Y, \vartheta^*) + h]'(HC^{-1}H')^{-1}[H\hat{\beta}(Y, \vartheta^*) + h],$$

where $C = X'\Sigma^{-1}(\vartheta^*)X$, possesses the central chi-square probability distribution with q degrees of freedom.

If $H\beta^* + h = \xi \neq 0$, then $T_H(Y, \vartheta^*)$ possesses a noncentral chi-square probability distribution with q degrees of freedom and the parameter of noncentrality

$$\delta = \xi'(HC^{-1}H')^{-1}\xi.$$

All statements given in the following have been proved in [6].

Statement 4.2. *Let*

$$T_H(Y, \vartheta) = [H\hat{\beta}(Y, \vartheta) + h]' \Big(H[X'\Sigma^{-1}(\vartheta)X]^{-1}H' \Big)^{-1} [H\hat{\beta}(Y, \vartheta) + h]$$

and

$$\delta T_H = \delta\vartheta' \partial T_H(Y, \vartheta^*)/\partial\vartheta.$$

Then

$$\begin{aligned}
\delta T_H = & -2\sum_{i=1}^{p} \delta\vartheta_i [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_i \Sigma^{-1}(\vartheta^*) v \\
& -2\sum_{i=1}^{p} \delta\vartheta_i [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_i F_H' C_H [H\hat{\beta}(Y, \vartheta^*) + h] \\
= & -2\delta\vartheta' \left[\begin{pmatrix} [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_1 \Sigma^{-1}(\vartheta^*) v \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_p \Sigma^{-1}(\vartheta^*) v \end{pmatrix} \right. \\
& \left. + \begin{pmatrix} [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_1 F_H' C_H [H\hat{\beta}(Y, \vartheta^*) + h] \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ [H\hat{\beta}(Y, \vartheta^*) + h]' C_H F_H V_p F_H C_H [H\hat{\beta}(Y, \vartheta^*) + h] \end{pmatrix} \right]
\end{aligned}$$

where $v = Y - X\hat{\beta}(Y, \vartheta^*)$, $F_H = HC^{-1}X'\Sigma^{-1}(\vartheta^*)$ and $C_H = (HC^{-1}H')^{-1}$. The mean value of the random variable δT_H is

$$\begin{aligned}
\mathrm{E}(\delta T_H | \beta^*, \vartheta^*) = & -[\mathrm{Tr}(U_H V_1), \dots, \mathrm{Tr}(U_H V_p)]' \delta\vartheta \\
& -[\xi' Z_1 \xi, \dots, \xi' Z_p \xi]' \delta\vartheta \\
= & -a_\xi' \delta\vartheta,
\end{aligned}$$

where

$$\begin{aligned}
a_\xi &= a_0 + [\xi' Z_1 \xi, \dots, \xi' Z_p \xi]' \\
a_0 &= \vartheta[\mathrm{Tr}(U_H V_1), \dots, \mathrm{Tr}(U_H V_p)]', \\
U_H &= F_H' C_H F_H, \\
Z_i &= C_H F_H V_i F_H' C_H, \ i = 1, \dots, p,
\end{aligned}$$

and

$$\xi = H\beta^* + h.$$

The variance of δT_H is

$$\begin{aligned}
&\operatorname{Var}(\delta T_H|\beta^*, \vartheta^*)\\
&= 4\sum_{i=1}^{p}\sum_{j=1}^{p}\delta\vartheta_i\delta\vartheta_j \operatorname{Tr}\Big(U_H V_i [M_X \Sigma(\vartheta^*) M_X]^+ V_j\Big)\\
&\quad + 2\sum_{i=1}^{p}\sum_{j=1}^{p}\delta\vartheta_i\delta\vartheta_j \operatorname{Tr}(U_H V_i U_H V_j)\\
&\quad + 4\sum_{i=1}^{p}\sum_{j=1}^{p}\delta\vartheta_i\delta\vartheta_j \xi' C_H F_H V_i \Big(U_H + [M_X\Sigma(\vartheta^*)M_X]^+\Big) V_j F_H' C_H \xi\\
&= \delta\vartheta'(A_0 + D_\xi)\delta\vartheta,
\end{aligned}$$

where

$$\{A_0\}_{i,j} = 2\operatorname{Tr}(U_H V_i U_H V_j) + 4\operatorname{Tr}\Big(U_H V_i [M_X\Sigma(\vartheta^*)M_X]^+ V_j\Big),\ i,j = 1,\dots,p,$$

and

$$\{D_\xi\}_{i,j} = \begin{cases} 0, & \textit{if } \xi = 0\\ \xi' C_H F_H V_i \Big(U_H + [M_X\Sigma(\vartheta^*)M_X]^+\Big) V_j F_H' C_H \xi, & \textit{if } \xi \neq 0, \end{cases}$$

$i, j = 1, \dots, p.$

The idea of solving the problems (i) and (ii) consists, analogously to the preceding case of the confidence region, in applying the Chebyshev inequality under the special conditions of this section mentioned in Statement 4.2.

Statement 4.3.

(i) *The boundary of the set $\mathscr{R}_\varepsilon$ is*

(4.1) $$\overline{\mathscr{R}}_\varepsilon = \left\{x\colon\ (x - x_0)'(t^2 A_0 - a_0 a_0')(x - x_0) = \frac{\delta_\varepsilon^2 t^2}{t^2 - a_0' A_0^- a_0}\right\},$$

where $x_0 = \delta_\varepsilon A_0^- a_0/(t^2 - a_0' A_0^- a_0)$ and $P\{\chi_q^2 \geqslant \chi_q^2(0; 1-\alpha) - \vartheta_\varepsilon\} = \alpha + \varepsilon$.

(ii) *The boundary of the set $\mathscr{H}_{\varepsilon,\xi}$ is*

(4.2) $$\overline{\mathscr{H}}_{\varepsilon,\xi} = \left\{y\colon\ (y + y_0)'(t_\xi^2 A_\xi - a_\xi a_\xi')(y + y_0) = \frac{\delta_{\varepsilon,\xi}^2 t^2}{t^2 - a_\xi' A_\xi^- a_\xi}\right\}$$

where $y_0 = \delta_{\varepsilon,\xi} A_\xi^- a_\xi/(t^2 - a'_\xi A_\xi^- a_\xi)$, $P\{\chi_q^2(\xi'[HC^{-1}H']^{-1}\xi) \geqslant \chi_q^2(1-\alpha) + \delta_{\varepsilon,\xi}\} = p(\xi) - \varepsilon$ *and* $p(\xi) = P\{\chi_q^2(\xi'[HC^{-1}H']^{-1}\xi) \geqslant \chi_q^2(1-\alpha)\}$, $\xi \in \mathbb{R}^q$.

Example 4.4. Let us return to Example 2.8. As far as the problem (i) is concerned, under the condition that $H = G$, the regions $\overline{\mathscr{K}}_\varepsilon$ and $\overline{\mathscr{R}}_\varepsilon$ are identical (compare (3.7) and (4.2)). Here $\alpha = 0.05$, $\varepsilon = 0.05$ and $t = 3, 4, 5$ (Figs 3.1–3.6).

As far as (ii) is concerned, Figs 4.1–4.4 demonstrate the nonsensitiveness regions $\overline{\mathscr{H}}_{\varepsilon,\xi}$ of the form (4.2) for $\alpha = 0.05$, $\varepsilon = 0.05$ and $t = 5$, namely Figs 4.1 and 4.2 for the parameter β_1 ($i = 1$) for designs $k = 1$ and $k = 2$ (the solid line corresponds to the design $k = 1$ with the design matrix $X_1 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ 5, & 10, & -5, & 0 \end{pmatrix}'$ and the dot-and-dash line to the design $k = 2$, where $X_2 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ -5, & 5, & 0, & 10 \end{pmatrix}'$) and Figs 4.3 and 4.4 for the parameter β_2 ($i = 2$) for the designs $k = 3$ and $k = 4$ (the solid line corresponds to the design $k = 3$ with the design matrix $X_3 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ 0, & 5, & -5, & 10 \end{pmatrix}'$ and the dot-and-dash line to the design $k = 4$, where $X_4 = \begin{pmatrix} 1, & 1, & 1, & 1 \\ -10, & 10, & 0, & 5 \end{pmatrix}'$). For the mentioned combinations of indices the parameter ξ of noncentrality of the form

$$\xi = \frac{j}{3} e_i \sqrt{\{C_k\}_{i,i}},\ j = 1, 2,$$

was considered (the variant $j = 0$, in accordance with the problem solved, is meaningless), where

$$C_k = X'_k \Sigma^{-1}(\vartheta^*) X_k,\ k = 1, 2, 3, 4.$$

For the parameter $\delta_{\varepsilon,\xi}$ for the given combinations of the indices i, j, k see Table 4.1.

	$i = 1$		$i = 2$	
	$k = 1$	$k = 2$	$k = 3$	$k = 4$
j=1	3.62784	4.89392	4.68243	4.89487
j=2	1.24392	3.95692	2.90514	3.96069

Table 4.1

Nonsensitiveness regions $\overline{\mathscr{H}_{\varepsilon,\xi}}$

$\alpha = 0.05$

$X_1, X_2 \quad (k = 1, k = 2)$ $\quad \varepsilon = 0.05 \quad$ $X_3, X_4 \quad (k = 3, k = 4)$

$i = 1$ $\quad t = 5 \quad$ $i = 2$

$j = 1$

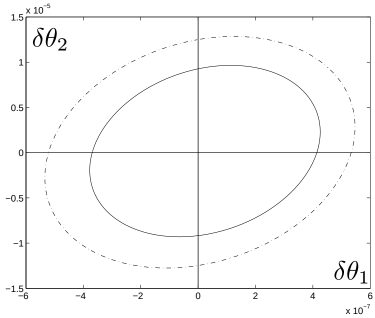

Fig. 4.1

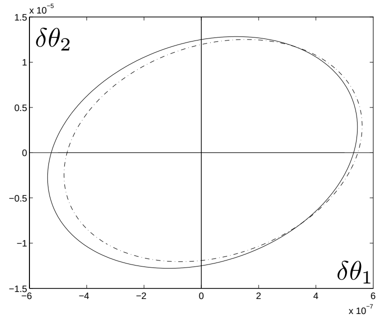

Fig. 4.3

$j = 2$

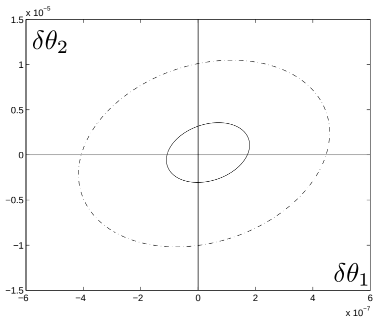

Fig. 4.2

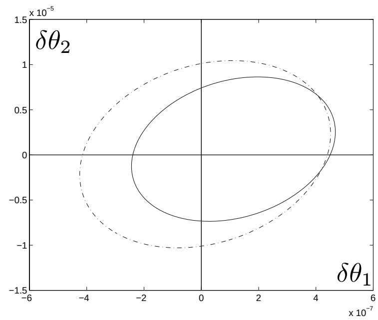

Fig. 4.4

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