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A REVIEW OF TWO DIFFERENT APPROACHES FOR SUPERCONVERGENCE ANALYSIS

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Abstract. In 1995, Wahbin presented a method for superconvergence analysis called "Interior symmetric method," and declared that it is universal. In this paper, we carefully examine two superconvergence techniques used by mathematicians both in China and in America. We conclude that they are essentially different.

Keywords: finite element method, superconvergence error estimates

MSC 2000: 65N30

Křížek and Neittaanmäki [11] have been systematically introducing many superconvergence techniques. In this paper, we want to stress two different approaches of treating superconvergence.

Consider a second order boundary value problem whose associated bilinear form is

$$a(u,v)=\int_\Omega\left(\sum_{i,j=1}^n a_{ij}D_iD_jv+quv\right)\,\mathrm{d}x,$$

where $q\geqslant 0$ and a_{ij} are sufficiently smooth functions, a_{ij} satisfy the well-known uniform ellipticity condition, Ω is a bounded domain in R^n with a smooth or polyhedral boundary. We will mostly deal with the two-dimensional case $n=2$.

Denote by S^h a finite element space consisting of continuous piecewise polynomial functions of degree k. Let $\{S^h\}_{h\to 0}$ be a family of finite element spaces satisfying

$$\inf\{h\|u-v\|_1+\|u-v\|_0\colon\ v\in S^h\}=o(h^{k+1}),$$

where u is the weak solution of the boundary value problem considered, and let $u^h\in S^h$ be the Galerkin solution satisfying

$$a(u-u^h,v)=0\quad\forall v\in S^h.$$

1.1. Green's function method. Green's function G_z associated to a node z is non-singular when $n = 1$. Recall that a function G_z^h with values in the space S^h is said to be a discrete Green's function if

$$a(v, G_z^h) = v(z) \quad \forall v \in S^h$$

(see [28, Chapt. 3.1]). Hence, we have

$$\|G_z - G_z^h\|_1 \leqslant Ch^k, \tag{1}$$

and thus the error can be estimated (see Douglas and Dupont (1973)) as follows:

$$|(u - u^h)(z)| = |a(u - u^h, G_z)| = |a(u - u^h, G_z - G_z^h)| \leqslant Ch^{2k}\|u\|_{k+1}.$$

Remarks:

1) It is interesting that we can obtain optimal convergence rates in the node z; unfortunately, we are not able to find any other superconvergence phenomenon in this way.

2) For n ($\geqslant 2$) dimensional problems we can only obtain L^∞ estimates instead of (1), since Green's function G_z is singular.

1.2. The tensor convolution method. In 1974, Douglas, Dupont and Wheeler used the tensor product space

$$S^h = S^h(I) \otimes S^h(I),$$

where $S^h(I)$ is a one dimensional finite element space. The error can be written as

$$u - u^h = u - w + w - u^h,$$

where $w = P \otimes Pu$ and the mapping $P\colon\ H_0^1(I) \to S^h(I)$ is Galerkin's projection.

Using the properties of the projection, we find that

$$\|w - u^h\|_0 = o(h^{k+3}), \quad k \geqslant 3.$$

Furthermore, by the inverse inequality we have

$$e(z) = |w(z) - u^h(z)| = o(h^{k+2}) \quad \text{and} \quad \|w - u^h\|_{0,\infty} = o(h^{k+2}),$$

where z is the node of rectangular apex.

Remarks:

1) These estimates hold only for $k \geqslant 3$, i.e., they are not true for $k \leqslant 2$.

2) They cannot be applied to equations with mixed terms.

3) An order of convergence is lost because of using the inverse estimate. In fact, we have $e(z) = o(h^{k+3})$.

4) There are also other superconvergence points, not only apexes.

5) The method cannot be applied to Serendipity families, for instance, 8 nodes element.

1.3. Averaging convolution method. In 1977, Bramble and Schatz, and also Thomée obtained by constructing a function K_h^α with a small support,

$$\|D^\alpha u - K_h^\alpha * u^h\|_0 = o(h^{2k}), \quad k \geqslant 1,$$

where $D^\alpha u$ is the α-th order derivative of the function u and $|\alpha| \leqslant 1$.

Remarks:

1) We can obtain the optimal convergence rate $O(h^{2k})$ for both u and its derivatives. However, there are no advantages for the first or second order elements.

2) Superconvergence occurs only in the interior of the domain.

2. Conventional methods in China and Europe

Conventional methods in China and Europe are based on the following two estimates. Construct a generalized interpolating function $W \in S^h$ such that

$$(A) \quad |a(u - W, v)| \leqslant Ch^{k+\tau} \|u\|_{k+1+\tau,p} \|v\|_{1,q} \quad \forall v \in S^h \quad (k \geqslant 1),$$

$$(B) \quad |a(u - W, v)| \leqslant Ch^{k+\tau+1} \|u\|_{k+1+\tau,p} \|v\|_{h,2,q} \quad \forall v \in S^h \quad (k \geqslant 2),$$

where $\|v\|_{h,2,q} = (\sum_e \|v\|_{2,q,e}^q)^{1/q}$, $\tau \in \{1, 2\}$ and $1 \leqslant q \leqslant 2$.

There are two approaches for constructing interpolate functions: In 1977, Chen points out that W can be chosen as the Lagrange interpolation function and expanded according to the method presented by Ding, Jiang, Lin and Luo (1977). Then it can be applied for the first and second order element. At the same time, Zhu points out that we may choose W as the projection interpolate by the orthogonal expansion (which is also called "point-line-plane interpolation"). It is proved to be efficient for k-order elements ($k \geqslant 2$).

In 1969, Oganesyan and Rukhovetz obtained (A) for linear triangular elements ($k = 1$, $\tau = 1$, $p = 2$).

Zlámal in 1977, Chen and Zhu in 1978 obtained (A) for rectangular elements ($k = 1, 2$, $\tau = 1$, $p = 2$).

In 1978, Chen obtained (A) for linear triangular elements ($k = 1$, $\tau = 1$, $p = 2$).

In 1978, Zhu obtained (A) for isoparametric quadratic elements ($k = 1, 2$, $\tau = 1$, $p = 2$).

In 1981, Chen obtained (A) and (B) ($k \geqslant 3$, $\tau = 2$, $p = 2$) for k-th order rectangular elements by the orthogonal expansion method. In 1990, Jia in his Master thesis obtained (A) and (B) ($k \geqslant 3$, $\tau = 2$, $2 \leqslant p \leqslant \infty$) for k-th order rectangular elements by the orthogonal expansion method.

Zhu (1981–1985) obtained (A) and (B) for 2nd order triangular element ($k = 2$, $\tau = 1$, $2 \leqslant q \leqslant \infty$).

Up to now, all superconveregence techniques are based on the above estimates.

2.1. Energy orthogonalization methods—averaging superconvergence estimates (Chen, Zhu (1978–1981)). Since $u^h - W \in S^h$, by estimates (A) and (B),

$$\begin{aligned} \|u^h - W\|_1^2 &\leqslant Ca(u^h - W, u^h - W) \\ &\leqslant Ca(u - W, u^h - W) \\ &= o(h^{k+\tau})\|u^h - W\|_1 \end{aligned}$$

and thus we have

$$\|u^h - W\|_1 = o(h^{k+\tau}) \qquad (k \geqslant 1).$$

Moreover, by means of Nitsche's technique and estimates (A) and (B), we obtain

$$\|u^h - W\|_0 = o(h^{k+1+\tau}) \quad (k \geqslant 2).$$

Remarks:

1) This method can be applied to triangular and rectangular elements; therefore it is universal without any limitation.

2) Different interpolate functions lead to different results.

3) Before 1981, no superconvergence results were obtained in this way except the above estimate.

2.2. Green's function—energy orthogonalization method. In 1982, Zhu obtained, by using discrete Green's function G_z^h and estimates (A) and (B),

$$\begin{aligned}(u^h - W)(z) &= a(u^h - W, G_z^h) \\ &= a(u - W, G_z^h) \\ &= o(h^{k+1+\tau})\|G_z^h\|_{h,2,1} \\ &= o(h^{k+1+\tau}|\ln h|).\end{aligned}$$

Let $\partial_z U$ be the derivative of U with respect to z. Then we have for $k \geqslant 2$

$$\begin{aligned}\partial_z(u^h - W)(z) &= a(u - W, \partial_z G_z^h) \\ &= o(h^{k+1+\tau})\|\partial_z G_z^h\|_{h,2,1} \\ &= o(h^{k+1+\tau})\end{aligned}$$

and for $k \geqslant 1$

$$\begin{aligned}\partial_z(u^h - W)(z) &= a(u - W, \partial_z G_z^h) \\ &= o(h^{k+\tau})\|\partial_z G_z^h\|_{1,1} \\ &= o(h^{k+\tau}|\ln h|).\end{aligned}$$

Remarks:

1) This method is used to estimate superconvergence at points which serve for an error expansion, extrapolation and postprocessing.

2) Since discrete Green's functions are singular, no optimal convergence rates can be obtained, because some norms are unbounded as h approaches 0.

2.3. Local superconvergence estimates. Let a locally smooth and "locally good" partition u be given. Using a local estimate technique, Zhu and Lin (1989) obtained the local superconvergence result

$$\|u^h - u^I\|_{0,\infty,D_0} + h\|u^h - u^I\|_{1,\infty,D_0} = o(h^{k+2})\|u\|_{k+2,\infty,D_1} + C\|u - u^h\|_{-s,D}$$

for $D_0 \subset\subset D_1 \subset\subset \Omega$, where $D_0 \subset\subset D_1$ means that $\overline{D}_0 \subset D_1$.

Remarks:

1) This method could be used to deal with both the interior and the boundary estimates. However, the superconvergence rates will be lower if the boundary is not smooth. But in the special case $k = 1$, even if the boundary is not smooth, the superconvergence rates will be almost the same as those for a smooth boundary. Especially, for a polygonal domain there is no loss of convergence rates.

2) Numerical experiments show that we can still obtain superconvergence results in case $k=1$ even if u has a singularity. Furthermore, the optimal extrapolation estimate also holds in this case ($k=1$).

2.4. The interpolate postprocessing. Since there are no complete "good derivative points" for second order triangular elements, Chen, H. S. (1986) (see Zhu and Lin (1989)) obtained complete "good derivate points" by using the interpolate postprocessing method in his Master thesis. Lin, Zhou, Yan et al. (1986–1994) (see also Lin and Zhu (1994)) obtained even a more general interpolate postprocessing method and global superconvergence results. The following is an outline of the construction of a high order interpolation operator Π_h

$$\begin{aligned} \Pi_h W &= \Pi_h u, \\ \|\Pi_h u\|_{s,q} &\leqslant C\|u\|_{s,q} \quad \text{and} \\ \|u - \Pi_h u\|_{s,q} &= o(h^{k+1+\tau-s}), \quad s = 0, 1. \end{aligned}$$

Then

$$\begin{aligned} \|u - \Pi_h u^h\|_{s,q} &\leqslant \|u - \Pi_h u\|_{s,q} + \|\Pi_h W - u^h\|_{s,q} \\ &= o(h^{k+1+\tau-s}), \quad q = 2, \infty. \end{aligned}$$

2.5. Asymptotic expansion for $u^h - W$. Lin, Lu and Shen (1993) pointed out that, for linear triangular elements, if the partition is uniform then not only estimates (A) and (B) hold, but also the following expansion is valid provided the given u is sufficiently smooth:

$$a(u - u^I, v) = h^2(D^4 u, v) + o(h^4)\|v\|_{0,1},$$

where D^4 is the 4-th order derivative operator

$$\begin{aligned} (u^h - u)(z) &= (u^h - u^I)(z) \\ &= a(u - u^I, G_z^h) \\ &= W(z)h^2 + O(h^4|\ln h|^2), \end{aligned}$$

$W(z) = (D^4 u, G_z)$ is independent of h, and where we have used the estimate

$$\|G_z - G_z^h\|_{0,1} = o(h^2|\ln h|^2).$$

Then we obtain

$$[(4u^{h/2} - u^h)/3 - u](z) = o(h^4|\ln h|^2),$$

which is the so-called extrapolating formula. Similarly, we get

$$\partial_z[(4u^{h/2} - u^h)/3 - u](z) = o(h^{4-\delta}),$$

where $\delta > 0$ is a small constant.

For interpolate processing, expansion, extrapolating, preprocessing and post-processing we refer to Lin and Zhu (1994) and Lin and Yan (1996).

2.6. Counter-example presented by B. Li. To check the superconvergence of higher order triangular elements, Li, B. (1990) found a counter-example for the 3rd order triangular element. He chose the function $u(x_1, x_2) = \frac{1}{2}(1-x_1^2)(1-x_2^2)$ on the unit square and proved that

$$\|u^h - u^I\|_0 \geqslant Ch^4 \quad \text{and} \quad \|u^h - u^I\|_1 \geqslant Ch^3,$$

where u^I is the Lagrange interpolation function of u. This shows that there exist many points z such that

$$|(u^h - u)(z)| = |(u^h - u^I)(z)| \geqslant Ch^4,$$

but none of them are superconvergence points. However, this does not mean that the superconvergence is not reachable for higher order triangular elements by conventional methods used by Chinese mathematicians. The reasonable explanation is that the choice $W = u^I$ is not suitable.

3. Interior symmetric method—challenged by Wahlbin (1995)

Let z be in the interior of a domain, $d \geqslant Ch$, and let B_d be the neighbourhood of z with d as its radius. Wahlbin (1995) obtained a sharp interior estimate

$$|e(z)| \leqslant Ch^{k+1}|u|_{k+1,\infty,B_d} + o(h^{k+1+\tau}d^{-s-2/q}),$$

where $0 \leqslant s \leqslant k-1$, $2 \leqslant q < \infty$.

Let the partition in B_d be symmetric with respect to the point z. Then by special techniques we get

$$|e(z)| \leqslant Ch^{k+1}d|u|_{k+2,\infty,B_d} + o(h^{k+1+\tau}d^{-s-2/q}).$$

Let $d = h^\sigma$ be such that

$$k + 1 + \sigma = k + 1 + \tau - (x + 2/q)\sigma.$$

This gives the superconvergence result

$$e(z) = o(h^{k+1+\sigma}),$$

where $\sigma = \tau/(s + 2/q + 1)$, k is even. Similarly, we have

$$\nabla e(z) = o(h^{k+\sigma'}), \quad \sigma' = \tau/(s + 2 + 2/q),$$

where k is odd, and ∇ is the averaging gradient operator.

Wahlbin pointed out that this method is a "universal principle."

Remarks:

1) No matter what k is, triangular elements possess some superconvergence and superconvergence points are located on six fixed points in every triangle (provided the partition is uniform)

2) Superconvergence rate σ is very small, for instance $1/3 \leqslant \sigma \leqslant 1/2$, for second order elements. No order of superconvergence rate can be obtained.

3) Superconvergence on the boundary cannot be obtained by this interior estimate method. However, we can obtain both global and local superconvergence estimates (see Zhu and Lin (1989)).

4) According to Chinese methods, we can always obtain superconvergence rate of order two for k-th order rectangular elements ($k \geqslant 3$) only if the partition is uniform. On the contrary, we cannot obtain the same results by Wahlbin's method.

5) This method cannot be used for postprocessing, for example, interpolate processing and extrapolation processing.

6) Combining Chinese methods with the interior symmetric method we can obtain deeper superconvergence results for the 2nd order triangular element. For instance, we can get the following result:

$$\nabla(u^h - u^I)(z) = o(h^{3+\sigma})$$

(z is a symmetric point)—see [9]. This method cannot be obtained by means of Wahlbin's method.

We conclude that the so-called "interior symmetric method" is not a "general principle." Moreover, it cannot be applied to piecewise quasi-uniform triangulations whereas Chinese methods can.

4. Review of superconvergence of higher order triangular elements

Zhu and Lin (1989) pointed out that for higher order triangular elements the optimal error estimate is

$$h\|u-u^h\|_1+\|u-u^h\|_0=o(h^{k+1})$$

and this estimate cannot be improved. However, we notice that

$$\int_\Omega (u-u^h)(x)\,\mathrm{d}x=o(h^{2k})\quad k\geqslant 2.$$

For higher order elements ($k\geqslant 3$), the latter estimate is much better than the previous one. This only shows that the positive part and negative part of the error $u-u^h$ have been mutually eliminated. Since the error $u-u^h$ is continuous on the domain Ω, it must have many zero points, from which we believe that superconvergence exists for higher order triangular elements. This can be seen partly in Wahlbin's work. In [25–29], we proved that the u and the tangential derivative of each side of the triangular element have the superconvergence

$$(u-u^h)(z)=o(h^{k+2}|\ln h|),$$

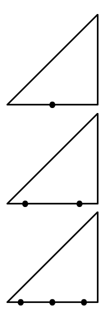
where z is the k-th order Lobatto point on each element side, $k=2$, and

$$\partial_z(u-u^h)(z)=o(h^{k+1}),$$

where z is the k-th order Gauss point on each side, $k=1,2$.

Anyway, we believe that these results hold also for $k\geqslant 3$ (see Figures 1 and 2).

Running through the Lagrange triangular element with $k=1,2,\ldots$ but considering only strictly natural superconvergence for $\partial_x u$, we find



$k=1$ midpoint of the side parallel to the x-axis

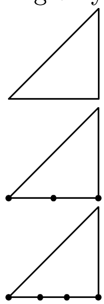
$k=2$ 2nd order Gauss points

$k=3$ 3rd order Gauss points

. . . .

Figure 1

Considering only strictly natural superconvergence for u, we find



$k = 1$ — no superconvergence

$k = 2$ — 2nd order Lobatto points

$k = 3$ — 3rd order Lobatto points

. . . .

Figure 2

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