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František Melkes; Alexander Ženíšek

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ON A CERTAIN TWO-SIDED SYMMETRIC CONDITION IN MAGNETIC FIELD ANALYSIS AND COMPUTATIONS

František Melkes, Alexander Ženíšek, Brno[1]

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Abstract. A special two-sided condition for the incremental magnetic reluctivity is introduced which guarantees the unique existence of both the weak and the approximate solutions of the nonlinear stationary magnetic field distributed on a region composed of different media, as well as a certain estimate of the error between the two solutions. The condition, being discussed from the physical as well as the mathematical point of view, can be easily verified and is fulfilled for various magnetic reluctivity models used in electrotechnical practice.

Keywords: magnetic field, variational formulation, two-sided existence and uniqueness condition, finite element method

MSC 2000: 65N30

1. Introduction

The knowledge of magnetic field distribution over a domain consisted of several parts with different physical properties is of great importance for electrotechnic practice because it enables us to investigate various, desirable as well as undesirable, physical phenomena. As a consequence, designers of electrical machinery are able to foresee run, behaviour and working characteristics of projected facilities in a better way. The magnetic field can be investigated in various ways, see for example [11] using the physical modelling or [3, 14] applying the finite difference method. Great attention in literature has been devoted to the magnetic field analysis using the finite element method (see e.g. references in [1, 4, 5, 6, 10, 12, 13, 15] or the COMPUMAG conference papers). From the finite element theoretical point of view the stationary

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magnetic field problem was studied first in [7]. In this thesis a special two-sided symmetric condition for the incremental magnetic reluctivity was introduced. This condition guarantees the unique existence of both the weak and the approximate solutions, as well as a certain estimate of the corresponding error. In view of narrow publicity of the result mentioned, the authors of this paper believe that a modernized republishing of the condition with an explanation of its physical meaning, as well as of the proof of the corresponding theorem, in a journal with wider range will be useful. At the end of the paper some practical aspects are mentioned.

2. Preliminaries

The proof of the assertion concerning stationary magnetic fields is based on the results from [8] where a nonlinear operator equation $F(x) = \theta$, $||\theta|| = 0$, has been studied, the operator F being defined on a real Banach space E. The operator is required to be potential, its potential being denoted by $f(x)$, i.e. $F(x) = \operatorname{grad} f(x)$, and to fulfil the condition

$$(1)\qquad \alpha(\|x_2 - x_1\|) \leqslant \big\langle F(x_2) - F(x_1), x_2 - x_1 \big\rangle \leqslant \beta(\|x_2 - x_1\|) \qquad \forall x_1, x_2 \in E,$$

$\alpha(t), \beta(t)$ being non-negative functions of the non-negative argument such that the functions

$$\bar{\alpha}(s) = \int_0^s \frac{\alpha(t)}{t}\,\mathrm{d}t, \quad \bar{\beta}(s) = \int_0^s \frac{\beta(t)}{t}\,\mathrm{d}t$$

are continuous and increasing for $s \geqslant 0$, $\lim\limits_{s\to\infty} \frac{\bar{\alpha}(s)}{s} = \infty$, and $\langle .,. \rangle$ represents the duality between E and the dual space E^*. The left-hand side of (1) expresses the uniform monotonicity of F while the right-hand side implies the uniform continuity of F. Under this notation the main result from [8] can be formulated in the following way:

Theorem 1. *Let a potential operator $F(x)$ fulfilling (1) be defined on E. Let $M \subset E$ be a nonempty closed convex set. Then there exist unique $x^* \in E$, $f(x^*) = \min\limits_{x\in E} f(x)$ and $\bar{x} \in M$, $f(\bar{x}) = \min\limits_{x \in M} f(x)$, and for their difference in the norm of E we have*

$$(2)\qquad \|\bar{x} - x^*\| \leqslant \gamma(\|x - x^*\|) \qquad \forall x \in M,$$

where γ is an increasing non-negative function of the non-negative argument and $\gamma(0) = 0$.

Let us note that γ is defined as a composition of β and α^{-1}. The elements x^* and $\bar{x}$ are called respectively *the weak* and *the approximate solutions* of the above operator equation. Observe also that for the unique existence of both solutions the uniform monotonicity of F is sufficient while the right-hand side inequality of (1) is necessary for the estimate (2). As (2) is valid for any $x \in M$ we can see that the error between the weak and the approximate solutions is, in a certain sense, not worse than that of the best approximation of x^* over M. An assertion similar to Theorem 1 was proved in [2], pp. 322–323.

3. Magnetic field problem formulation

The two-dimensional nonlinear stationary magnetic field problem can be mathematically formulated in the following way:

Problem 1. A bounded planar domain Ω with a Lipschitz continuous boundary $\partial\Omega$ is given. Assume Ω to be divided into a finite number of mutually disjoint subregions Ω_i, $i = 1, \dots, N$, with Lipschitz continuous boundaries $\partial\Omega_i$. Suppose the physical medium in Ω to be described by a given current density $J = J(x, y)$ and a positive magnetic reluctivity $\nu = \nu(x, y, B)$ so that both quantities are continuous on each Ω_i. Across the boundaries $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i - \partial\Omega$ between different media, these quantities can be discontinuous. We look for a function $u = u(x, y)$ satisfying the equation

$$\frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial u}{\partial y}\right) = -J \qquad in \;\; \Omega - \Gamma \tag{3}$$

which is continuous together with the function $\nu \frac{\partial u}{\partial n}$ across Γ, n denoting the normal to the common boundary oriented in a unique way, and which satisfies the boundary conditions

$$u = g_1 \quad on \;\; \Gamma_1, \qquad\qquad \nu \frac{\partial u}{\partial n} = g_2 \quad on \;\; \Gamma_2 = \partial\Omega - \Gamma_1, \tag{4}$$

n being the outer normal to the boundary $\partial\Omega$ and $\Gamma_1 \subseteq \partial\Omega$ denoting a nonempty measurable set. The relationship between u and B is given by $B \equiv B(u) = |\operatorname{grad} u|$.

Let us point out that (3) has a wider meaning in practice, for one can also meet it in the dual problem of the magnetic field where the permeability appears in the role of ν, or in the investigation of a potential gas flow where $\nu(B) = 1 - B^a/a$, $a > 0$, or in the minimum surface problem where $\nu(B) = (1 + B^2)^{-0.5}$, or in the elasticity theory when searching the elastic plastic bar twisting where $\nu(B)$ is an empiric function.

In the sequel we shall assume that g_1 is the trace of a function from $W^{1,p}(\Omega)$, $p > 2$, $g_2 \in L^2(\Gamma_2)$ and $J \in L^2(\Omega)$. Concerning the magnetic reluctivity we shall suppose that $\nu \equiv \nu(x, y, B)$ is measurable on Ω as a function of space variables x, y and that the derivative $\nu_d \equiv \nu_d(x, y, B) = \frac{\partial}{\partial B}\big[B\,\nu(x, y, B)\big]$ exists for almost all $(x, y) \in \Omega$(this guarantees the measurability of ν with respect to B) and fulfils the condition

$$\alpha_0 \leqslant \nu_d(x, y, B) \leqslant \beta_0, \tag{5}$$

where α_0, β_0 are positive constants.

To equation (3) the Dirichlet form

$$a(u, v) = \int_\Omega \big[\nu(x, y, B(u))\,\mathrm{grad}\,u\;\;\mathrm{grad}\,v - J\,v\big]\,\mathrm{d}\Omega - \int_{\Gamma_2} g_2\,v\,\mathrm{d}\Gamma \tag{6}$$

can be adjoint. This form, having sense for all $u, v \in H^1(\Omega)$, enables us to formulate our problem in a variational way:

Problem 2. Let $V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1\}$. Choose $u_0 \in W^{1,p}(\Omega)$, $p > 2$ so that $\mathrm{tr}\,u_0 = g_1$ on Γ_1. We look for such a function u, $u - u_0 \in V$, that

$$a(u, v) = 0 \qquad \forall v \in V. \tag{7}$$

4. Two-sided symmetric condition

The two-sided symmetric condition (5) will play the main role in our further considerations, above all in problems concerning the existence and uniqueness of the weak and approximate solutions as well as the convergence of the approximate solution. Therefore, we shall be concerned with it in more detail.

Let us start with the physical meaning of (5). Magnetic reluctivities, as functions of space variables x, y, are usually piecewise constant in practice. Let us restrict our further considerations on one medium. Here the magnetic reluctivity is a function of the third argument only, i.e. $\nu(x, y, B) = \nu(B)$, and can be easily determined from the magnetization characteristic. For an isotropic medium with negligible magnetic hysteresis the magnetization characteristic (the initial magnetic curve) is a single-valued function $B = B(H)$, where B and H denote the magnitudes of the flux density and the magnetic intensity, respectively. This function, defined for $H \in \langle 0, \infty)$, is continuous, increasing and satisfying the restrictions $\mu_0 H \leqslant B \leqslant \mu_0 H + B_0$, where $\mu_0 = 4\pi\,10^{-7}\,\mathrm{H/m}$ denotes the permeability of vacuum and B_0 is known. From the magnetization characteristic the magnetic permeability μ, the incremental magnetic permeability μ_d and the correlation between them are defined, and given respectively by

$$\mu \equiv \mu(H) = \frac{B}{H}, \qquad \mu_d \equiv \mu_d(H) = \frac{\mathrm{d}B}{\mathrm{d}H} \qquad \text{and} \qquad \mu_d = \mu + H\frac{\mathrm{d}\mu}{\mathrm{d}H}.$$

However, it is more convenient for magnetic field analysis to use the inverse magnetization characteristic model $H = H(B)$ where the function $H = H(B)$ is also continuous, increasing and fulfilling $\nu_0\,(B - B_0) \leqslant H \leqslant \nu_0\,B, \quad \nu_0\,\mu_0 = 1$. In this case the magnetic reluctivity ν, the incremental magnetic reluctivity ν_d and their mutual relation are given by

$$\nu \equiv \nu(B) = \frac{H}{B}, \qquad \nu_d \equiv \nu_d(B) = \frac{\mathrm{d}H}{\mathrm{d}B} \qquad \text{and} \qquad \nu_d = \nu + B\frac{\mathrm{d}\nu}{\mathrm{d}B}.$$

The basic condition (5), rewritten now in the form

$$\alpha_0 \leqslant \nu_d(B) \leqslant \beta_0 \qquad \text{or} \qquad \alpha_0 \leqslant \frac{\mathrm{d}}{\mathrm{d}B}\big[B\,\nu(B)\big] \leqslant \beta_0, \tag{8}$$

says that the incremental magnetic reluctivity must be bounded from both sides by positive constants. The main condition understood in the physical terms means that the theoretical aspects of nonlinear magnetic field analysis require a two-sided bound of the incremental magnetic reluctivity.

Let us turn our attention to the mathematical conclusions of (5). In the first place notice that integrating (5) with respect to B from B_1 to $B_2 \geqslant B_1$, we obtain

$$\alpha_0\,(B_2 - B_1) \leqslant B_2\,\nu(x, y, B_2) - B_1\,\nu(x, y, B_1) \leqslant \beta_0\,(B_2 - B_1) \tag{9}$$

and as a result for $B_1 = 0$, $B_2 = B$ also

$$\alpha_0 \leqslant \nu(x, y, B) \leqslant \beta_0. \tag{10}$$

Thus, bounding the incremental magnetic reluctivity ν_d implies the same bounding of the magnetic reluctivity ν.

Before mentioning some more general mathematical conclusions of (5) let us present the following lemma:

Lemma 1. *Let X be a linear space with a scalar product. Choose $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in X$ so that $\mathbf{a}\,\mathbf{d} = \mathbf{b}\,\mathbf{c} = 0$ and simultaneously at least one of the triplets $\mathbf{a}$, $\mathbf{c}$, $\mathbf{e}$ or $\mathbf{b}$, $\mathbf{d}$, $\mathbf{f}$ is linearly dependent. Then*

$$(\mathbf{a}\,\mathbf{f})\,(\mathbf{b}\,\mathbf{e})\,(\mathbf{c}\,\mathbf{d}) + (\mathbf{c}\,\mathbf{f})\,(\mathbf{de})\,(\mathbf{a}\,\mathbf{b}) = (\mathbf{a}\,\mathbf{b})\,(\mathbf{c}\,\mathbf{d})\,(\mathbf{e}\,\mathbf{f}). \tag{11}$$

P r o o f. In order to prove (11) begin with the linear dependence of the given elemets, let for example $\xi\mathbf{a}+\eta\mathbf{c}+\zeta\mathbf{e}=0$ where ξ, η, ζ mean real numbers, at least one of them being non-zero. Multiplying this relation successively by $\mathbf{b}$, $\mathbf{d}$, $\mathbf{f}$ and using the above orthogonalities, we arrive at

$$\begin{aligned} \xi\,(\mathbf{a}\,\mathbf{b}) \qquad\qquad &+ \zeta\,(\mathbf{b}\,\mathbf{e}) = 0,\\ \eta\,(\mathbf{c}\,\mathbf{d}) &+ \zeta\,(\mathbf{d}\,\mathbf{e}) = 0,\\ \xi\,(\mathbf{a}\,\mathbf{f}) + \eta\,(\mathbf{c}\,\mathbf{f}) &+ \zeta\,(\mathbf{e}\,\mathbf{f}) = 0. \end{aligned} \tag{12}$$

This homogeneous system of linear equations has a non-trivial solution ξ, η, ζ and therefore its determinant must be equal to zero. This fact implies (11). □

Let us note that respecting the above orthogonalities, the linear dependence of the first and the second triplets implies the respective relations

$$\mathbf{c}=\gamma\,[(\mathbf{a}\,\mathbf{b})\,\mathbf{e}-(\mathbf{b}\,\mathbf{e})\,\mathbf{a}] \qquad \text{and} \qquad \mathbf{d}=\delta\,[(\mathbf{a}\,\mathbf{b})\,\mathbf{f}-(\mathbf{a}\,\mathbf{f})\,\mathbf{b}], \tag{13}$$

$\gamma,\delta\neq 0$ being real. Indeed, when determining e.g. $\mathbf{c}$, we start from the linear dependence relation used in the above proof. If $\eta=0$ then (13) is valid, for $\mathbf{c}$ can be arbitrary. If $\eta\neq 0$ then $\mathbf{c}=-\frac{1}{\eta}\,(\xi\mathbf{a}+\zeta\mathbf{e})$ and thus

$$0=\mathbf{bc}=\frac{1}{\eta}\begin{vmatrix} -\xi & \zeta \\ \mathbf{be} & \mathbf{ab} \end{vmatrix}.$$

This identity results in $\xi=\gamma\eta\mathbf{be}$, $\zeta=-\gamma\eta\mathbf{ab}$, γ being real, and consequently in (13). Notice also that the relation (11) is symmetric with respect to $\mathbf{a}\leftrightarrow\mathbf{c}$, $\mathbf{b}\leftrightarrow\mathbf{d}$ as well as with respect to $\mathbf{a}\leftrightarrow\mathbf{b}$, $\mathbf{c}\leftrightarrow\mathbf{d}$, $\mathbf{e}\leftrightarrow\mathbf{f}$. Moreover, for $\mathbf{b}=\mathbf{a}$, $\mathbf{d}=\mathbf{c}$, $\mathbf{aa}=\mathbf{cc}=1$ the simpler relation

$$(\mathbf{a}\,\mathbf{e})\,(\mathbf{a}\,\mathbf{f})+(\mathbf{c}\,\mathbf{e})\,(\mathbf{c}\,\mathbf{f})=\mathbf{e}\,\mathbf{f} \tag{14}$$

is true. This relation in fact is a special case of the well-known Parseval identity in the more general form.

By means of Lemma 1 and an idea of [15] the assertion of Lemma 2 from [15] can be generalized. To this end denote by l_2 the space of all infinite sequences $\mathbf{a}=\{a_1,a_2,\ldots\}$ with the scalar product $\mathbf{a}\,\mathbf{b}=\sum\limits_{i=1}^{\infty}a_i\,b_i$ and the induced norm $\|\mathbf{a}\|=(\mathbf{a}\,\mathbf{a})^{\frac{1}{2}}$.

Lemma 2. *Let* $\mathbf{a},\mathbf{e},\mathbf{f}\in l_2$, $\mathbf{a}\neq 0$. *Let* $\nu=\nu(a)$ *be a function fulfilling the condition* (8), $a=\|\mathbf{a}\|$. *Then there exists* $\mathbf{c}\in l_2$ *orthogonal to* $\mathbf{a}$, $\mathbf{c}\neq 0$, *such that*

$$\sum_{i=1}^{\infty}\frac{\partial}{\partial a_i}\big[\nu(a)\,\mathbf{af}\big]\,e_i=\nu(a)\,\frac{(\mathbf{c}\,\mathbf{e})\,(\mathbf{c}\,\mathbf{f})}{\mathbf{c}\,\mathbf{c}}+\nu_d(a)\,\frac{(\mathbf{a}\,\mathbf{e})\,(\mathbf{a}\,\mathbf{f})}{\mathbf{a}\,\mathbf{a}}. \tag{15}$$

P r o o f. First suppose that at least one of the elements $\mathbf{e}, \mathbf{f}$ differs from $\mathbf{a}$. Let for instance $\mathbf{e} \neq \mathbf{a}$. Then we put $\mathbf{c} = \gamma\,[(\mathbf{a}\,\mathbf{a})\,\mathbf{e} - (\mathbf{a}\,\mathbf{e})\,\mathbf{a}]$, $\gamma \neq 0$ being real. Evidently $\mathbf{a}\,\mathbf{c} = 0$ and $\mathbf{c}\,\mathbf{c} = \gamma^2\,(\mathbf{a}\,\mathbf{a})\,[(\mathbf{a}\,\mathbf{a})\,(\mathbf{e}\,\mathbf{e}) - (\mathbf{a}\,\mathbf{e})^2] > 0$. Let us calculate

$$\sum_{i=1}^{\infty} \frac{\partial}{\partial a_i}\big[\nu(a)\,\mathbf{af}\big]\,e_i \equiv \sum_{i,j=1}^{\infty} \frac{\partial}{\partial a_i}\big[a_j\,\nu(a)\big]\,e_i\,f_j = \nu(a)\,\mathbf{e}\,\mathbf{f} + \frac{1}{a}\nu'(a)\,(\mathbf{a}\,\mathbf{e})\,(\mathbf{a}\,\mathbf{f})$$
$$= \nu(a)\,\mathbf{ef} + a\,\nu'(a)\,\frac{(\mathbf{ae})\,(\mathbf{af})}{\mathbf{a}\,\mathbf{a}} = \nu(a)\,\frac{(\mathbf{c}\,\mathbf{e})\,(\mathbf{c}\,\mathbf{f})}{\mathbf{c}\,\mathbf{c}} + \nu_d(a)\,\frac{(\mathbf{a}\,\mathbf{e})\,(\mathbf{a}\,\mathbf{f})}{\mathbf{a}\,\mathbf{a}}.$$

The last identity is a consequence of (11). Now suppose that $\mathbf{e} = \mathbf{f} = \mathbf{a}$. Then

$$\sum_{i=1}^{\infty} \frac{\partial}{\partial a_i}\big[\nu(a)\,\mathbf{aa}\big]\,a_i \equiv \sum_{i,j=1}^{\infty} \frac{\partial}{\partial a_i}\big[a_j\,\nu(a)\big]\,a_i\,a_j = \nu(a)\,\mathbf{a}\,\mathbf{a} + \frac{1}{a}\nu'(a)\,(\mathbf{a}\,\mathbf{a})^2 = \nu_d(a)\,\mathbf{a}\,\mathbf{a}.$$

In this case the element $\mathbf{c} \neq 0$ can be chosen arbitrarily so that $\mathbf{a}\,\mathbf{c} = 0$. Such an element always exists. Both terms on the right-hand side of (15) are finite because of bounds of ν and ν_d and therefore the sum on the left-hand side converges. □

The above mentioned generalization of this lemma goes in three directions: First, (15) is written in the form of an equality, the corresponding inequality from [15] being obtained by using (5), (10) and then (11). Second, the equality holds generally for infinite-dimensional vectors and not only for two-dimensional ones. Third, it is generally valid for $\mathbf{f} \neq \mathbf{e}$ and not only for $\mathbf{f} = \mathbf{e}$ as the generalized inequality. Notice also that the sum on the left-hand side of (15) can be interpreted as $\mathrm{grad}_{\mathbf{a}}[\nu(a)\,\mathbf{af}]\,\mathbf{e}$.

Hereinafter, the element $\mathbf{a}$ will be a function of a real parameter. Then the following lemma will be useful in the proof of Theorem 2.

Lemma 3. *Let* $\mathbf{a}$, a, $\nu(a)$ *and* $\mathbf{f}$ *fulfil the assumptions of Lemma* 2. *Assume that* $\mathbf{a}$ *depends on a real parameter* t, *i.e.* $a_i = a_i(t)$, $i = 1, 2, \ldots$ *and suppose all* a_i *to be differentiable. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\nu(a)\,\mathbf{a}\,\mathbf{f}\big] = \nu(a)\frac{(\mathbf{c}\,\mathbf{f})\,(\mathbf{c}\,\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t})}{\mathbf{c}\,\mathbf{c}} + \nu_d(a)\frac{(\mathbf{a}\,\mathbf{f})\,(\mathbf{a}\,\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t})}{\mathbf{a}\,\mathbf{a}}$$

where $\mathbf{f}$ *is constant and* $\mathbf{c} = \gamma\,a^3\,\frac{\mathrm{d}}{\mathrm{d}t}\big(\frac{\mathbf{a}}{a}\big) = \gamma\,[a^2\,\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} - \frac{\mathbf{a}}{2}\,\frac{\mathrm{d}a^2}{\mathrm{d}t}]$, $\gamma \neq 0$ *being real.*

P r o o f. The proof is based on the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\nu(a)\,\mathbf{a}\,\mathbf{f}\big] = \mathrm{grad}_{\mathbf{a}}[\nu(a)\,\mathbf{af}]\,\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t}$$

and the use of Lemma 2. □

In planar magnetic field applications only two-component vectors from l_2 are needed, i.e. we restrict ourselves to the elements $\mathbf{a} = \{a_1, a_2\}$ the components $a_1 = a_1(x, y)$, $a_2 = a_2(x, y)$ of which are functions of two variables. In further considerations we shall use the notation

$$\operatorname{grad} u = \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big), \qquad \operatorname{grad}_1 u = \frac{\operatorname{grad} u}{|\operatorname{grad} u|},$$
$$\operatorname{curl} u = \Big(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\Big), \qquad \operatorname{curl}_1 u = \frac{\operatorname{curl} u}{|\operatorname{curl} u|}. \tag{16}$$

Now choose $u, v, w \in H^1(\Omega)$, $u \neq 0$ arbitrarily and put

$$\mathbf{a} = \mathbf{b} = \operatorname{grad}_1 u, \qquad \mathbf{c} = \mathbf{d} = \operatorname{curl}_1 u, \qquad \mathbf{e} = \operatorname{grad} v, \qquad \mathbf{f} = \operatorname{grad} w.$$

Then (14) implies that

$$(\operatorname{grad}_1 u \ \operatorname{grad} v)(\operatorname{grad}_1 u \ \operatorname{grad} w) + (\operatorname{curl}_1 u \ \operatorname{grad} v)(\operatorname{curl}_1 u \ \operatorname{grad} w) = \operatorname{grad} v \ \operatorname{grad} w \tag{17}$$

is valid for any $u, v, w \in H^1(\Omega)$ when the expressions have sense. Let us emphasize two important properties of (17): the identity is symmetric with respect to $v \leftrightarrow w$ and contains only non-negative terms for $w = v$.

5. Magnetic field problem solution

The results of the previous sections will be used to prove the basic theorem.

Theorem 2. *Let all the above assumptions be fulfilled. Choose a nonempty finite-dimensional subspace $V_n \subset V$. Then Problem 2 has a unique weak solution $u^* \in V$ minimizing the magnetic energy in the whole V and a unique approximate solution $u \in V_n$ minimizing the same energy in V_n only. In the norm of $H^1(\Omega)$ the error between these two solutions satisfies the estimate*

$$\|u - u^*\| \leqslant \kappa \, \|v - u^*\| \qquad \forall v \in V_n,$$

$\kappa > 1$ being a real constant.

P r o o f. The Dirichlet form (6) represents a bounded linear functional with respect to the second argument because of $|a(u, v)| \leqslant K_u \|v\|$. According to the Riesz Theorem, for any $u \in V$ there exists $F(u) \in V$ such that

$$\langle F(u), v\rangle = a(u, v) \qquad \forall v \in V.$$

We have to show that the operator F fulfils the assumptions of Theorem 1.

First we determine the Gâteaux derivative of the Dirichlet form

$$a'_u(v,w) = \frac{\mathrm{d}}{\mathrm{d}t}\, a(u+t\,v,w)\Big|_{t=0} = \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\left[\nu(x,y,B(u+t\,v))\,\mathrm{grad}(u+t\,v)\,\mathrm{grad}\,w\right]\Big|_{t=0}\mathrm{d}\Omega$$

After some rearrangements and simultaneous use of Lemma 3, putting $\mathbf{f} = \mathrm{grad}\,w$, $\mathbf{a} = \mathrm{grad}(u + t\,v)$ and consequently $\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \mathrm{grad}\,v$ one obtains

$$\begin{aligned} a'_u(v,w) = \int_\Omega \big[\, & \nu\ (\mathrm{curl}_1\, u\ \ \mathrm{grad}\,v)\,(\mathrm{curl}_1\, u\ \ \mathrm{grad}\,w) \\ + \quad & \nu_d\ (\mathrm{grad}_1\, u\ \ \mathrm{grad}\,v)\,(\mathrm{grad}_1\, u\ \ \mathrm{grad}\,w)\,\big]\,\mathrm{d}\Omega. \end{aligned} \tag{18}$$

From (18) both the existence of $a'_u(v,w)$ for all $u,v,w \in H^1(\Omega)$ and the relation of symmetry $a'_u(v,w) = a'_u(w,v)$ follow. Using these results, one can prove, in the same way as in [8], that F is a potential operator with the potential

$$f(u) = \int_0^1 a(tu,u)\,\mathrm{d}t = \int_\Omega \left[\int_0^{B(u)} \nu(x,y,b)\,b\,db - J\,u\right]\mathrm{d}\Omega - \int_{\Gamma_2} g_2\,u\,\mathrm{d}\Gamma$$

representing the residual magnetic energy.

In order to verify (1) notice that the equalities

$$\langle F(v) - F(u), v-u\rangle = a(v,v-u) - a(u,v-u) = \int_0^1 a'_{u+t(v-u)}\,(v-u,v-u)\,\mathrm{d}t \tag{19}$$

are valid for any $u,v \in V$. To this end it is sufficient to estimate $a'_u(v,v)$ only. Applying (18) for $w = v$, one can express this Gâteaux derivative in the advantageous form

$$a'_u(v,v) = \int_\Omega \left[\nu\,(\mathrm{curl}_1\, u\ \ \mathrm{grad}\,v)^2 + \nu_d\,(\mathrm{grad}_1\, u\ \ \mathrm{grad}\,v)^2\right]\mathrm{d}\Omega$$

where the integrand is expressed in the form of a linear combination of ν and ν_d with non-negative coefficients, both reluctivities being bounded from both sides by positive constants. Respecting (5), (10) and (17), we conclude that

$$\alpha_0 \int_\Omega \mathrm{grad}^2\, v\,\mathrm{d}\Omega \leqslant a'_u(v,v) \leqslant \beta_0 \int_\Omega \mathrm{grad}^2\, v\,\mathrm{d}\Omega.$$

In order to change the seminorm in $H^1(\Omega)$, appearing in these inequalities, to the corresponding norm we use the Friedrichs inequality

$$\int_\Omega v^2\,\mathrm{d}\Omega \leqslant c\int_\Omega \mathrm{grad}^2\, v\,\mathrm{d}\Omega \qquad \forall v \in V,$$

where $c > 0$ is a constant. Then we have in the norm of the space $H^1(\Omega)$ that

$$\frac{\alpha_0}{1+c}\,\|v\|^2 \leqslant a'_u(v,v) \leqslant \beta_0\,\|v\|^2.$$

Combining it with (19) we arrive at

$$\frac{\alpha_0}{1+c}\,\|v-u\|^2 \leqslant \langle F(v)-F(u), v-u\rangle \leqslant \beta_0\,\|v-u\|^2$$

and thus (1) is satisfied for $\alpha(t) = \frac{\alpha_0}{1+c}\,t^2$ and $\beta(t) = \beta_0\,t^2$. It is evident that $\kappa = \sqrt{(1+c)\,\frac{\beta_0}{\alpha_0}} > 1$. □

So, if e.g. the domain Ω has a polygonal boundary and linear triangular finite elements are used which approximate u^* with the accuracy of the first order then also the error between u and u^* is at worst of the first order.

6. Main condition in practice

The magnetization characteristic, given as a rule empirically, can be modelled by various functions. Let us mention some models of the type $H = H(B)$ from [9] and study the validity of (5) or, which is the same, of (8). It is sufficient to investigate the incremental reluctivity in the bounded interval $[0, B_{\max}]$ only because the magnitude of the real flux density is finite and relatively not very high (usually $B_{\max} \doteq 2.5\,\mathrm{T}$ in the heavy current electrical engineering). This fact enables us to determine α_0, β_0 for most of the magnetization characteristic models. The survey of the results is presented in Table 1 where a, b, c, d, e denote real numbers chosen so that the corresponding function represents a real magnetization characteristic, n means a natural number and $\omega = \nu_d(B_{\max})$. Similarly in the case when only the model $B = B(H)$ is at one's disposal, one calculates positive bounds $\hat\alpha_0$, $\hat\beta_0$ of the incremental magnetic permeability and puts $\alpha_0 = 1/\hat\beta_0$, $\beta_0 = 1/\hat\alpha_0$, see Table 2 where $\kappa = 1/\mu_d(H_{\max})$.

7. Conclusion

In solving the nonlinear stationary magnetic field distributed over a planar region composed of different isotropic materials, an important role is played by the incremental magnetic reluctivity (or permeability). Its positive bound from below guarantees the unique existence of the weak and approximate solutions. If it is also bounded from above then the error estimate between these solutions is given. We point out that on the basis of this theory and the use of the finite element method

$H = H(B)$	α_0	β_0	Notes
$\frac{a}{b-B}$	$\frac{a}{b^2}$	ω	$a > 0,\ B_{\max} < b$
$\frac{aB}{b-B}$	$\frac{a}{b}$	ω	$a > 0,\ B_{\max} < b$
$\frac{aB+b}{B+c}\,B$	$\min(a, \frac{b}{c})$	$\max(a, \frac{b}{c})$	$a, b, c > 0$
$a + bB + cB^n$	b	ω	$c > 0,\ n > 1$
$\sqrt{a^2 + 2bB + c^2B^2} +$	$d + \min\left(\frac{b}{\lvert a\rvert}, \lvert c\rvert\right)$	$d + \max\left(\frac{b}{\lvert a\rvert}, \lvert c\rvert\right)$	$a \neq 0,\ \lvert c\rvert + d > 0$
$+\,\mathrm{d}B + e$			$b + \lvert a\rvert d > 0$
$a\,\exp(bB)$	ab	ω	$a, b > 0$
$a\,[1 - \exp(-bB)]$	$\min(ab, \omega)$	$\max(ab, \omega)$	$ab > 0$
$-a\,\ln(1 - bB)$	$\min(ab, \omega)$	$\max(ab, \omega)$	$ab > 0,\ bB_{\max} < 1$
$a\,\mathrm{sh}\,(bB)$	ab	ω	$ab > 0$
$a\,\mathrm{Arth}\,(bB)$	ω	ab	$ab > 0,\ b^2B_{\max}^2 < 1$
$a\,\mathrm{tg}\,(bB) + cB + d$	$ab + c$	ω	$2bB_{\max} < \pi,\ ab > 0$

Table 1. Determination of the condition coefficients for $H = H(B)$

$B = B(H)$	α_0	β_0	Notes
$a\,\mathrm{Arth}\,(bH) + c\,H$	κ	$\frac{1}{ab+c}$	$b^2H_{\max}^2 < 1$
			$ab + c > \max(cb^2H_{\max}^2, 0)$
$a + bH + cH^n$	κ	$\frac{1}{b}$	$b, c > 0$
$\exp(\frac{H}{a+b\,H})$	$\min(a, q)$	$\max(a, q)$	$a, b > 0,\ q = \frac{1}{a}(a + b\,H)^2$

Table 2. Determination of the condition coefficients for $B = B(H)$

with a triangular mesh and linear splines, an algorithm for the planar nonlinear stationary magnetic field computation has been constructed. The system of nonlinear discretization equations has been solved by the Newton method, each Newton iterate having been inverted by the Gaussian elimination respecting symmetry and the band structure of the system matrix. Within years a large number of practical problems, mostly from the heavy current electrical engineering, has been resolved (see e.g. [1]), one of the solved problems being illustrated on the case of a synchronous machine in an on-load state (see Figs. 1–3). The convergence in spite of strong nonlinearity of the problem, material discontinuity and geometric complexity of the region has been relatively very fast, for in nearly all solved cases the number of Newton iterates has fluctuated somewhere between 8 and 12 to get a numerical fixed point.

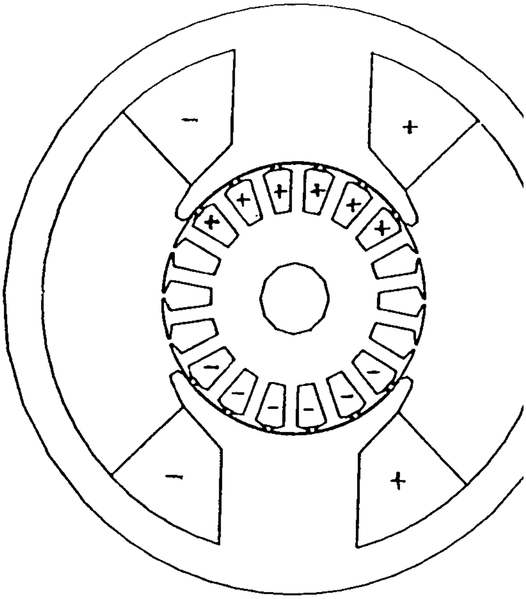

Fig. 1

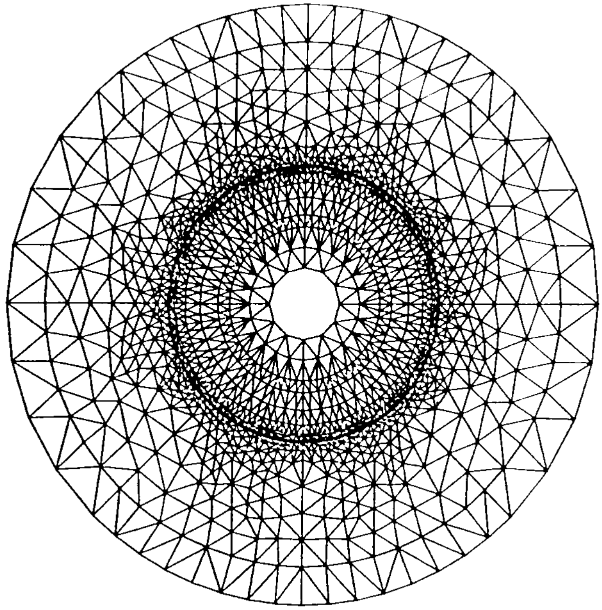

Fig. 2

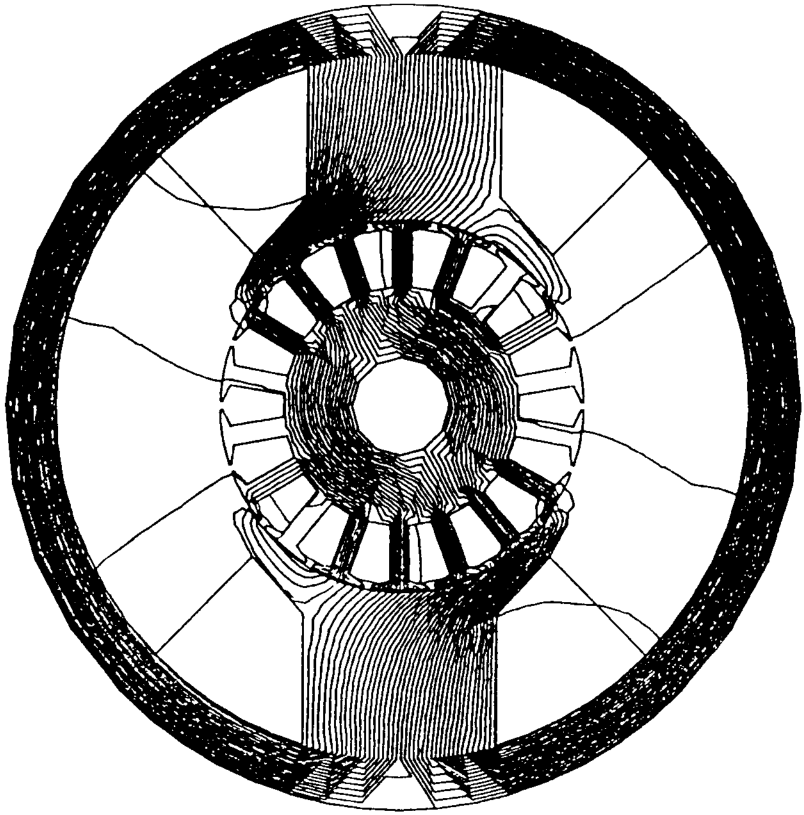

Fig. 3

References

[1] *K. Chrobáček, F. Melkes, L. Rak*: Stationary magnetic field computation of electrical machines. TES 1977, theoretical number, 22–29. (In Czech.)

[2] *P.G. Ciarlet*: The Finite Element Method for Elliptic Problems. North-Holland, Amsterodam, 1978.

[3] *E.A. Erdelyi, E.F. Fuchs, (D.H. Binkley)*: Nonlinear magnetic field analysis of DC machines I, II, III. IEEE Trans., PAS-89 (1970), 7, 1546–1583.

[4] *A. Foggia, J.C. Sabonnadière, P. Silvester*: Finite element solution of saturated travelling magnetic field problems. IEEE Trans., PAS-94 (1975), 866–871.

[5] R. *Glowinski, A. Marrocco*: Analyse numerique du champ magnetique d'un alternateur par elements finis et sur-relaxation ponctuelle non lineaire. Comput. Methods Appl. Mech. Engrg. *3* (1974), 55–85.

[6] *B. Lencová, M. Lenc*: A finite element method for the computation of magnetic electron lenses. Scanning Electron Microscopy 1986/III, SEM Inc., AMF O'Hare. Chicago, 1986, pp. 897–915.

[7] *F. Melkes*: Solving the magnetic field by the finite element method. PhD. Thesis, Czechoslovak Academy of Sciences, Prague, 1970. (In Czech—see also report of VÚES Brno, TZ 1481.)

[8] *F. Melkes*: The finite element method for non-linear problems. Apl. Mat. *15* (1970), 177–189.

[9] *F. Melkes*: Magnetic energy computation using piecewise linear approximations. Acta Tech. ČSAV (1990), 365–373.

[10] *J. Polák*: Variational Principles and Methods of Electromagnetic Theory. Academia, Prague, 1988. (In Czech.)

[11] *H. Reiche*: Die Ermittlung stationärer magnetischer Felder in elektrischen Maschinen. IX. Internat. Kolloquium TH. Ilmenau, 1966.

[12] *P. Silvester, H.S. Cabayan, B.T. Browne*: Efficient techniques for finite element analysis of electric machines. IEEE Trans., PAS-92 (1973), 1274–1281.

[13] *H. Tsuboi, F. Kobayashi, T. Misaki*: Two-dimensional magnetic field analysis using edge elements. Proc. of the Third Japanese-Czech-Slovak Joint Seminar on Applied Electromagnetics, Prague. 1995, pp. 53–56.

[14] *A.M. Winslow*: Numerical solution of the quasilinear Poisson equation in a non-uniform triangle mesh. LRL Livermore California, 1967, pp. 149–172.

[15] *A. Ženíšek*: The maximum angle condition in the finite element method for monotone problems with applications in magnetostatics. Numer. Math. *71* (1995), 399–417.

Authors' addresses: *František Melkes*, Technical University, FEI UMAT, Technická 8, 616 00 Brno, e-mail: `melkes@dmat.fee.vutbr.cz`; *Alexander Ženíšek*, Technical University, FS UMAT, Technická 2, 616 69 Brno, e-mail: `zenisek@kinf.fme.vutbr.cz`.