Petr Tichý; Jan Zítko

Derivation of BiCG from the conditions defining Lanczos' method for solving a system of linear equations

Applications of Mathematics, Vol. 43 (1998), No. 5, 381–388

Persistent URL: `http://dml.cz/dmlcz/134394`

DERIVATION OF BICG FROM THE CONDITIONS DEFINING LANCZOS' METHOD FOR SOLVING A SYSTEM OF LINEAR EQUATIONS

Petr Tichý, Jan Zítko, Praha

(Received December 12, 1997)

Abstract. Lanczos' method for solving the system of linear algebraic equations $Ax = b$ consists in constructing a sequence of vectors x_k in such a way that $r_k = b - Ax_k \in r_0 + A\mathcal{K}_k(A, r_0)$ and $r_k \perp \mathcal{K}_k(A^T, \widetilde{r}_0)$. This sequence of vectors can be computed by the BiCG (BiOMin) algorithm. In this paper is shown how to obtain the recurrences of BiCG (BiOMin) directly from this conditions.

Keywords: biorthogonalization, linear equations, biconjugate gradient method

MSC 2000: 65F10, 65F25

1. Introduction

The application of recursive biorthogonalization to the numerical solution of eigenvalue problems and linear systems goes back to Lanczos ([Lancz–50], [Lancz–52]) and is therefore referred to as the *Lanczos process*. In its basic form [Lancz–50], the process generates a pair of biorthogonal bases for a pair of Krylov spaces, one generated by the matrix A and the other by the matrix A^T. This process is characterized by a three-term recurrence and is here called the *Lanczos biorthogonalization* (BiO) *algorithm* [Gutkn–97]. A variation of it, described already in the second Lanczos paper [Lancz–52] under the section heading "*The Complete Algorithm for Minimized Iterations*", applies instead a pair of coupled two-term recurrences and is here referred to as BiOC algorithm [Gutkn–97], because it produces additionaly a second pair of biconjugate bases.

This paper was supported by the Grant Agency of the Czech Republic under Grant No 201/96/0918.

An application of the BiOC for solving linear algebraic systems was already presented in the above mentioned Lanczos paper. This algorithm was later reformulated by many authors and is named BiOMin [Gutkn–97] or *Biconjugate gradient* (BiCG) *algorithm* [Fletch–76]. Block formulation of BiCG is due to [Leary–80].

Let $\mathcal{K}_k(A, r_0) = \text{span}\{r_0, Ar_0, \ldots, A^{k-1}r_0\}$ be the *Krylov space* spanned by a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and a vector $r_0 \in \mathbb{R}^n$. Let x_0 be any initial guess for the solution of the system $Ax = b$, $r_0 = b - Ax_0$ the starting residual and $\widetilde{r}_0$ an arbitrary nonzero vector. The *Lanczos' method* for solving this system consists in the construction of a sequence of vectors x_k and residuals $r_k = b - Ax_k$ $(k \geqslant 1)$ such that

- $x_k \in x_0 + \mathcal{K}_k(A, r_0)$,
- $r_k \perp \mathcal{K}_k(A^T, \widetilde{r}_0)$.

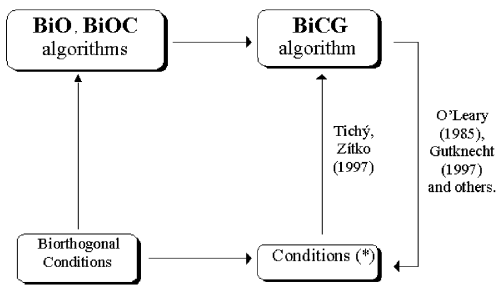

Figure 1: In 1950, Lanczos introduced an algorithm (BiO) that generates a pair of biorthogonal vector sequences. Lanczos (1952) suggested under the section heading "The Complete Algorithm for Minimized Iterations" an alternative algorithm (BiOC) for computing these sequences of vectors generated by the BiO algorithm. In the same paper an algorithm for solving linear algebraic systems was presented. Many authors ([Leary–80], [Gutkn–97]) showed that the vectors x_k and r_k generated by the BiCG algorithm fulfil the conditions $(*)$ $x_k \in x_0 + \mathcal{K}_k(A, r_0)$ and $r_k \perp \mathcal{K}_k(A^T, \widetilde{r}_0)$. In this paper we show how to obtain the recurrences of BiCG directly from these conditions.

For deriving BiCG we need to define auxiliary vektors $\widetilde{r}_k$ $(k \geqslant 1)$ under conditions

- $\widetilde{r}_k \in \widetilde{r}_0 + A^T\mathcal{K}_k(A^T, \widetilde{r}_0)$,
- $\widetilde{r}_k \perp \mathcal{K}_k(A, r_0)$.

If we substitute the residual r_{i+1} in the form

$$r_{i+1} = r_0 + A\sum_{j=0}^{i} b_j^{(i)} A^j r_0 \quad (i = 0, \ldots, k-1)$$

into conditions $r_{i+1}^T A^{T^j}\widetilde{r}_0) = 0$ $(j = 0, \dots, i)$ we find that the coefficients $b_j^{(i)}$ fulfil the system of linear equations

$$-\begin{pmatrix} (\widetilde{r}_0, r_0) \\ (\widetilde{r}_0, Ar_0) \\ \vdots \\ (\widetilde{r}_0, A^i r_0) \end{pmatrix} = \begin{pmatrix} (\widetilde{r}_0, Ar_0) & (\widetilde{r}_0, A^2 r_0) & \dots & (\widetilde{r}_0, A^{i+1} r_0) \\ (\widetilde{r}_0, A^2 r_0) & (\widetilde{r}_0, A^3 r_0) & \dots & (\widetilde{r}_0, A^{i+2} r_0) \\ \vdots & & & \\ (\widetilde{r}_0, A^{i+1} r_0) & (\widetilde{r}_0, A^{i+2} r_0) & \dots & (\widetilde{r}_0, A^{2i+1} r_0) \end{pmatrix} \begin{pmatrix} b_0^{(i)} \\ b_1^{(i)} \\ \vdots \\ b_i^{(i)} \end{pmatrix}.$$

The determinant of the matrix of this system is called the *Hankel determinant* and is denoted by d_{i+1}. If $d_{i+1} \neq 0$ then the residual r_{i+1} exists and is unique. By analogy, it can be shown that if $d_{i+1} \neq 0$, the vektor $\widetilde{r}_{i+1}$ exists, is defined uniquely and can be written in the form

$$\widetilde{r}_{i+1} = \widetilde{r}_0 + A^T \sum_{j=0}^{i} b_j^{(i)} (A^T)^j \widetilde{r}_0 \quad (i = 0, \dots, k-1).$$

Moreover, if $\widetilde{r}_j^T r_j \neq 0$ $(j = 0, \dots, i)$ then the vectors r_j and the vectors $\widetilde{r}_j$ are linearly independent,

$$\text{span}\{r_0, \dots, r_i\} = \mathcal{K}_{i+1}(A, r_0), \ \text{span}\{\widetilde{r}_0, \dots, \widetilde{r}_i\} = \mathcal{K}_{i+1}(A^T, \widetilde{r}_0) \tag{1}$$

and the vectors r_{i+1} and $\widetilde{r}_{i+1}$ can then be written as

$$r_{i+1} = r_i + A \sum_{j=0}^{i} \gamma_j^{(i)} r_j \quad \text{and} \quad \widetilde{r}_{i+1} = \widetilde{r}_i + A^T \sum_{j=0}^{i} \gamma_j^{(i)} \widetilde{r}_j \quad (i = 0, \dots, k-1).$$

It will be shown in Lemma 1 and in Theorem 1 that if $d_{j+1} \neq 0$ and $\widetilde{r}_j^T r_j \neq 0$ for $j = 0, \dots, i$ then all coefficients $\gamma_j^{(i)}$ are different from zero and the vectors r_{i+1} can be written in the form

$$r_{i+1} = r_i + \gamma_i^{(i)} A \left(r_i + \frac{\gamma_{i-1}^{(i)}}{\gamma_i^{(i)}} \left(r_{i-1} + \frac{\gamma_{i-2}^{(i)}}{\gamma_{i-1}^{(i)}} \left(r_{i-2} + \dots \frac{\gamma_1^{(i)}}{\gamma_2^{(i)}} \left(r_1 + \frac{\gamma_0^{(i)}}{\gamma_1^{(i)}} r_0 \right) \dots \right) \right) \right)$$

and the ratio $\gamma_{j-1}^{(i)}/\gamma_j^{(i)}$ does not depend on i. If we set $\alpha_i = -\gamma_i^{(i)}$ and $\beta_{i-1} = \gamma_{i-1}^{(i)}/\gamma_i^{(i)}$, then r_{i+1} can be written as

$$r_{i+1} = r_i - \alpha_i A(r_i + \beta_{i-1}(r_{i-1} + \beta_{i-2}(r_{i-2} + \dots \beta_1(r_1 + \beta_0 r_0) \dots))) \ (i = 0, \dots, k-1).$$

If we define $p_0 = r_0,\ p_{j+1} = r_{j+1} + \beta_j p_j$ for $j = 0, \dots, i-1$, then

$$r_{i+1} = r_i - \alpha_i A p_i,$$
$$x_{i+1} = x_i + \alpha_i p_i.$$

Likewise, if we define $\widetilde{p}_0 = \widetilde{r}_0$, $\widetilde{p}_{j+1} = \widetilde{r}_{j+1} + \beta_j \widetilde{p}_j$, we obtain recurrences for vectors $\widetilde{r}_{i+1}$ in the form

$$\widetilde{r}_{i+1} = \widetilde{r}_i - \alpha_i A^T \widetilde{p}_i.$$

The classical form of coefficients α_i and β_i will be obtained by using biorthogonal relations.

2. Recurrences for the BiCG iterates

Lemma 1. *Let us suppose that $d_j \neq 0$ for $j = 1, \ldots, k+1$ and $\widetilde{r}_i^T r_i \neq 0$ for $i = 0, \ldots, k$. Then there exist real numbers $\gamma_j^{(i)} \neq 0$, $0 \leqslant j \leqslant i \leqslant k$, such that*

$$r_{i+1} = r_i + A(\gamma_i^{(i)} r_i + \gamma_{i-1}^{(i)} r_{i-1} + \ldots + \gamma_1^{(i)} r_1 + \gamma_0^{(i)} r_0), \tag{2}$$

$$\widetilde{r}_{i+1} = \widetilde{r}_i + A^T(\gamma_i^{(i)} \widetilde{r}_i + \gamma_{i-1}^{(i)} \widetilde{r}_{i-1} + \ldots + \gamma_1^{(i)} \widetilde{r}_1 + \gamma_0^{(i)} \widetilde{r}_0). \tag{3}$$

Moreover, the numbers $\gamma_j^{(i)}$ are determined uniquely.

Proof. The vectors r_{i+1} and r_i fulfil

$$\begin{aligned} r_{i+1} &= r_0 + b_1^{(i+1)} A r_0 + b_2^{(i+1)} A^2 r_0 + \ldots + b_{i+1}^{(i+1)} A^{i+1} r_0, \\ r_i &= r_0 + b_1^{(i)} A r_0 + b_2^{(i)} A^2 r_0 + \ldots + b_i^{(i)} A^i r_0, \end{aligned}$$

and thus

$$r_{i+1} = r_i + A((b_1^{(i+1)} - b_1^{(i)}) r_0 + \ldots + (b_i^{(i+1)} - b_i^{(i)}) A^{i-1} r_0 + b_{i+1}^{(i+1)} A^i r_0).$$

According to (1), there exist uniquely determined real numbers $\gamma_0^{(i)}, \ldots, \gamma_i^{(i)}$ such that

$$r_{i+1} = r_i + A(\gamma_i^{(i)} r_i + \gamma_{i-1}^{(i)} r_{i-1} + \ldots + \gamma_1^{(i)} r_1 + \gamma_0^{(i)} r_0). \tag{4}$$

Analogously we obtain equality (3).

Now, we have to prove that the real numbers $\gamma_j^{(i)}$ are different from zero. Let us define $c_{i,j} = \widetilde{r}_i^T A r_j$. It is easy to see that $c_{i,j} = 0$ when $|i-j| > 1$. For $i = 0, \ldots, k$ we have

$$r_i = r_{i-1} + A(\gamma_{i-1}^{(i-1)} r_{i-1} + \ldots + \gamma_0^{(i-1)} r_0). \tag{5}$$

If we multiply the equation (5) from the left by the vector $\widetilde{r}_i^T$ we obtain

$$\widetilde{r}_i^T r_i = \gamma_{i-1}^{(i-1)} \widetilde{r}_i^T A r_{i-1}.$$

Since $\widetilde{r}_i^T r_i \neq 0$ we have $\widetilde{r}_i^T A r_{i-1} \neq 0$ and hence $c_{i,i-1} \neq 0$. Likewise, if we write $\widetilde{r}_i$ in the form (3), we get $c_{i-1,i} \neq 0$. Let us multiply the equation

$$r_{i+1} = r_i + A(\gamma_i^{(i)} r_i + \ldots + \gamma_0^{(i)} r_0)$$

from the left by the vectors $\widetilde{r}_i^T, \ldots, \widetilde{r}_0^T$. Then we get for the real numbers $\gamma_j^{(i)}$ the identity

$$(6)\qquad -\begin{pmatrix} \widetilde{r}_i^T r_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_{i,i} & c_{i,i-1} & & & & \\ c_{i-1,i} & c_{i-1,i-1} & c_{i-1,i-2} & & & \\ & c_{i-2,i-1} & c_{i-2,i-2} & c_{i-2,i-3} & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{1,2} & c_{1,1} & c_{1,0} \\ & & & c_{0,1} & c_{0,0} \end{pmatrix} \begin{pmatrix} \gamma_i^{(i)} \\ \gamma_{i-1}^{(i)} \\ \vdots \\ \gamma_0^{(i)} \end{pmatrix}.$$

The matrix in (6) is nonsingular because if a vector $(\hat{\gamma}_i, \hat{\gamma}_{i-1}, \ldots, \hat{\gamma}_0)^T$ fulfils the identity (6) then the vector $\hat{r}_{i+1} = r_i + A(\hat{\gamma}_i r_i + \ldots + \hat{\gamma}_0 r_0)$ is orthogonal to $\widetilde{r}_0, \ldots, \widetilde{r}_i$ and lies in $r_0 + A\mathcal{K}_{i+1}(A, r_0)$. But the uniqueness implies that $\hat{r}_{i+1} = r_{i+1}$ and thus $\hat{\gamma}_j = \gamma_j^{(i)}$, $j = 0, \ldots, i$. The matrix in the identity (6) has to be nonsingular. If we denote the three-diagonal matrix in identity (6) as V_i, then $\det(V_i) \neq 0$ for $i = 0, \ldots, k$. Let us denote by $V_i^{(j)}$ the matrix that arises from the matrix V_i if we substitute the jth column of the matrix V_i by the left hand side of identity (6). Then according to the Cramer rule for $j = 0, \ldots, i$ we have

$$\gamma_j^{(i)} = \frac{\det(V_i^{(i+1-j)})}{\det(V_i)}.$$

If we define $V_{-1} = 1$, then we can write $\det(V_i^{(j)})$ in the form

$$(7)\qquad \det(V_i^{(j)}) = (-1)^j \cdot \widetilde{r}_i^T r_i \cdot \prod_{l=i-j+1}^{i-1} c_{l,l+1} \cdot \det(V_{i-j}), \quad j = 1, \ldots, i+1.$$

From (7) it follows that $\det(V_i^{(i+1-j)}) \neq 0$ and hence $\gamma_j^{(i)} \neq 0$ for $j = 0, \ldots, i$. □

Theorem 1. *Let us suppose that $d_i \neq 0$ for $i = 1, \ldots, k+1$, $\widetilde{r}_i^T r_i \neq 0$ for $i = 0, \ldots, k$ and $p_0 = r_0$, $\widetilde{p}_0 = \widetilde{r}_0$. Then we can compute the vectors r_{i+1}, $\widetilde{r}_{i+1}$ and x_{i+1} for $i = 0, \ldots, k$ from the recurrences*

$$(8)\qquad r_{i+1} = r_i - \alpha_i A p_i,$$

$$(9)\qquad \widetilde{r}_{i+1} = \widetilde{r}_i - \alpha_i A^T \widetilde{p}_i,$$

$$(10)\qquad p_{i+1} = r_{i+1} + \beta_i p_i,$$

$$(11)\qquad \widetilde{p}_{i+1} = \widetilde{r}_{i+1} + \beta_i \widetilde{p}_i,$$

$$(12)\qquad x_{i+1} = x_i + \alpha_i p_i,$$

where

$$\alpha_i = \frac{\widetilde{r}_i^T r_i}{\widetilde{p}_i^T A p_i}, \quad \beta_i = \frac{\widetilde{r}_{i+1}^T r_{i+1}}{\widetilde{r}_i^T r_i}. \tag{13}$$

P r o o f. According to relation (5) (we know that $\gamma_j^{(i)} \neq 0$ for $j \leqslant i \leqslant k$) we can write

(14)

$$\begin{aligned} r_{i+1} &= r_i + \gamma_i^{(i)} A \left(r_i + \frac{\gamma_{i-1}^{(i)}}{\gamma_i^{(i)}} r_{i-1} + \frac{\gamma_{i-2}^{(i)}}{\gamma_i^{(i)}} r_{i-2} + \ldots + \frac{\gamma_0^{(i)}}{\gamma_i^{(i)}} r_0 \right) \\ &= r_i + \gamma_i^{(i)} A \left(r_i + \frac{\gamma_{i-1}^{(i)}}{\gamma_i^{(i)}} \left(r_{i-1} + \frac{\gamma_{i-2}^{(i)}}{\gamma_{i-1}^{(i)}} r_{i-2} + \ldots + \frac{\gamma_0^{(i)}}{\gamma_{i-1}^{(i)}} r_0 \right) \right) \\ &= r_i + \gamma_i^{(i)} A \left(r_i + \frac{\gamma_{i-1}^{(i)}}{\gamma_i^{(i)}} \left(r_{i-1} + \frac{\gamma_{i-2}^{(i)}}{\gamma_{i-1}^{(i)}} \left(r_{i-2} + \ldots \frac{\gamma_1^{(i)}}{\gamma_2^{(i)}} \left(r_1 + \frac{\gamma_0^{(i)}}{\gamma_1^{(i)}} r_0 \right) \ldots \right) \right) \right). \end{aligned}$$

We can rewrite all residuals ($i = 0, \ldots, k$) in the form (14). Let us prove that

$$\frac{\gamma_{i-1}^{(l)}}{\gamma_i^{(l)}} = \frac{\gamma_{i-1}^{(j)}}{\gamma_i^{(j)}} \quad \text{for} \quad 1 \leqslant i \leqslant l \leqslant k, \quad 1 \leqslant i \leqslant j \leqslant k \tag{15}$$

holds. We have

$$\begin{aligned} \frac{\gamma_{i-1}^{(l)}}{\gamma_i^{(l)}} &= \frac{\det(V_l^{(l+2-i)})}{\det(V_l^{(l+1-i)})} = \frac{(-1)^{l+2-i} \cdot \widetilde{r}_l^T \widetilde{r}_l \cdot \prod\limits_{j=i-1}^{l-1} c_{j,j+1} \det(V_{i-2})}{(-1)^{l+1-i} \cdot \widetilde{r}_l^T \widetilde{r}_l \cdot \prod\limits_{j=i}^{l-1} c_{j,j+1} \det(V_{i-1})} \\ &= -\, c_{i-1,i} \, \frac{\det(V_{i-2})}{\det(V_{i-1})}. \end{aligned}$$

We can see that the right-hand side does not depend on the index l. Likewise we can write

$$\frac{\gamma_{i-1}^{(j)}}{\gamma_i^{(j)}} = -c_{i-1,i} \cdot \frac{\det(V_{i-2})}{\det(V_{i-1})}$$

and (15) holds. Let us define real numbers α_i and β_i as

$$\alpha_i = -\gamma_i^{(i)} \quad \text{and} \quad \beta_{i-1} = \frac{\gamma_{i-1}^{(i)}}{\gamma_i^{(i)}}.$$

Then according to relations (15) and (14), we can rewrite the residuals r_{i+1} ($i = 0, \ldots, k$) in the form

(16) $$r_{i+1} = r_i - \alpha_i A(r_i + \beta_{i-1}(r_{i-1} + \beta_{i-2}(r_{i-2} + \ldots \beta_1(r_1 + \beta_0 r_0) \ldots))).$$

If we define

$$p_0 = r_0, \quad p_{j+1} = r_{j+1} + \beta_j p_j \quad \text{for} \quad j = 0, \ldots, i-1,$$

we can rewrite (16) in the form

(17) $$r_{i+1} = r_i - \alpha_i A p_i \quad \text{for} \quad i = 0, \ldots, k.$$

Since $\alpha_i \neq 0$ we have that $\widetilde{p}_i^T A p_i = -\widetilde{r}_i^T r_i / \alpha_i \neq 0$. Let us derive now the well-known forms of the real numbers α_i and β_i. We find that

$$\widetilde{p}_i^T r_{i+1} = \widetilde{p}_i^T r_i - \alpha_i \widetilde{p}_i^T A p_i$$

and thus we obtain for α_i the formula

$$\alpha_i = \frac{\widetilde{p}_i^T r_i}{\widetilde{p}_i^T A p_i}.$$

Since

$$\widetilde{p}_i^T r_i = \widetilde{r}_i^T r_i + \beta_{i-1} \widetilde{p}_{i-1}^T r_i = \widetilde{r}_i^T r_i,$$

we obtain

$$\alpha_i = \frac{\widetilde{r}_i^T r_i}{\widetilde{p}_i^T A p_i}.$$

Let us carry on the derivation of β_i. We have

$$\widetilde{r}_i^T p_{i+1} = \widetilde{r}_i^T r_{i+1} + \beta_i \widetilde{r}_i^T p_i$$

and therefore

$$\beta_i = \frac{\widetilde{r}_i^T p_{i+1}}{\widetilde{r}_i^T r_i}.$$

Since

$$\widetilde{r}_{i+1}^T r_{i+1} = \widetilde{r}_{i+1}^T p_{i+1} = \widetilde{r}_i^T p_{i+1} - \alpha_i \widetilde{p}_i^T A p_{i+1} = \widetilde{r}_i^T p_{i+1} + \frac{\alpha_i}{\alpha_{i+1}} \widetilde{p}_i^T (r_{i+2} - r_{i+1}) = \widetilde{r}_i^T p_{i+1},$$

we get

$$\beta_i = \frac{\widetilde{r}_{i+1}^T r_{i+1}}{\widetilde{r}_i^T r_i}.$$

The forms (9) and (11) can be proved analogously. The form (12) we get from

$$b - A x_{i+1} = b - A x_i - \alpha_i A p_i.$$

$\square$

According to the previous theorem, we can formulate Algorithm of BiCG.

Algorithm BiCG

Input $x_0,\ A,\ b,\ \widetilde{r}_0 \neq \mathrm{o},\ \varepsilon;$

$r_0 = b - Ax_0;$

$p_0 = r_0;\ \widetilde{p}_0 = \widetilde{r}_0;$

$k = 0;$

while $\frac{\|r_k\|}{\|r_0\|} > \varepsilon$ **do**

begin

$\alpha_k \ = \frac{\widetilde{r}_k^T r_k}{\widetilde{p}_k^T A p_k};$

$x_{k+1} = x_k + \alpha_k p_k;$

$r_{k+1} = r_k - \alpha_k A p_k;$

$\widetilde{r}_{k+1} = \widetilde{r}_k - \alpha_k A^T \widetilde{p}_k;$

$\beta_k \ = \frac{\widetilde{r}_{k+1}^T r_{k+1}}{\widetilde{r}_k^T r_k};$

$p_{k+1} = r_{k+1} + \beta_k p_k;$

$\widetilde{p}_{k+1} = \widetilde{r}_{k+1} + \beta_k \widetilde{p}_k;$

$k \ = k + 1$

end;

$x^* = x_k.$

References

[Brezn–94] *C. Brezinski, M. Redivo-Zaglia*: Treatment of near-breakdown in the CGS algorithm. Numerical Algorithms *7* (1994), 33–73.

[Fletch–76] *R. Fletcher*: Conjugate gradient methods for indefinite systems. Numerical Analysis, Dundee, 1975 (G. A. Watson, ed.). Vol. 506 of Lecture Notes in Mathematics, Springer, Berlin, 1976.

[Gutkn–97] *M. H. Gutknecht*: Lanczos-type Solvers for Nonsymmetric Linear Systems of Equations. Technical Report TR-97-04, Swiss Center for Scientific Computing ETH-Zentrum, Switzerland, 1997.

[Lancz–50] *C. Lanczos*: An iteration method for the solution of eigenvalue problem of linear differential and integral operators. J. Res. Nat. Bureau Standards *45* (1950).

[Lancz–52] *C. Lanczos*: Solution of systems of linear equations by minimized iterations. J. Res. Nat. Bureau Standards *49* (1952).

[Leary–80] *D. P. O'Leary*: The block conjugate gradient algorithm. Linear Algebra Appl. *99* (1980), 293–322.

[Tichý–97] *P. Tichý*: Behaviour of BiCG and CGS algorithms. Mgr. thesis, Department of Numerical Mathematics, Faculty of Mathematics and Physics Praha, 1997.

[Weiss–95] *R. Weiss*: A theoretical overview of Krylov subspace methods. Applied Numerical Mathematics *19* (1995), 207–233.

Authors' address: *Petr Tichý*, *Jan Zítko*, Department of Numerical Mathematics, Faculty of Mathematics and Physics, Charles University of Prague, Malostranské náměstí 25, 118 00 Praha 1, Czech Republic, e-mail: `tichy@menza.mff.cuni.cz`, `zitko@ms.mff.cuni.cz`.