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RELIABLE SOLUTION OF AN ELASTO-PLASTIC REISSNER-MINDLIN BEAM FOR HENCKY'S MODEL WITH UNCERTAIN YIELD FUNCTION

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Abstract. We apply the method of reliable solutions to the bending problem for an elasto-plastic beam, considering the yield function of the von Mises type with uncertain coefficients. The compatibility method is used to find the moments and shear forces. Then we solve a maximization problem for these quantities with respect to the uncertain input data.

Keywords: elasto-plastic beams, Hencky's model of plasticity, Mindlin-Timoshenko beam, uncertain data

MSC 2000: 73K05, 73E99, 49A29

Introduction

Employing the method of reliable solutions ("worst scenario" method) for problems with uncertain input data (see [2–6]), we consider a bending problem for an elasto-plastic beam, according to the Reissner-Mindlin model in combination with Hencky's model of perfect plasticity. We assume that the yield function is uncertain, so that its coefficients are given within some intervals.

Passing to the dual variational formulation in terms of bending moments and shear forces, we obtain an analogue of the Haar-Kármán principle (cf. [8]), i.e., a modification of the Castigliano principle of minimum complementary energy. Thus the well-known compatibility method [10] can be employed to reduce the bending problem to a finite-dimensional one.

We consider the yield function of the von Mises type, i.e., a homogeneous quadratic function in terms of the components of the stress tensor deviator, with uncertain coefficients.

Let the maximal absolute value of the bending moment or of the shear force, respectively, be the main goal of computations. Then we introduce a maximization problem with respect to the uncertain coefficients, which are assumed to belong to a given compact set in $\mathbb{R}^2$. On the basis of methods of Optimal Design, the solvability of the maximization problem is proved. The theory is illustrated by a simple example of a beam with the von Mises yield function.

1. Setting of the state problem in terms of bending moments and shear forces

Let us consider a homogeneous elastic beam of the length l. The Reissner-Mindlin (or Mindlin-Timoshenko) model leads to the potential energy (see [9, 11])

$$\pi(\beta, w) = \frac{1}{2}\int_0^l EI(\beta')^2\,\mathrm{d}x + \frac{1}{2}\int_0^l kGA(w' - \beta)^2\,\mathrm{d}x - \langle f, w\rangle, \tag{1.1}$$

where E is Young's modulus, $G = E(1+\nu)^{-1}/2$, ν is Poisson's ratio, I the moment of inertia and A the area of the cross-section, k is the shear correction factor, β is the rotation angle of the cross-section and w is the deflection. Henceforth the prime denotes the derivative with respect to x. The last term denotes the work of external loads:

$$\langle f, w\rangle = \sum_{j=1}^{p} P_j w(x_j) + \int_0^l f_0(x)w(x)\,\mathrm{d}x,$$

where P_j are given constants (loads) and $f_0 \in L^1(0,l)$ a given loading, $x_j \in [0,l]$ are given points.

The *primal* problem of an *elastic* beam is to find

$$\min_{\beta\in V_1, w\in V_2} \pi(\beta, w) \tag{1.2}$$

where V_i, $i = 1, 2$, are some subspaces of the Sobolev space $H^1(\Omega)$, $\Omega = (0,l)$, corresponding to the essential boundary conditions.

Using the saddle-point approach (see e.g. [1]) or the Friedrichs transform (see e.g. [8]), we pass from the primal problem (1.2) to the *dual* problem

$$\min_{\lambda\in E(f)} \varphi(\lambda) \tag{1.3}$$

for the complementary energy

$$\varphi(\lambda) = \frac{1}{2}\int_0^l \Big(\frac{1}{EI}\lambda_1^2 + \frac{1}{kGA}\lambda_2^2\Big)\,\mathrm{d}x,$$

where

(1.4)

$$E(f) = \Big\{\lambda \in [L^2(\Omega)]^2 \colon \int_0^l (\lambda_1\beta' + \lambda_2(w' - \beta))\,\mathrm{d}x = \langle f, w\rangle \quad \forall\ (\beta, w) \in V_1 \times V_2\Big\}.$$

The solutions of the problems (1.2) and (1.3) are linked as follows:

$$\lambda_1 = EI\beta';\ \ \lambda_2 = kGA(w' - \beta),$$

so that λ_1 represents the bending moment and λ_2 the shear force.

Now let us consider Hencky's model of *elasto-plasticity*, based on the criterion

$$\mathscr{F}(\sigma) \leqslant 1 \tag{1.5}$$

with the yield function

$$\mathscr{F}(\sigma) = b_1(\sigma^D_{xx})^2 + b_2(\sigma^D_{xz})^2,$$

where b_1, b_2 are some positive constants and

$$\sigma^D_{xx} = \frac{2}{3}\sigma_{xx}, \quad \sigma^D_{xz} = \sigma_{xz}$$

are the only nonzero components of the stress deviator.

R e m a r k 1.1. In case of the von Mises yield function we have $b_1 = b_2 = 1/(2K^2)$, where K is a given constant.

Since

$$\begin{aligned}\sigma_{xx}(x,z) &= -Ez\beta' = -z\lambda_1/I, \quad z \in [-t,t],\\ \sigma_{xz}(x,z) &= kG(w' - \beta) = \lambda_2/A,\end{aligned}$$

the condition (1.5), evaluated in the extreme fibers of the cross-section, can be expressed as follows:

$$F(a,\lambda) \equiv \sum_{i=1}^{2} a_i\lambda_i^2(x) \leqslant 1 \quad \text{for a.a.} \quad x \in [0,l], \tag{1.6}$$

where $a = (a_1, a_2)$,

$$a_1 = \frac{4}{9}(t/I)^2 b_1, \quad a_2 = b_2/A^2 \tag{1.7}$$

are positive constants.

Hencky's model of perfect plasticity implies the following *analogue of the Haar-Kármán principle* (see [8—§13.2]) for the actual vector $\lambda(a) = (\lambda_1(a), \lambda_2(a))$:

$$\lambda(a) = \arg \min_{E(f)\cap \mathscr{P}(a)} \varphi(\lambda), \tag{1.8}$$

where $E(f)$ is defined in (1.4) and

$$\mathscr{P}(a) = \{\lambda \in [L^2(\Omega)]^2 \colon\ F(a, \lambda(x)) \leqslant 1 \quad \text{for a.a.} \quad x \in [0, l]\}. \tag{1.9}$$

Assume that the coefficients b_1, b_2 of the yield function are *uncertain*, as we can prescribe them in some intervals only. (The intervals may result from experimental measurements and an inverse identification problem.)

Therefore, we introduce the set of admissible coefficients

$$U_{\text{ad}} = \{a \in \mathbb{R}^2 \colon\ a^{(i)}_{\min} \leqslant a_i \leqslant a^{(i)}_{\max},\ i = 1, 2\},$$

where $a^{(i)}_{\min}$, $a^{(i)}_{\max}$ are given positive constants.

Proposition 1.1. *Assume that*

$$K(a) = E(f) \cap \mathscr{P}(a) \neq \emptyset \qquad \forall\ a \in U_{\text{ad}}. \tag{1.10}$$

Then there exists a unique solution $\lambda(a)$ of the problem (1.8) *for any $a \in U_{\text{ad}}$.*

Proof. It is easy to find that the set $E(f)$ is closed and convex in $H := [L^2(\Omega)]^2$. By the Lebesgue Theorem, we can verify that the set $\mathscr{P}(a)$ is closed in H. Since the function $F(a, \cdot)$ is convex for any $a \in U_{\text{ad}}$, the set $\mathscr{P}(a)$ is convex. Altogether, the set $K(a)$ is closed, convex and nonempty by assumption for any $a \in U_{\text{ad}}$. Moreover, $K(a)$ is bounded, since

$$\lambda \in \mathscr{P}(a) \Rightarrow \|\lambda_i\|_{L^\infty(\Omega)} \leqslant a_i^{-1/2}, \quad i = 1, 2.$$

The function φ is strictly convex, quadratic, so that it is weakly lower semicontinuous in H. As a consequence, there exists a unique minimizer in (1.8). $\square$

Theorem 1.2. *Assume that*

$$K(a_{\max}) \neq \emptyset,\ \textit{where}\ a_{\max} = (a^{(1)}_{\max}, a^{(2)}_{\max}) \tag{1.11}$$

and let

$$a^n \in U_{\text{ad}}, \quad a^n \to a \quad (\textit{in}\ \mathbb{R}^2), \quad \textit{as} \quad n \to \infty. \tag{1.12}$$

Then a unique solution $\lambda(a^n)$ of the problem (1.8) *exists for all* n *and*

$$\lambda(a^n) \to \lambda(a) \quad \textit{in} \quad [L^2(\Omega)]^2.$$

For the proof we shall need the following definition: we say that

$$K(a) = \lim_{n\to\infty} K(a^n)$$

if

(i) for any $\lambda \in K(a)$ there exists a sequence $\{\lambda^n\}$, $\lambda^n \in K(a^n)$, such that $\lambda^n \to \lambda$ in H;

(ii) if $\lambda^n \in K(a^n)$ and $\lambda^n \rightharpoonup \lambda$ (weakly) in H, then $\lambda \in K(a)$.

Lemma 1.3. *Let the assumptions (1.11), (1.12) be satisfied. Then*

$$K(a) = \lim_{n\to\infty} K(a^n).$$

Proof. *ad* (i): Let any $\lambda \in K(a)$ be given. From (1.11) we obtain that a $\lambda_0 \in E(f) \cap \mathscr{P}(a_{\max})$ exists. Since

$$\mathscr{P}(a_{\max}) \subset \mathscr{P}(a) \qquad \forall a \in U_{\mathrm{ad}},$$

$\lambda_0 \in K(a)$ for any $a \in U_{\mathrm{ad}}$. Let us denote $\bar\lambda := \lambda - \lambda_0$ and define

$$t_n = \max\{t \in [0,1] \colon \lambda_0 + t\bar\lambda \in \mathscr{P}(a^n)\}; \ \lambda^n = \lambda_0 + t_n\bar\lambda.$$

Obviously, we have $t_n \to 1$ as $n \to \infty$,

$$\begin{aligned} &\lambda^n \in \mathscr{P}(a^n), \ \lambda^n \in E(f) \quad (\text{since } \bar\lambda \in E(0)),\\ &\lambda^n \to \lambda_0 + \bar\lambda = \lambda \quad \text{in} \quad H. \end{aligned}$$

ad (ii): Let $\lambda^n \in K(a^n)$, $\lambda^n \rightharpoonup \lambda$ (weakly) in H. The set $E(f)$ is weakly closed, being closed and convex. As a consequence, $\lambda \in E(f)$.

To verify that $\lambda \in \mathscr{P}(a)$, we choose an arbitrary function $\varphi \in C_0^\infty(\Omega)$, $\varphi \geqslant 0$ in Ω and prove that

$$\int_0^l \varphi(1 - F(a,\lambda))\,\mathrm{d}x \geqslant 0. \tag{1.13}$$

In fact, from $\lambda^n \in \mathscr{P}(a^n)$ we obtain

$$\int_0^l \varphi(1 - F(a^n, \lambda^n))\,\mathrm{d}x \geqslant 0,$$

so that

$$\int_0^l \varphi\,\mathrm{d}x \geqslant \int_0^l \varphi \sum_{i=1}^2 a_i^n (\lambda_i^n)^2\,\mathrm{d}x := J_n. \tag{1.14}$$

On the other hand, we may write

$$\begin{aligned} J_n &= \int_0^l \varphi \sum_i [a_i^n(\lambda_i^n)^2 - a_i(\lambda_i^n)^2]\,\mathrm{d}x + \int_0^l \varphi \sum_i a_i(\lambda_i^n)^2\,\mathrm{d}x \\ &= I_{1n} + I_2(\lambda^n), \end{aligned} \tag{1.15}$$

$$|I_{1n}| \leqslant \sum_i |a_i^n - a_i| \int_0^l \varphi(\lambda_i^n)^2\,\mathrm{d}x \to 0 \tag{1.16}$$

since the sequence $\{\lambda^n\}$ is bounded in H and (1.12) holds. The functional $I_2(\lambda)$ is weakly lower semicontinuous, so that

$$\lim_{n\to\infty} \inf I_2(\lambda^n) \geqslant I_2(\lambda) = \int_0^l \varphi \sum_i a_i \lambda_i^2\,\mathrm{d}x. \tag{1.17}$$

Combining (1.15), (1.16) and (1.17), we arrive at

$$\liminf J_n \geqslant \lim I_{1n} + \liminf I_{2n} \geqslant \int_0^l \varphi F(a,\lambda)\,\mathrm{d}x.$$

Passing to $\liminf$ on both sides of (1.14), we obtain

$$\int_0^l \varphi\,\mathrm{d}x \geqslant \int_0^l \varphi F(a,\lambda)\,\mathrm{d}x,$$

which is (1.13). It follows from (1.13) that

$$1 \geqslant F(a,\lambda) \qquad \text{a.e. in} \quad (0,l),$$

so that $\lambda \in \mathscr{P}(a)$. $\square$

P r o o f o f T h e o r e m 1.2 is a simplified version of that for a more general Theorem 1.1 in [7]. Let us introduce the bilinear form

$$[\lambda, \mu] = D\varphi(\lambda, \mu) = \int_0^l (e_1\lambda_1\mu_1 + e_2\lambda_2\mu_2)\,\mathrm{d}x, \quad e_i = \text{const} > 0,$$

where $D\varphi(\lambda, \mu)$ denotes the Gateaux differential. Then $[\lambda, \mu]$ represents a scalar product in the space H and the problem (1.8) is equivalent to the following variational inequality for $\lambda(a) \in K(a)$:

$$[\lambda(a), \mu - \lambda(a)] \geqslant 0 \qquad \forall\ \mu \in K(a). \tag{1.18}$$

By (1.11), (1.12) and Proposition 1.1, the inequality (1.18) has a unique solution for any $a \in U_{\mathrm{ad}}$.

Denoting $\lambda^n := \lambda(a^n)$, we have $\lambda^n \in K(a^n)$ and

$$[\lambda^n, \lambda^n - \mu] \leqslant 0 \qquad \forall\ \mu \in K(a^n). \tag{1.19}$$

By (1.11), there is a $\lambda_0 \in K(a^n)$ for all n (cf. the proof of Lemma 1.3) and we have

$$\|\lambda^n\|^2 := [\lambda^n, \lambda^n] \leqslant [\lambda^n, \lambda_0] \leqslant \|\lambda^n\|\ \|\lambda_0\|.$$

Therefore $\{\lambda^n\}$ is bounded and there exist $\lambda \in H$ and a subsequence $\{\lambda^m\} \subset \{\lambda^n\}$ such that

$$\lambda^m \rightharpoonup \lambda \quad \text{(weakly) in } H \quad \text{as} \quad m \to \infty. \tag{1.20}$$

We are going to show that λ coincides with the solution $\lambda(a)$. By Lemma 1.3, $\lambda \in K(a)$ and there exists a sequence $\{\omega^n\}$, $\omega^n \in K(a^n)$, such that $\omega^n \to \lambda$ in H. Inserting $\mu := \omega^n$ into (1.19), we obtain

$$\limsup_{n\to\infty} \|\lambda^m\|^2 \leqslant \lim_{m\to\infty} [\lambda^m, \omega^m] = [\lambda, \lambda],$$

using also (1.20). On the other hand,

$$\liminf_{m\to\infty} \|\lambda^m\|^2 \geqslant \|\lambda\|^2,$$

since the functional $\|\lambda\|^2$ is weakly lower semicontinuous. Combining these results, we arrive at

$$\lim_{m\to\infty} \|\lambda^m\|^2 = \|\lambda\|^2. \tag{1.21}$$

Given any $\mu \in K(a)$, by Lemma 1.3 there exists a sequence $\{\mu^n\}$, $\mu^n \in K(a^n)$, such that $\mu^n \to \mu$ in H. Inserting the subsequence $\{\mu^m\}$ into (1.19) and using (1.21), we may write

$$[\lambda, \lambda] = \lim[\lambda^m, \lambda^m] \leqslant \lim[\lambda^m, \mu^m] = [\lambda, \mu]$$

so that

$$[\lambda, \lambda - \mu] \leqslant 0 \qquad \forall\, \mu \in K(a).$$

Since the solution $\lambda(a)$ of (1.18) is unique by Proposition 1.1, $\lambda = \lambda(a)$ and the whole sequence $\{\lambda^n\}$ tends weakly to $\lambda(a)$ in H.

To verify the strong convergence, we realize that

$$\|\lambda^n\|^2 \to \|\lambda\|^2 \quad \text{as} \quad n \to \infty, \tag{1.22}$$

as the proof of (1.21) can be repeated for the whole sequence $\{\lambda^n\}$. Then

$$\|\lambda^n - \lambda\|^2 = \|\lambda^n\|^2 - 2[\lambda^n, \lambda] + \|\lambda\|^2 \to 0$$

follows from (1.22) and the weak convergence. □

2. A maximization problem with respect to uncertain yield function

Let us assume that the main goal of computations is a functional $\Phi(a, \lambda(a))$. For instance, the maximal absolute value of the bending moment or of the shear force, respectively, over the whole interval $[0, l]$ can be sought. Then we formulate the following Maximization Problem: find

$$a^0 = \arg\max_{a \in U_{\mathrm{ad}}} \Phi(a, \lambda(a)). \tag{2.1}$$

In this way a "worst scenario" approach is adopted, since we want to be always "on a safe side", taking into account the possible error in the determination of the input data $a \in U_{\mathrm{ad}}$.

Theorem 2.1. *If $K(a_{max}) \neq \emptyset$ and if the function $a \to \Phi(a, \lambda(a))$ is continuous in U_{ad}, then there exists at least one solution of the Maximization Problem* (2.1).

P r o o f. Let $\{a^n\}$, $a^n \in U_{\mathrm{ad}}$, be a maximizing sequence, i.e.,

$$\lim_{n\to\infty} \Phi(a^n, \lambda(a^n)) = \max_{a \in U_{\mathrm{ad}}} \Phi(a, \lambda(a)). \tag{2.2}$$

Since U_{ad} is compact in $\mathbb{R}^2$, there exist $a^0 \in U_{\mathrm{ad}}$ and a subsequence $\{a^m\}$ such that $a^m \to a^0$ as $m \to \infty$. By assumption,

$$\Phi(a^m, \lambda(a^m)) \to \Phi(a^0, \lambda(a^0)). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$\Phi(a^0, \lambda(a^0)) = \max_{a \in U_{\mathrm{ad}}} \Phi(a, \lambda(a)),$$

so that a^0 is a solution of the problem (2.1). □

Example 2.1. If $\lambda_1 \in C([0,l])$, we can define

$$\Phi_1(\lambda) = \max_{1 \leqslant i \leqslant N} |\lambda_1(X_i)|,$$

where $X_i \in [0,l]$ are some points chosen a priori (e.g., the places where the maximum of $|\lambda_1(x)|$ over $[0,l]$ could be attained).

Let us consider a statically indeterminate beam with a finite number of redundancies. Using the well-known Compatibility Method (see e.g. [10]), we find

$$\lambda(a) \in E(f) \Rightarrow \lambda_1(a) = \sum_{j=1}^{J} C_j(a)\varphi_j + \lambda_1(f). \tag{2.4}$$

Here $\lambda_1(f)$ and φ_j denote the bending moment on a reference statically determinate beam, corresponding to the loading f and to the unit redundance $C_j(a) = 1$, respectively; $C_j(a)$ are constants to be determined. Assume that the functions $\{\varphi_j\}$, $j = 1, \ldots, J$ are linearly independent on $[0,l]$, $\varphi_j \in C([0,l])$.

If $a^n \to a$ in $\mathbb{R}^2$, $a^n \in U_{\mathrm{ad}}$, we may write

$$\begin{aligned} \|\lambda_1(a^n) - \lambda_1(a)\|^2_{L^2(\Omega)} &= \Big\| \sum_{j=1}^{J} \alpha_j^n \varphi_j \Big\|^2_{L^2(\Omega)} = \sum_{i,j=1}^{J} G_{ij}\alpha_i^n \alpha_j^n \\ &\geqslant g_0 \sum_{j=1}^{J} (\alpha_j^n)^2, \end{aligned} \tag{2.5}$$

where

$$\alpha_j^n := C_j(a^n) - C_j(a), \quad G_{ij} = \int_0^l \varphi_i \varphi_j \,\mathrm{d}x, \quad g_0 = \mathrm{const} > 0.$$

By Theorem 1.2 (assume that $K(a_{\max}) \neq \emptyset$), the left-hand side of (2.5) tends to zero as $n \to \infty$, so that

$$\sum_{j=1}^{J} (\alpha_j^n)^2 \to 0 \quad \text{and} \quad C_j(a^n) \to C_j(a) \tag{2.6}$$

follows. As a consequence of (2.4) and (2.6), the function $a \mapsto |\lambda_1(a)(X_i)|$ is continuous in U_{ad} for all $i \leqslant N$. Next, we have

$$\lim_{n\to\infty} \Phi_1(\lambda(a^n)) = \lim_{n\to\infty} \max_{i\leqslant N} |\lambda_1(a^n)(X_i)|$$
$$= \max_{i\leqslant N} \lim_{n\to\infty} |\lambda_1(a^n)(X_i)| = \max_{i\leqslant N} |\lambda_1(a)(X_i)| = \Phi_1(\lambda(a)),$$

i.e., the function $a \mapsto \Phi_1(\lambda(a))$ is continuous in U_{ad}. By Theorem 2.1, there exists at least one solution of the Maximization Problem (2.1) for $\Phi(a,\lambda) := \Phi_1(\lambda)$.

Example 2.2. If λ_2 is piecewise continuous, we can define

$$\Phi_2(\lambda) = \max_{1\leqslant i\leqslant N} \max\{|\lambda_2(x_i+)|; |\lambda_2(x_i-)|\},$$

where x_i are the points where the pointwise loads or reaction forces act; (x_i+) and (x_i-) denote the limits for $x \to x_i+$ and $x \to x_i-$, respectively.

Considering the same beam as in Example 1, we use the Compatibility Method to obtain

$$\lambda_2(a) = \sum_{k=1}^{K} C_k(a)\psi_k + \lambda_2(f),$$

where $\lambda_2(f)$ and ψ_k denote the shear force on a reference beam for the load f and the unit redundance force $C_k(a) = 1$, respectively. Assuming that $\{\psi_k\}_{k=1}^K$ are linearly independent on $[0,l]$ and piecewise continuous, we may argue like in Example 1 to verify the continuity of $C_k(a)$, $k = 1,\dots,K$ in U_{ad}. As a consequence, the continuity of $\lambda_2(a)(x_i\pm)$ and $\Phi_2(\lambda(a))$ in U_{ad} follows. By Theorem 2.1, the Maximization Problem (2.1) with $\Phi \equiv \Phi_2$ has at least one solution.

3. A numerical example for the von Mises yield function

Let us consider a cantilever beam completed with a simple support on the free end under a unit pointwise load at the center. Taking the cantilever for the reference configuration and setting $\Omega = (0,2)$, we find

$$\lambda_1(f) = (x-1)^+, \quad \lambda_2(f) = -H(x-1)$$

where $H(\cdot)$ denotes the Heaviside function. If C denotes the reaction force at $x = 0$ (the single redundancy), then

$$\lambda \in E(f) \Rightarrow \lambda_1 = -Cx + (x-1)^+, \quad \lambda_2 = C - H(x-1). \tag{3.1}$$

R e m a r k 3.1. Recalling Examples 2.1 and 2.2, we realize that $J = 1$, $\varphi_1(x) = -x$, $K = 1$, $\psi_1(x) = 1$. Here $X_1 = 1$ and $X_2 = 2$, $N = 2$ should be chosen in the definition of Φ_i, $i = 1,2$.

Next, we find that

$$\lambda \in \mathscr{P}(a) \Leftrightarrow 1 \geqslant a_1(-Cx+(x-1)^+)^2 + a_2(C-H(x-1))^2 \quad \text{for a.a.} \quad x \in [0,2]. \tag{3.2}$$

This condition is equivalent to the following one:

$$\max\{(a_1+a_2)C^2;\ a_1 \max[C^2; 4(C-\tfrac{1}{2})^2] + a_2(C-1)^2\} \leqslant 1. \tag{3.3}$$

To simplify the analysis, we will assume henceforth that

$$a_1 = a_2\kappa, \quad \kappa = \text{const} > 0. \tag{3.4}$$

This assumption corresponds to the *von Mises yield function*, i.e., to $b_1 = b_2$ in the formulae (1.5) (cf. Remark 1.1) and (1.7). For a rectangular cross-section of the height $2t$ we obtain $\kappa = (2/t)^2$.

Let us set

$$\kappa = 100 \tag{3.5}$$

and $a := a_2$, $U_{\mathrm{ad}} = [a_{\min}, a_{\max}]$ in what follows. Then the condition (3.3) reduces to the inequality

$$\max\{101C^2;\ 100\max[C^2; 4(C-\tfrac{1}{2})^2] + (C-1)^2\} \leqslant 1/a. \tag{3.6}$$

Obviously, the resulting $\lambda_1(a)$ should be positive at the point $x = 2$. Therefore, we restrict ourselves to the interval $C < 1/2$.

By a thorough analysis, we find that (3.6) and $C < 1/2$ hold iff $C \in \mathscr{K}(a)$, where

$$\begin{aligned} &\mathscr{K}(a) = [c(a), \mathscr{G}(a)] \quad \text{for} \quad a_0 \leqslant a \leqslant \bar{a}, \\ &\mathscr{K}(a) = \emptyset \quad \text{for} \quad \bar{a} < a, \end{aligned}$$

where $a_0 = 0.0396604$, $\bar{a} = 0.086538$,

$$\begin{aligned} c(a) &= 201/401 - (4.01/a - 1)^{1/2}/40.1, \\ \mathscr{G}(a) &= 1/101 + (1.01/a - 1)^{1/2}/10.1 \quad \text{for} \quad a_0 \leqslant a \leqslant \underline{a} = 0.085682, \\ \mathscr{G}(a) &= 1/3 \quad \text{for} \quad \underline{a} < a \leqslant \bar{a}. \end{aligned}$$

Inserting the formulae (3.1) into (1.8), the problem (1.8) can be reduced to the following one: find

$$C(a) = \arg \min_{C \in \mathscr{K}(a)} S(C), \tag{3.7}$$

where

$$\begin{aligned} S(C) &= (8e_1/3 + 2e_2)C^2 - (5e_1/3 + 2e_2)C, \\ e_1 &= (EI)^{-1}, \quad e_2 = (kGA)^{-1} \quad (\text{cf. } (1.3)). \end{aligned} \tag{3.8}$$

Let

$$C_e = \frac{\frac{5}{3}e_1 + 2e_2}{\frac{16}{3}e_1 + 4e_2}$$

denote the minimizer of S over the whole interval $(-\infty, \infty)$. This parameter corresponds to the pure *elastic* Reissner-Mindlin (Mindlin-Timoshenko) model.

Remark 3.1. Note that always $C_e < 1/2$. If the Poisson's ratio is 0.25 and the shear correction factor $k = 5/6$ (cf. [11]), we obtain

$$e_1/e_2 = \kappa/4 = 25, \quad C_e = 0.31796.$$

Using the classical Bernoulli model of beam bending, we have $e_2 \equiv 0$ and $C_e = 0.3125$, independent of Poisson's ratio.

Henceforth we assume that

$$a_0 \leqslant a_{\min} < a_{\max} \leqslant \bar{a},$$

which implies $K(a_{\max}) \neq \emptyset$ (cf. (1.11)).

We shall distinguish the following three cases:

(i) $C_e = 1/3$.
Then $C(a) = C_e = 1/3$, independent of $a \in U_{\mathrm{ad}}$, since $1/3 \in \mathscr{K}(a)$ for all $a \in U_{\mathrm{ad}}$.

(ii) $C_e > 1/3$.
Then denoting $a_g = \mathscr{G}^{-1}(C_e)$, we find that: if $a_{\min} \leqslant a_g \leqslant a_{\max}$,

$$\begin{aligned} &C(a) = C_e \quad \text{for} \quad a \in [a_{\min}, a_g] \\ &C(a) = \max\{\mathscr{G}(a); 1/3\} \quad \text{for} \quad a \in]a_g, a_{\max}]; \end{aligned}$$

if $a_g < a_{\min}$, $C(a) = \max\{\mathscr{G}(a), 1/3\}$; if $a_{\max} < a_g$, $C(a) = C_e$.

(iii) $C_e < 1/3$.
Denoting $a_e = c^{-1}(C_e)$, we find: if $a_{\min} \leqslant a_e \leqslant a_{\max}$

$$\begin{aligned} &C(a) = C_e \quad \text{for} \quad a \in [a_{\min}, a_e] \\ &C(a) = c(a) \quad \text{for} \quad a \in]a_e, a_{\max}]; \end{aligned}$$

if $a_e < a_{\min}$, $C(a) = c(a)$; if $a_{\max} < a_e$, $C(a) = C_e$.

Let us consider the criterion of Example 2.1, i.e.,

$$\Phi_1(\lambda(a)) = \max_{1 \leqslant i \leqslant 2} |\lambda_1(a)(X_i)|, \quad X_1 = 1, \quad X_2 = 2.$$

Since

$$0 < c(a_0) \leqslant C(a) < 1/2,$$

we have

$$\Phi_1(\lambda(a)) = \max\{C(a); 1 - 2C(a)\}$$

and we may write

$$\max_{a \in U_{\mathrm{ad}}} \Phi_1(\lambda(a)) = \max\Big\{\max_{U_{\mathrm{ad}}} C(a); 1 - 2 \min_{U_{\mathrm{ad}}} C(a)\Big\}.$$

The case (i) with $C_e = 1/3$ being uninteresting, we arrive at the following result:

(ii) $C_e > 1/3$

$$\max_{a\in U_{\rm ad}} \Phi_1(\lambda(a)) = \begin{cases} C_e & \text{if} \quad a_{\min} \leqslant a_g, \\ \max\{\mathscr{G}(a_{\min}); 1/3\} & \text{if} \quad a_g < a_{\min}; \end{cases}$$

(iii) $C_e < 1/3$

$$\max_{a\in U_{\rm ad}} \Phi_1(\lambda(a)) = \begin{cases} 1-2C_e & \text{if} \quad a_{\min} \leqslant a_e, \\ 1-2c(a_{\min}) & \text{if} \quad a_e < a_{\min}. \end{cases}$$

In both cases (ii) and (iii) the "greatest" maximal elasto-plastic bending moment for $a_g < a_{\min}$ or $a_e < a_{\min}$ is less than the elastic one. If $C_e > 1/3$, the maximal moment is under the pointwise load, whereas if $C_e < 1/3$, it is at the clamped end of the beam.

For Example 2.2, we obtain the following result:

(ii) $C_e > 1/3$

$$\max_{a\in U_{\rm ad}} \Phi_2(\lambda(a)) = \begin{cases} 1-\max\{\mathscr{G}(a_{\max}); 1/3\} & \text{if} \quad a_g \leqslant a_{\max}, \\ 1-C_e & \text{if} \quad a_g > a_{\max}; \end{cases}$$

(iii) $C_e < 1/3$

$$\max_{a\in U_{\rm ad}} \Phi_2(\lambda(a)) = \begin{cases} 1-C_e & \text{if} \quad a_{\min} \leqslant a_e, \\ 1-c(a_{\min}) & \text{if} \quad a_e < a_{\min}. \end{cases}$$

Here the "greatest" maximal elasto-plastic shear force is less than the elastic one iff $C_e < 1/3$, whereas the opposite holds if $C_e > 1/3$.

R e m a r k. The case $C_e \geqslant 1/3$, however, is not realistic. Indeed, it occurs if and only if

$$(*) \qquad \frac{e_1}{e_2} = \frac{k}{2(1+\nu)}\frac{A}{I} \leqslant 6.$$

Considering the real beams with ratio $1/t > 10$, the condition $(*)$ can be satisfied only for cross-sections of the I-form with either too thin web or too small shear correction factor k.

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