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DOMINANT EIGENVALUE PROBLEM FOR POSITIVE INTEGRAL OPERATORS AND ITS SOLUTION BY MONTE CARLO METHOD

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Abstract. In this paper, a method of numerical solution to the dominant eigenvalue problem for positive integral operators is presented. This method is based on results of the theory of positive operators developed by Krein and Rutman. The problem is solved by Monte Carlo method constructing random variables in such a way that differences between results obtained and the exact ones would be arbitrarily small. Some numerical results are shown.

Keywords: Monte Carlo method, integral operators, positive operators

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INTRODUCTION

Let $L(\mathbf{M})$ be a Banach space of real functions defined on a set $\mathbf{M} \subset \mathbf{E}_n$, $\mathbf{K}$ a cone of nonnegative functions belonging to $L(\mathbf{M})$ with $\emptyset \neq \mathbf{K}^0$ (the interior of $\mathbf{K}$). Consider linear integral operators $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}\colon\ L(\mathbf{M}) \to L(\mathbf{M})$, $\mathbf{B}\colon\ L(\mathbf{M}) \to L(\mathbf{M})$,

$$\mathbf{A}\Phi = \int_{\mathbf{M}} A(x,y)\Phi(y)\,\mathrm{d}y, \qquad \mathbf{B}\Phi = \int_{\mathbf{M}} B(x,y)\Phi(y)\,\mathrm{d}y, \qquad \Phi \in L(\mathbf{M})$$

with nonnegative kernels $A(x,y)$, $B(x,y)$ and suppose that

(i) $\mathbf{A}$ and $\mathbf{B}$ are bounded, $\mathbf{AK} \subset \mathbf{K}$, $\mathbf{BK} \subset \mathbf{K}$ while for any $\Phi \in \mathbf{K} \setminus \mathbf{K}^0$ there is a positive integer $m = m(\Phi)$ such that $\mathbf{B}^m\Phi \in \mathbf{K}^0$;

(ii) $\mathbf{BA}^n\mathbf{B}$ are compact for any $n = 0, 1, 2, \ldots$;

(iii) spectral radii $r(\mathbf{A})$ and $r(\mathbf{B})$ of operators $\mathbf{A}$ and $\mathbf{B}$ satisfy the inequalities $r(\mathbf{A}) < 1$ and $r(\mathbf{B}) > 0$.

Next, let us consider an operator $\mathbf{T}\colon\ L(\mathbf{M}) \to L(\mathbf{M})$,

$$\mathbf{T} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \sum_{i=0}^{\infty} \mathbf{A}^i\mathbf{B} \tag{1}$$

and let us search for an eigensolution to the problem

$$\lambda\Phi = \mathbf{T}\Phi \tag{2}$$

such that the eigenvalue λ is dominant.

Clearly, by assumptions (i)–(iii) $\mathbf{T}^2$ is a positive compact operator while the spectral radius $r(\mathbf{T})$ of the operator $\mathbf{T}$ is positive. So the theory of Krein and Rutman [1] can be employed according to which the following assertion is true:

There is one and only one eigenfunction φ_0, $\|\varphi_0\| = 1$ to the problem (2) *($\|\cdot\|$ means the norm of the space $L(\mathbf{M})$). This solution is nonnegative, the eigenvalue λ_0 corresponding to it being positive and $\lambda_0 = r(\mathbf{T})$. Next, for any function $F \in L(\mathbf{M})$ and any positive integer n, the relation*

$$\mathbf{T}^n F = \lambda_0^n \varphi_0 \langle \psi_0, F\rangle + \mathbf{T}_1^n F \tag{3}$$

holds where ψ_0 is a positive eigensolution to the problem adjoint to the problem (2) *(for $\lambda = \lambda_0$), $\langle \psi_0, F\rangle \equiv \psi_0(F)$ and $\mathbf{T}_1$ is a linear operator such that*

$$r(\mathbf{T}_1) < \lambda_0.$$

Usually, the problem (2) is solved iteratively as follows: Choose a nonnegative element $f \in L(\mathbf{M})$, $\|f\| = 1$. Set $\varphi_0 = f$, $\alpha_0 = 1$ and calculate

$$\varphi_n = \mathbf{T}\varphi_{n-1}/\|\mathbf{T}\varphi_{n-1}\|, \qquad \alpha_n = \|\mathbf{T}\varphi_{n-1}\| \tag{4}$$

recurrently for $n = 1, 2, 3, \dots$

Under the conditions (i)–(iii), it follows from (3) that the process (4) is convergent,

$$\lim_{n\to\infty} \alpha_n = \lambda_0 \qquad \text{and} \qquad \lim_{n\to\infty} \varphi_n = \varphi_0.$$

The use of this iteration process to the numerical solution of the problem (2) brings some difficulties:

- the function $\mathbf{T}\varphi_n$ and the number $\|\mathbf{T}\varphi_n\|$ must be expressed numerically in any step n

- the usual form the operator $\mathbf{T}$ is an infinite series of integral terms (see (1)) and, moreover, we cannot expect that the expression in digits of each of them is accurate.
- by (4), inaccuracy of computation in step n transfers to step $n+1$ and, unfortunately, it need not decrease as n tends to infinity.

Of course, (3) implies

$$\lambda_0 = \lim_{n\to\infty} \|\mathbf{T}^n F\|^{1/n}, \qquad \varphi_0(x) = \lim_{n\to\infty} (\mathbf{T}^n F)(x)/\|\mathbf{T}^n F\|. \tag{5}$$

In principle, by (5) the quantities λ_0 and $\varphi_0(x)$ can be determined with accuracy arbitrarily large (depending on the number n chosen). But in practice we cannot avoid the difficulty of expressing formulae (5) numerically with sufficient accuracy if n is large. In such a case, the use of the Monte Carlo method can be advantageous.

Random sequences and the solution to the problem

Let g be a nontrivial real nonnegative function such that the inequality

$$\int_{\mathbf{M}} \mathrm{d}x g(x)|\varphi(x)| < \infty$$

holds for any $\varphi \in L(\mathbf{M})$. Then by (3)

$$\lambda_0 = \lim_{n\to\infty} \left(\int_{\mathbf{M}} \mathrm{d}x g(x)(\mathbf{T}^n F)(x) \right)^{1/n} \tag{6}$$

and

$$\int_{\mathbf{M}} \mathrm{d}x g(x)\varphi_0(x) = \lim_{n\to\infty} \int_{\mathbf{M}} \mathrm{d}x g(x)(\mathbf{T}^n F)(x)/\|\mathbf{T}^n F\|. \tag{7}$$

The random sequence of points of the set $\mathbf{M}$ is defined as follows: The first point x_1 is chosen randomly according to the probability density p_{in},

$$\int_{\mathbf{M}} \mathrm{d}x p_{in}(x) = 1.$$

The next point x_2 is chosen randomly according to the density $p(x_1, x')$ of the transition probability (i.e. $p(x, x')\,\mathrm{d}x'$ is the probability that if x is a point of the random sequence then the next one will belong to the neighbourhood $\mathrm{d}x'$ of the point x'). Similarly, the point x_3 is chosen randomly according to the probability density

$p(x_2, x'), \ldots$ etc. Termination of the random sequence at a point y (point of termination) is governed by the termination probability $p_t(y)$. For any $x \in \mathbf{M}$ the quantities p and p_t are bounded by the equation

$$p_t(x) + \int_{\mathbf{M}} \mathrm{d}y p(x,y) = 1.$$

A random sequence terminated at a point z can be prolonged. This possibility is governed by the probability density $q(z,z')$ of the prolongation ($q(z,z')\,\mathrm{d}z'$ is the probability that the random sequence terminated at the point z is prolonged and its next point ν (point of prolongation) belongs to the neighbourhood $\mathrm{d}z'$ of z'). In general,

$$\int_{\mathbf{M}} \mathrm{d}y q(x,y) \leqslant 1 \quad \text{for all} \quad x \in \mathbf{M}.$$

So the random sequence has the form

$$\alpha^m \equiv \{\beta^1, \beta^2, \ldots, \beta^m\}$$

where $m > 0$ is an integer, β^i, $i = 1, 2, \ldots, m$ are subsequences of α^m,

$$\beta^i \equiv \{x_1^i, x_2^i, \ldots, x_{n(i)}^i\}$$

while $x_j^i \in \mathbf{M}$, $j = 1, 2, \ldots, n(i)$, are points of the random sequence. In particular, x_1^1 denotes the first point, x_1^i ($i = 2, 3, \ldots, m$) are the points of prolongation and $x_{n(i)}^i$ ($i = 1, 2, \ldots, m$) the points of termination.

From now on, we will suppose that the number of terms of any subsequence β^i, $i = 1, \ldots, m$, is finite with probability 1, i.e.

$$\lim_{n\to\infty} \int_{\mathbf{M}} \mathrm{d}x_1 \ldots \int_{\mathbf{M}} \mathrm{d}x_n p(x, x_1) \prod_{j=1}^{n-1} p(x_j, x_{j+1}) = 0 \tag{9}$$

for any $x \in \mathbf{M}$.

Using relations (8) and (9), we get the identity

$$\begin{aligned} p_t(x) &+ \int_{\mathbf{M}} \mathrm{d}y_1 p(x,y_1) p_t(y_1) + \int_{\mathbf{M}} \mathrm{d}y_1 p(x,y_1) \int_{\mathbf{M}} \mathrm{d}y_2 p(y_1,y_2) p_t(y_2) + \ldots \\ &= 1 - \int_{\mathbf{M}} \mathrm{d}y_1 p(x,y_1) + \int_{\mathbf{M}} \mathrm{d}y_1 p(x,y_1) \\ &\quad - \int_{M} \mathrm{d}y_1 p(x,y_1) \int_{\mathbf{M}} \mathrm{d}y_2 p(y_1,y_2) + \ldots = 1. \end{aligned} \tag{10}$$

Let G be a nonnegative nontrivial element of $L(\mathbf{M})$.

Considering the sequence α^m define numbers $\zeta^{(m)}$ and $\eta^{(m)}$,

$$\zeta^{(m)} = H(x_1^1) \prod_{i=1}^{m-1} \Big(\prod_{i=1}^{n(i)-1} u(x_j^i, x_{j+1}^i) \Big) w(x_{n(i)}^i, x_1^{i+1}) \times \Big(\prod_{k=1}^{n(m)-1} u(x_k^m, x_{k+1}^m) \Big) h(x_{n(m)}^m), \tag{11}$$

$$\eta^{(m)} = H(x_1^1) \prod_{i=1}^{m-1} \Big(\prod_{j=1}^{n(i)-1} u(x_j^i, x_{j+1}^i) \Big) w(x_{n(i)}^i, x_1^{i+1}) \times \sum_{k=1}^{n(m)} \Big(\prod_{l=1}^{k-1} u(x_l^m, x_{l+1}^m) \Big) g(x_k^m), \tag{12}$$

where it is understood that

$$\prod_{i=j}^{k} f(x_i) \equiv 1 \quad \text{for} \quad j > k.$$

and where

$$H(x) = \begin{cases} G(x)/p_{in}(x) & \\ 0 & \text{if} \quad p_{in}(x) = 0, \end{cases}$$

$$u(x,y) = \begin{cases} A(x,y)/p(x,y) & \\ 0 & \text{if} \quad p(x,y) = 0, \end{cases}$$

$$w(x,y) = \begin{cases} B(x,y)/q(x,y) & \\ 0 & \text{if} \quad q(x,y) = 0 \end{cases}$$

and

$$h(x) = \begin{cases} g(x)/p_t(x) & \\ 0 & \text{if} \quad p_t(x) = 0. \end{cases}$$

Clearly, if to any sequence α^m a number $\zeta^{(m)}(\eta^{(m)})$ is attached by formula (11), ((12)) we have defined a random variable $\zeta^m(\eta^m)$ as a real function the domain of which is the set of all random sequences α^m.

Let us simulate the behaviour of these random variables by N mutually independent trials. Each of the trials consists of the construction of a random sequence α_i^m and of the computation of the number $\zeta^m(\alpha_i^m)(\eta^m(\alpha_i^m))$, $i = 1, \dots, N$, according to (11) ((12)).

Denote by $\mathbf{E}\zeta^m(\mathbf{E}\eta^m)$ the expectation and by $\mathbf{D}\zeta^m(\mathbf{D}\eta^m)$ the dispersion of the random variable $\zeta^m(\eta^m)$.

Theorem 1. *Let the following implications hold:*

$$G(x) \neq 0 \Rightarrow p_{in}(x) \neq 0,$$
$$g(x) \neq 0 \Rightarrow p_t(x) \neq 0$$

and

$$A(x,y) \neq 0 \Rightarrow p(x,y) \neq 0,$$
$$B(x,y) \neq 0 \Rightarrow q(x,y) \neq 0.$$

Then, for any $\delta > 0$,

$$\lim_{N\to\infty} P\Bigg(\sqrt[m-1]{\int_{\mathbf{M}} \mathrm{d}x g(x)\varphi(x) + \delta} \geqslant \sqrt[m-1]{\sum_{i=1}^{N} \zeta^m(\alpha_i^m)/N} \geqslant \sqrt[m-1]{\left|\int_{\mathbf{M}} \mathrm{d}x g(x)\varphi(x) - \delta\right|}\Bigg) = 1 \tag{13}$$

where P denotes the probability and $\varphi \equiv \mathbf{T}^{m-1}(\mathbf{I}-\mathbf{A})^{-1}G$. If, moreover, $\mathbf{D}\zeta^m < \infty$ then, for any $x > 0$,

$$\lim_{N\to\infty} P\Bigg(\sqrt[m-1]{\int_{\mathbf{M}} \mathrm{d}y g(y)\varphi(y) + x\sqrt{\mathbf{D}\zeta^m/N}} > \sqrt[m-1]{\sum_{i=1}^{N} \zeta^m(\alpha_i^m)/N} > \sqrt[m-1]{\left|\int_{\mathbf{M}} \mathrm{d}y g(y)\varphi(y) - x\sqrt{\mathbf{D}\zeta^m/N}\right|}\Bigg) = \sqrt{2/\pi}\int_0^x \mathrm{d}t \exp(-t^2/2). \tag{14}$$

P r o o f. We can write

$$\mathbf{E}\zeta^m = \sum_{\alpha^m} P(\alpha^m)\zeta^m(\alpha^m)$$

where the summation is taken over all random sequences corresponding to a given m, $P(\alpha^m)$ is the probability of the occurrence of the random sequence α^m.

Expressing $P(\alpha^m)$ by P_{in}, p, q, p_t and using (11) for $\zeta^m(\alpha^m)$ we get

(15)

$$\mathbf{E}\zeta^m = \sum_{n(1)=1}^{\infty} \cdots \sum_{n(m)=1}^{\infty} \int_{\mathbf{M}} \mathrm{d}x_1^1 \ldots \int_{\mathbf{M}} \mathrm{d}x_{n(1)}^1 \cdots \int_{\mathbf{M}} \mathrm{d}x_1^m \ldots \int_{\mathbf{M}} \mathrm{d}x_{n(m)}^m$$

$$p_{in}(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} p(x_j^i, x_{j+1}^i) \right) q(x_{n(i)}^i, x_1^{i+1}) \left(\prod_{k=1}^{n(m)-1} p(x_k^m, x_{k+1}^m) \right) p_t(x_{n(m)}^m) \zeta^m(\alpha^m)$$

$$= \sum_{n(1)=1}^{\infty} \cdots \sum_{n(m)=1}^{\infty} \int_{\mathbf{M}} \mathrm{d}x_1^1 \ldots \int_{\mathbf{M}} \mathrm{d}x_{n(1)}^1 \cdots \int_{\mathbf{M}} \mathrm{d}x_1^m \ldots \int_{\mathbf{M}} \mathrm{d}x_{n(m)}^m$$

$$G(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} A(x_j^i, x_{j+1}^i) \right) B(x_{n(i)}^i, x^{i+1}) \left(\prod_{k=1}^{n(m)-1} A(x_k^m, x_{k+1}^m) \right) g(x_{n(m)}^m)$$

$$= \int_{\mathbf{M}} \mathrm{d}x g(x) (\mathbf{T}^{m-1} (\mathbf{I} - \mathbf{A})^{-1} G)(x).$$

But, by assumptions (i)–(iii),

$$\int_{\mathbf{M}} \mathrm{d}x g(x) \varphi(x) \equiv \int_{\mathbf{M}} \mathrm{d}x g(x) (\mathbf{T}^{m-1} (\mathbf{I} - \mathbf{A})^{-1} G)(x) < \infty \tag{16}$$

so that, according to the Kchinchin theorem ([2], §32), the relation

$$\lim_{N \to \infty} P\left(\left| \sum_{i=1}^{N} \zeta^m(\alpha_i^m)/N - \mathbf{E}\zeta^m \right| \leqslant \delta \right) = 1 \tag{17}$$

holds for any $\delta > 0$.

In the case $\mathbf{D}\zeta^m < \infty$, Lyapunov's theorem ([2], §42) can be applied by which

$$\lim_{N \to \infty} P\left(\left| \sum_{i=1}^{N} \zeta^m(\alpha_i^m)/N - \mathbf{E}\zeta^m \right| < x \sqrt{\mathbf{D}\zeta^m/N} \right) = \sqrt{2/\pi} \int_0^x \mathrm{d}t \exp(-t^2/2), \qquad x \in (0, \infty). \tag{18}$$

Obviously, the relation (13) follows from (15), (16) and (17) while the relation (14) is a consequence of (15), (16) and (18). □

N o t e 1. Condition $\mathbf{D}\zeta^m < \infty$ of the theorem can be fulfilled if, for example, the inequalities

$$C_1 p_{in}(x) \geqslant G(x), \;\; C_2 p(x,y) \geqslant A(x,y), \;\; C_3 q(x,y) \geqslant B(x,y) \text{ and } C_4 p_t(x) \geqslant g(x)$$

hold where C_i, $i = 1, 2, 3, 4$ are finite positive constants, $C_2 \leqslant 1$. Indeed, according to (11) and the assumption of the theorem, we have

$$\zeta^{(m)} = C_1 \left[\prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} C_2 \right) C_3 \right] \left(\prod_{k=1}^{n(m)-1} C_2 \right) C_4 \leqslant C_1 C_4 C_3^{m-1}$$

and therefore, by (16),

$$\mathbf{D}\zeta^m \equiv \mathbf{E}(\zeta^m)^2 - (\mathbf{E}\zeta^m)^2 \leqslant C_1 C_4 C_3^{m-1} \mathbf{E}\zeta^m - (\mathbf{E}\zeta^m)^2 < \infty.$$

Theorem 2. *Let the following implications hold:*

$$G(x) \neq 0 \Rightarrow p_{in}(x) \neq 0,\ A(x,y) \neq 0 \Rightarrow p(x,y) \neq 0 \text{ and } B(x,y) \neq 0 \Rightarrow q(x,y) \neq 0.$$

Then, for any $\delta > 0$,

$$\lim_{N\to\infty} P\left(\sqrt[m-1]{\int_{\mathbf{M}} \mathrm{d}x g(x)\varphi(x) + \delta} \geqslant \sqrt[m-1]{\sum_{i=1}^{N} \eta^m(\alpha_i^m)/N} \right.$$
$$\left. \geqslant \sqrt[m-1]{\left| \int_{\mathbf{M}} \mathrm{d}x g(x)\varphi(x) - \delta \right|} \right) = 1.$$

If, moreover, $\mathbf{D}\eta^m < \infty$ *then, for any* $x > 0$,

$$\lim_{N\to\infty} P\left(\sqrt[m-1]{\int_{\mathbf{M}} \mathrm{d}y g(y)\varphi(y) + x\sqrt{\mathbf{D}\eta^m/N}} > \sqrt[m-1]{\sum_{i=1}^{N} \eta^m(\alpha_i^m)/N} \right.$$
$$\left. > \sqrt[m-1]{\left| \int_{\mathbf{M}} \mathrm{d}y g(y)\varphi(y) - x\sqrt{\mathbf{D}\eta^m/N} \right|} \right)$$
$$= \sqrt{2/\pi} \int_0^x \mathrm{d}t \exp(-t^2/2).$$

P r o o f. Using the definition (12) and the identity (10) we can write

$$\mathbf{E}\eta^m = \sum_{n(1)=1}^{\infty} \dots \sum_{n(m)=1}^{\infty} \int_{\mathbf{M}} \mathrm{d}x_1^1 \dots \int_{\mathbf{M}} \mathrm{d}x_{n(1)}^1 \dots \int_{\mathbf{M}} \mathrm{d}x_1^m \dots \int_{\mathbf{M}} \mathrm{d}x_{n(m)}^m$$

$$p_{in}(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} p(x_j^i, x_{j+1}^i) \right) q(x_{n(i)}^i, x_1^{i+1}) \left(\prod_{k=1}^{n(m)-1} p(x_k^m, x_{k+1}^m) \right) p_t(x_{n(m)}^m)$$

$$H(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} u(x_j^i, x_{j+1}^i) \right) w(x_{n(i)}^i, x_1^{i+1}) \prod_{k=1}^{n(m)} \left(\prod_{l=1}^{k-1} u(x_l^m, x_{l+1}^m) \right) g(x_k^m)$$

$$= \sum_{n(1)=1}^{\infty} \dots \sum_{k=1}^{\infty} \dots \sum_{n(m)=k}^{\infty} \int_{\mathbf{M}} \mathrm{d}x_1^1 \dots \int_{\mathbf{M}} \mathrm{d}x_{n(1)}^1 \dots \int_{\mathbf{M}} \mathrm{d}x_1^m \dots \int_{\mathbf{M}} \mathrm{d}x_{n(m)}^m$$

$$G(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} A(x_j^i, x_{j+1}^i) \right) B(x_{n(i)}^i, x_1^{i+1}) \left(\prod_{l=1}^{k-1} u(x_l^m, x_{l+1}^m) \right) g(x_k^m)$$

$$\left(\prod_{l=1}^{n(m)-1} p(x_l^m, x_{l+1}^m) \right) p_t(x_{n(m)}^m) = \sum_{n(1)=1}^{\infty} \dots \sum_{n(m-1)=1}^{\infty} \dots \sum_{k=1}^{\infty} \int_{\mathbf{M}} \mathrm{d}x_1^1 \dots \int_{\mathbf{M}} \mathrm{d}x_{n(1)}^1 \dots$$

$$\int_{\mathbf{M}} \mathrm{d}x_1^m \dots \int_{\mathbf{M}} \mathrm{d}x_{n(m)}^m G(x_1^1) \prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} A(x_j^i, x_{j+1}^i) \right) B(x_{n(i)}^i, x_1^{i+1}) \left(\prod_{l=1}^{k-1} A(x_l^m, x_{l+1}^m) \right)$$

$$g(x_k^m) \left\{ p_t(x_k^m) + \int_{\mathbf{M}} \mathrm{d}x_{k+1}^m p(x_k^m, x_{k+1}^m) p_t(x_{k+1}^m) + \dots \right\}$$

$$= \int_{\mathbf{M}} \mathrm{d}x g(x) (\mathbf{T}^{m-1} (\mathbf{I} - \mathbf{A})^{-1} G)(x).$$

The next part of the proof is the same as in the case of Theorem 1. □

N o t e 2. Condition $\mathbf{D}\eta^m < \infty$ of Theorem 2 can be fulfilled if the inequalities

$$C_1 p_{in}(x) \geqslant G(x), \ C_2 p(x, y) \geqslant A(x, y), \ C_3 q(x, y) \geqslant B(x, y) \text{ and } C_4 \geqslant g(x)$$

hold where C_i, $i = 1, 2, 3, 4$ are finite positive constants, $C_2 < 1$. Indeed, according to (12) and the assumption of Theorem 2 we have

$$\eta^{(m)} \leqslant C_1 \left[\prod_{i=1}^{m-1} \left(\prod_{j=1}^{n(i)-1} C_2 \right) C_3 \right] C_4 / (1 - C_2) \leqslant C_1 C_4 C_3^{m-1} / (1 - C_2)$$

and therefore by (16)

$$\mathbf{D}\eta^m \equiv \mathbf{E}(\eta^m)^2 - (\mathbf{E}\eta^m)^2 \leqslant C_1 C_4 C_3^{m-1} / (1 - C_2) \mathbf{E}\eta^m - (\mathbf{E}\eta^m)^2 < \infty.$$

Theorems 1 and 2 can serve as a basis for numerical solution to the dominant eigenvalue problem (2) by the Monte Carlo method. Indeed, in accordance with (6) we have

$$\lim_{n\to\infty}\left[\int_{\mathbf{M}} \mathrm{d}x g(x)\varphi(x)\right]^{1/(m-1)} = \lim_{n\to\infty}[\mathbf{E}\zeta^m]^{1/(m-1)} = \lim_{m\to\infty}[\mathbf{E}\eta^m]^{1/(m-1)} = \lambda_0$$

so that λ_0 can be approximated by the number

$$\sqrt[m-1]{\sum_{i=1}^{N}\zeta^m(\alpha_i^m)/N} \qquad \text{or} \qquad \sqrt[m-1]{\sum_{i=1}^{N}\eta^m(\alpha_i^m)/N}$$

for N, m large. Similarly, knowing λ_0, then according to (3) and (7) the detailed behaviour of the eigensolution $\varphi_0(x)$ can be found out by computing $\mathbf{E}\Omega_j^{kl}$-the expectation of the random variable Ω_j^{kl},

$$\Omega_j^{kl} = \sum_{m=k+1}^{l} \lambda_0^{-m}/(l-k)\Gamma^{mj}.$$

Here k and l are large integers and Γ^{mj} are the random variables defined as ζ^m or η^m by formulae (11) or (12) for the set of functions $g = g_j$, $j = 1, 2, \ldots$ appropriately chosen.

Of course, the use of expression (3) for practical computations depends on the value of the dominance ratio $r(\mathbf{T}_1)/r(\mathbf{T})$. So it is necessary to ensure that the number m is sufficiently large. It may be a matter of an appropriate choice of the probability functions p_{in}, p, p_t and q.

Applications

On the basic of Theorems 1 and 2, the computing code MOCA 2 has been developed. It is aimed at the solution of the criticality problem (of the form (2)) for the neutron transport equation in the case of the multigroup and spatially three-dimensional formulation. Two test problems were computed by this code and the resulting values of the multiplication coefficient k_{eff} (the dominant eigenvalue) are shown in the following table. Problem 1 corresponds to the case of the fuel assembly VVER-1000 placed in an infinite regular hexagonal lattice of the same assemblies. The height of the assembly is infinite and the presence of absorbing elements is supposed. Formulation of the second problem is the same but the absorbing elements are replaced by water. Both calculations were performed with four energy groups and we have put $m = 100$.

For comparison, the same problems were computed by the method of successive generations [3] using code MOCA [4]. In this case, we have considered 100 particles per one generation, 5 initial generations were omitted.

	MOCA2			MOCA		
Var.	N	k_{eff}	Err. (%)	N. of gen.	k_{eff}	Err. (%)
1	600	.9585	.19	1245	.9517	.31
2	700	1.1547	.16	1774	1.1434	.24

Table. Results of two test problems

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