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CAUCHY PROBLEM FOR THE NON-NEWTONIAN VISCOUS INCOMPRESSIBLE FLUID

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Summary. We study the Cauchy problem for the non-Newtonian incompressible fluid with the viscous part of the stress tensor $\tau^V(\mathbf{e}) = \tau(\mathbf{e}) - 2\mu_1 \Delta \mathbf{e}$, where the nonlinear function $\tau(\mathbf{e})$ satisfies $\tau_{ij}(\mathbf{e})e_{ij} \geqslant c|\mathbf{e}|^p$ or $\tau_{ij}(\mathbf{e})e_{ij} \geqslant c(|\mathbf{e}|^2 + |\mathbf{e}|^p)$. First, the model for the bipolar fluid is studied and existence, uniqueness and regularity of the weak solution is proved for $p > 1$ for both models. Then, under vanishing higher viscosity μ_1, the Cauchy problem for the monopolar fluid is considered. For the first model the existence of the weak solution is proved for $p > \frac{3n}{n+2}$, its uniqueness and regularity for $p \geqslant 1 + \frac{2n}{n+2}$. In the case of the second model the existence of the weak solution is proved for $p > 1$.

Keywords: non-Newtonian incompressible fluids, Navier-Stokes equations, Cauchy problem

AMS classification: 35Q30, 76A05

1. INTRODUCTION

a. Equations and constitutive laws.

Let $n = 2$ or 3. The motion of incompressible viscous fluid in $\mathbb{R}^n$ is described by the system of equations

$$\operatorname{div} \mathbf{u} \equiv \frac{\partial u_i}{\partial x_i} = 0, \tag{1.1}$$

$$\varrho \frac{\partial u_i}{\partial t} + \varrho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} + \varrho f_i, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Here the equations (1.1)–(1.2) express the balance of mass and the balance of momentum, respectively. In the equations $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ represents the velocity field, $\varrho = \text{const} > 0$ the density, $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ the specific body force and τ_{ij} are the components of the stress tensor. All quantities are evaluated at $(\mathbf{x}, t)$, where

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is the actual position and t the present time. When no misunderstanding can occur, we will write only $\mathbf{u}$ instead of the correct $\mathbf{u}(\mathbf{x}, t)$. Hereafter, for simplicity in writing, we put $\varrho = 1$ and use summation convention.

In order to make the system of equations complete it is necessary to prescribe the constitutive relation for the stress tensor. Due to physical considerations, the stress tensor is decomposed as

$$\tau_{ij} = -\pi\delta_{ij} + \tau_{ij}^V, \tag{1.3}$$

where π is the pressure, δ_{ij} is the Kronecker delta and τ^V is the viscous part of the stress, which must be defined by a set of constitutive relations.

In the present work we will assume the stress tensor τ^V of the form

$$\tau^V = \tau(\mathbf{e}) \tag{1.4}$$

with τ a symmetric tensor, where the components of the deformation velocity tensor $\mathbf{e}$ are given by

$$e_{ij} = e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{1.5}$$

In our considerations the polynomial growth

$$\begin{aligned} &|\tau_{ij}(\mathbf{e})| \leqslant c_1(|\mathbf{e}| + |\mathbf{e}|^{p-1}), \quad c_1 > 0, p \geqslant 2, \\ &|\tau_{ij}(\mathbf{e})| \leqslant c_1|\mathbf{e}|^{p-1}, \quad 1 < p < 2 \end{aligned} \tag{1.6}$$

as well as the strong coercivity condition

$$\tau_{ij}(\mathbf{e})e_{ij} \geqslant c_2|\mathbf{e}|^p, \quad 1 < p < \infty, \ c_2 > 0 \tag{1.7}$$

will play an important role. Here $|\mathbf{e}|$ means the Euclidean norm of the tensor $\mathbf{e}$, i.e.

$$|\mathbf{e}| = (e_{ij}e_{ij})^{\frac{1}{2}}. \tag{1.8}$$

We will assume the existence of the scalar potential ϑ for the stress tensor

$$\tau_{ij}(\mathbf{e}) = \frac{\partial\vartheta(\mathbf{e})}{\partial e_{ij}} \tag{1.9}$$

with $\vartheta(\cdot)$ twice continuously differentiable in $\mathbb{R}^{n^2}$, $\vartheta \geqslant 0$, $\vartheta(\mathbf{o}) = 0$ such that we have for all $\xi \in \mathbb{R}^{n^2}_{\text{sym}}$:

$$\begin{aligned} &\frac{\partial^2\vartheta(\mathbf{e})}{\partial e_{ij}\partial e_{kl}}\xi_{ij}\xi_{lk} \geqslant c_3(1 + |\mathbf{e}|^{p-2})\xi_{ij}\xi_{ij}, \quad p \geqslant 2, \\ &\frac{\partial^2\vartheta(\mathbf{e})}{\partial e_{ij}\partial e_{kl}}\xi_{ij}\xi_{lk} \geqslant c_3|\mathbf{e}|^{p-2}\xi_{ij}\xi_{ij}, \quad p < 2. \end{aligned} \tag{1.10}$$

It is possible to show that (1.7) is a direct consequence of (1.9), (1.10) and the fact that $\vartheta(\mathbf{o}) = 0$.

There are several phenomena which appear studying non-Newtonian fluids: shear thinning and shear thickenning, ability of a creep, ability to relax stresses, presence of normal stress differences in simple shear flow, presence of yield stress. For more detailed description see [17]. Our model includes shear thinning ($p < 2$) and shear thickenning ($p > 2$).

1.11. Remark. Generally it is possible to assume that τ^V is a function of $D\mathbf{u}$. However the principle of material frame indifference (see [9]) implies that τ^V can depend only on the symmetric part of the velocity gradient.

We have in mind two examples: first, for $p > 2$

$$\tau_{ij}(\mathbf{e}) = (\mu_0 + \mu_1|\mathbf{e}|^{p-2})e_{ij} \tag{1.12}$$

with μ_0, μ_1 positive constants and second, for $p \in (1,2)$

$$\tau_{ij}(\mathbf{e}) = |\mathbf{e}|^{p-2}e_{ij}. \tag{1.13}$$

It is an easy matter to check that the potentials

$$\vartheta(\mathbf{e}) = \frac{1}{2}\int_0^{e_{ij}e_{ij}} (\mu_0 + \mu_1 s^{\frac{p-2}{2}})\,\mathrm{d}s$$

for $p > 2$ and

$$\vartheta(\mathbf{e}) = \frac{1}{2}\int_0^{e_{ij}e_{ij}} s^{\frac{p-2}{2}}\,\mathrm{d}s$$

for $p < 2$ satisfy the assumptions (1.9)–(1.10).

We will also study separately the model (1.12) for $p < 2$ for which we will be able to prove the existence of a weak solution for all $p > 1$. Of course, we have to modify the conditions (1.6), (1.7) and (1.10). The condition (1.6) will be the same for both $p < 2$ and $p \geqslant 2$, instead of (1.7) we have to use $\tau_{ij}(\mathbf{e})e_{ij} \geqslant c(|\mathbf{e}|^p + |\mathbf{e}|^2)$. The condition (1.10) must be replaced by $\frac{\partial^2\vartheta(\mathbf{e})}{\partial e_{ij}\partial e_{kl}}\xi_{ij}\xi_{lk} \geqslant c_3(1 + |\mathbf{e}|^{p-2})\xi_{ij}\xi_{ij}$. We will call this model the perturbated linear model.

b. Problem formulation and survey of results.

1.14. Definition. Let $\mathbf{u}_0\colon \mathbb{R}^n \mapsto \mathbb{R}^n$, $\mathbf{f}\colon Q_T \mapsto \mathbb{R}^n$ be given functions. The problem (CMN) denotes the following: to find $\mathbf{u}(\mathbf{x},t)$, $\pi(\mathbf{x},t)$ solving (1.1), (1.2), (1.3)–(1.7), where $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$. The letters (CMN) abreviate the Cauchy problem for the Monopolar Non-Newtonian incompressible fluid.

In Chapter 3 we will assume the viscous part of the stress tensor in the form

$$\tau^V = \tau(\mathbf{e}) - 2\mu_1 \Delta \mathbf{e}, \quad \mu_1 > 0, \tag{1.15}$$

where τ is supposed to satisfy all the assumptions (1.6)–(1.10). Such fluids are called bipolar viscous fluids. The theory of bipolar fluids is compatible with the basic principles of thermodynamics, including the Clausius-Duhem inequality and the material frame indifference. The thermodynamical principles also imply that the other higher (third order) stress tensor τ_{ijk} must be considered. See [15], [3] for a detailed description of multipolar fluids. Here we suppose

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}. \tag{1.16}$$

1.17. Definition. Let $\mathbf{u}_0 \colon \mathbb{R}^n \mapsto \mathbb{R}^n$, $\mathbf{f} \colon Q_T \mapsto \mathbb{R}^n$ be given functions. The problem (CBN) denotes the following: to find $\mathbf{u}(\mathbf{x}, t)$, $\pi(\mathbf{x}, t)$ solving (1.1), (1.2), (1.3), (1.5)–(1.7), (1.15), where $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$. The letters (CBN) abreviate Cauchy problem for the Bipolar Non-Newtonian incompressible fluid.

In Chapter 3 we will prove the existence, uniqueness and regularity of a weak solution of the problem (CBN). In Chapter 4 we will study the limiting process $\mu_1 \to 0^+$ in order to prove the existence of a weak solution of the problem (CMN). We will get the existence for $p > \frac{3n}{n+2}$ and its regularity and uniqueness for $p \geqslant 1 + \frac{2n}{n+2}$.

The mathematical theory of the problem for the monopolar fluid was introduced for the first time by O.A.Ladyzhenskaya (for bounded domains). She proved the existence of a weak solution for $p \geqslant \frac{11}{5} (n = 3)$ and its uniqueness for $p \geqslant \frac{5}{2} (n = 3)$. For details see [8]. The same results were been proved in [10] for the p-laplacian, i.e. the existence for $p \geqslant 1 + \frac{2n}{n+2}$ and uniqueness for $p \geqslant \frac{n+2}{n}$, $n \leqslant 4$. The limiting passage from the bipolar fluids to the monopolar ones was done for the first time in [14] and [11].

This paper follows up with the papers [2] and [12]. The former uses a similar method as the present work, i.e. the authors first solved the problem for the bipolar fluid and letting $\mu_1 \to 0^+$ they obtained a solution for the monopolar case (both Young measure-valued and weak). In the latter the results were obtained directly using the Galerkin method. In both papers the following results were proved: the existence of a Young measure-valued solution for the Dirichlet problem for $p > \frac{2n}{n+2}$, the existence of a weak solution for the space periodic problem for $p > \frac{3n}{n+2}$, its regularity and uniqueness for $p \geqslant 1 + \frac{2n}{n+2}$. The aim of this paper is to show that the same holds also for the Cauchy problem, i.e. $\Omega = \mathbb{R}^n$ is unbounded.

As far as it is known to the authors, there are up to now no results in the case of a general unbounded domain.

Let $n = 2$ or 3. Denote $I = (0, T)$ with $T > 0$, $Q_T = \mathbb{R}^n \times I$. The standard notation is used for both scalar ($u\colon \mathbb{R}^n \mapsto \mathbb{R}$ or $Q_T \mapsto \mathbb{R}$) and vector-valued functions ($\mathbf{u}\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ or $Q_T \mapsto \mathbb{R}^n$).

We denote by $C(\mathbb{R}^n)$ and $C^k(\mathbb{R}^n)$ ($k \in \mathbb{N}$ or $k = \infty$) *the space of real continuous functions* on $\mathbb{R}^n$ and *the space of k-times continuously differentiable functions* on $\mathbb{R}^n$, respectively. The space of real C^∞ functions on $\mathbb{R}^n$ with a compact support in $\mathbb{R}^n$ is denoted by $\mathscr{D}(\mathbb{R}^n)$ and its dual by $\mathscr{D}'(\mathbb{R}^n)$. Under $D^{(k)}u$ we understand the vector which consists of all possible derivatives of the k-th order with respect to the space variables, $Du = D^{(1)}u$.

The *Lebesgue spaces* of scalar and vector-valued functions are denoted by $L^q(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)^n$, respectively ($q \in [1, \infty]$). The spaces are equipped with the standard norm denoted by $\|\cdot\|_q$. The *Sobolev spaces* $W^{m,p}(\mathbb{R}^n)$ and $W^{m,p}(\mathbb{R}^n)^n$ are the sets of all measurable functions, for which the functions and all their generalized derivatives up to the order m belong to $L^p(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)^n$, respectively. The spaces are equipped with the standard norms and seminorms denoted by $\|\cdot\|_{m,q}$ and $|\cdot|_{m,q}$. For more detailed descriptions see e.g. [1].

Let s be a noninteger positive number, $s = [s] + \{s\}$, where $[s]$ is the integer and $\{s\}$ the fractional part of s. Let $1 \leqslant p < \infty$. Then the *Slobodeckij spaces* $W^{s,p}(\mathbb{R}^n)$ ($W^{s,p}(\mathbb{R}^n)^n$) are subsets of the Sobolev spaces $W^{[s],p}(\mathbb{R}^n)$ ($W^{[s],p}(\mathbb{R}^n)^n$), where

$$\|u\|_{s,p} = \|u\|_{[s],p} + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^p}{|\mathbf{x} - \mathbf{y}|^{n+\{s\}p}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{y} \right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

We will use the following imbeddings and interpolations which hold between Slobodeckij and Sobolev spaces:

2.2. Lemma (imbeddings). *Let $1 < p < q < \infty$. Let $0 \leqslant s_2 < s_1 < \infty$ be integer or non-integer. Then $W^{s_1,p}(\mathbb{R}^n) \hookrightarrow W^{s_2,q}(\mathbb{R}^n)$ if*

$$\frac{1}{q} = \frac{1}{p} - \frac{s_1 - s_2}{n}. \tag{2.3}$$

Proof. See [19, p. 129]. □

2.4. Lemma (interpolation in s). *Let $\mathbf{u} \in W^{s_1,p}(\mathbb{R}^n)^n$, $0 \leqslant s_2 \leqslant s \leqslant s_1 < \infty$, s non-integer. Then there exists a constant $c > 0$ such that*

$$\|\mathbf{u}\|_{s,p} \leqslant c \|\mathbf{u}\|_{s_1,p}^{\alpha} \|\mathbf{u}\|_{s_2,p}^{1-\alpha}, \tag{2.5}$$

where

(2.6) $$s = \alpha s_1 + (1-\alpha)\, s_2, \qquad \alpha \in \langle 0, 1\rangle.$$

P r o o f. See [20, pp. 181–186]. □

Korn inequality will be used for estimates of the nonlinear term:

2.7. Lemma (generalized Korn inequality). *Let $\varphi \in W^{1,q}(\mathbb{R}^n)^n \cap W^{1,2}(\mathbb{R}^n)^n$, $q > 1$. Then*

(2.8) $$\left(\int_{\mathbb{R}^n} |\mathbf{e}(\varphi)|\, \mathbf{dx}\right)^{\frac{1}{q}} \geqslant K_q |\varphi|_{1,q},$$

where $K_q > 0$, $2e_{ij}(\varphi) = \frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_i}$.

P r o o f. See [16, pp. 47–48]. □

The following classical lemma will be used for the limiting passages in the nonlinear term:

2.9. Lemma. *Let $Q_T \subset \mathbb{R}^{n+1}$ be bounded. Let $f_N \colon Q_T \mapsto \mathbb{R}$ be integrable for every N and let*

(i) $\lim_{N\to\infty} f_N(\mathbf{y})$ *exist and be finite for a.e.* $\mathbf{y} \in Q_T$
(ii) $\forall \varepsilon > 0\ \exists \delta > 0$ *such that*

$$\sup_N \int_H |f_N(\mathbf{y})|\, dy < \varepsilon \qquad \forall H \subset Q_T;\ |H| < \delta.$$

Then

(2.10) $$\lim_{N\to\infty} \int_{Q_T} f_N(\mathbf{y})\, \mathbf{dy} = \int_{Q_T} \lim_{N\to\infty} f_N(\mathbf{y})\, \mathbf{dy}.$$

P r o o f. See [5]. □

3. Weak solution for the bipolar fluid

In this part we will deal with the problem (CBN). Our goal is to prove existence, uniqueness and regularity of the system

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{3.1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} - 2\mu_1 \frac{\partial}{\partial x_j} \Delta e_{ij} + f_i, \tag{3.2}$$

$$u_i(\mathbf{x}, 0) = u_{0i}(\mathbf{x}), \tag{3.3}$$

where the nonlinear tensor function $\tau(\cdot)$ fulfils the conditions (1.6)–(1.10).

We denote

$$H = \{\varphi \in L^2(\mathbb{R}^n)^n;\ \operatorname{div} \varphi = 0\}, \tag{3.4}$$

$$V_p = \left\{\varphi \in \mathscr{D}'(\mathbb{R}^n)^n;\ D\varphi \in L^p(\mathbb{R}^n)^{n^2};\ \operatorname{div} \varphi = 0\right\}. \tag{3.5}$$

The latter is equipped with the usual seminorm of the Sobolev space $W^{1,p}(\mathbb{R}^n)^n$, i.e. $|\cdot|_{V_p} = |\cdot|_{1,p}$. Hereafter, $\mathbf{u} \in L^p(I; V_p)$ means that $D\mathbf{u} \in L^p(I; L^p(\mathbb{R}^n)^{n^2})$ and $|\mathbf{u}|_{L^p(I;V_p)} = \|D\mathbf{u}\|_{L^p(I;L^p(\mathbb{R}^n)^{n^2})}$. We denote

$$U = W^{2,2}(\mathbb{R}^n)^n \cap V_p. \tag{3.6}$$

We will assume the following about the data of our problem:

$$\begin{aligned} &\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^n)^n \cap H, \\ &\mathbf{f} \in L^2(I; L^2(\mathbb{R}^n)^n). \end{aligned} \tag{3.7}$$

3.8. Definition. The function $\mathbf{u} \in L^p(I; V_p) \cap C(I; H) \cap L^2(I; W^{2,2}(\mathbb{R}^n)^n)$ with $\frac{\partial \mathbf{u}}{\partial t} \in L^2(I; L^2(\mathbb{R}^n)^n)$ is called a weak solution of the problem (CBN) if

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\partial u_i}{\partial t} \varphi_i \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\varphi) \,\mathrm{d}\mathbf{x} \\ + 2\mu_1 \int_{\mathbb{R}^n} \frac{\partial e_{ij}(\mathbf{u})}{\partial x_k} \frac{\partial e_{ij}(\varphi)}{\partial x_k} \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} f_i \varphi \,\mathrm{d}\mathbf{x} \end{aligned} \tag{3.9}$$

is satisfied a.e. in I for every $\varphi \in U$.

In order to be able to use the Galerkin method, we need to find a countable dense subset of the space U with special properties. In fact, we need the functions of this subset to be smooth, have compact support and zero divergence in $\mathbb{R}^n$. The existence of such a subset is ensured by the following lemma.

We denote

$$\mathscr{D}_0(\mathbb{R}^n)^n = \{\varphi \in \mathscr{D}(\mathbb{R}^n)^n;\ \operatorname{div}\varphi = 0\}.$$

3.10. Lemma. *There exists a countable subset of the space $\mathscr{D}_0(\mathbb{R}^n)^n$ which is dense in U.*

Proof. As $\mathscr{D}(\mathbb{R}^n)^n$ is dense in $W^{2,2}(\mathbb{R}^n)^n$ and V_p, we have that $\mathscr{D}(\mathbb{R}^n)^n$ is dense in U. The separability of $\mathscr{D}(\mathbb{R}^n)^n$ yields the existence of a countable subset of $\mathscr{D}(\mathbb{R}^n)^n$ which is dense in U. We denote its elements $\{\varphi_n\}_{n=1}^{\infty}$. These functions have generally a non-zero divergence.

We denote $g_n = \operatorname{div}\varphi_n$, where evidently $g_n \in \mathscr{D}(\mathbb{R}^n)$. Let us solve the problem

$$\operatorname{div}\psi_n = g_n. \tag{3.11}$$

In [4] it is shown that there exists a solution $\psi_n \in \mathscr{D}(\mathbb{R}^n)$ such that

$$\|\psi_n\|_{2,2} \leqslant c_1\|g_n\|_{1,2}, \tag{3.12}$$

$$\|\psi_n\|_{1,p} \leqslant c_2\|g_n\|_p. \tag{3.13}$$

We denote

$$\mathbf{w}^n = \varphi_n - \psi_n. \tag{3.14}$$

Now, let $\mathbf{v}$ be an arbitrary element of U and ε a positive number. Then there exists $\varphi_n \in \mathscr{D}(\mathbb{R}^n)^n$ such that

$$\|\varphi_n - \mathbf{v}\|_U = \|\varphi_n - \mathbf{v}\|_{2,2} + |\varphi_n - \mathbf{v}|_{1,p} \leqslant \frac{\varepsilon}{1 + c_1 + c_2}, \tag{3.15}$$

see (3.12), (3.13). Then ($\operatorname{div}\mathbf{v} = 0$)

$$\begin{aligned}\|\mathbf{w}_n - \mathbf{v}\|_U &= \|\varphi_n - \mathbf{v} - \psi_n\|_U \\ &\leqslant \|\varphi_n - \mathbf{v}\|_U + \|\psi_n\|_U \\ &\leqslant \frac{\varepsilon}{1 + c_1 + c_2} + c_1\|\operatorname{div}(\varphi_n - \mathbf{v})\|_{1,2} + c_2\|\operatorname{div}(\varphi_n - \mathbf{v})\|_p \leqslant \varepsilon\end{aligned}$$

and the set $\{\mathbf{w}_n\}_{n=1}^{\infty}$ is dense in U. □

The next two simple lemmas will be used for the apriori estimates. Their proofs are direct consequences of the fact that $\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 |\xi|^{2k}\, d\xi$ is an equivalent seminorm on $W^{k,2}(\mathbb{R}^n)$, which can be found e.g. in [16]. Here $\widehat{u}(\xi)$ is the Fourier transform of u.

3.16. Lemma. *Let $u \in L^2(\mathbb{R}^n)$, $D^{(2)}u \in L^2(\mathbb{R}^n)^{n^2}$. Then $Du \in L^2(\mathbb{R}^n)^n$ and*

$$\|Du\|_2^2 \leqslant c_3 \|u\|_2 \|D^{(2)}u\|_2. \tag{3.17}$$

3.18. Lemma. *Let $u \in W^{2,2}(\mathbb{R}^n)$, $n \leqslant 3$. Then $u \in L^\infty(\mathbb{R}^n)$, i.e. there exists $c_4 > 0$ such that*

$$\operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^n} |u(\mathbf{x})| \leqslant c_4 \|u\|_{2,2}. \tag{3.19}$$

Now let $\{\mathbf{w}^n\}_{n=1}^\infty$ be our countable dense subset from Lemma 3.10 (after eliminating zero and linearly depending functions).

3.20. Definition. We say that $\mathbf{u}^N(\mathbf{x},t) = \sum_{i=1}^N c_i^N(t)\mathbf{w}^i(\mathbf{x})$ is the Galerkin approximation of the solution of the problem (CBN) if

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\sum_{l=1}^N \frac{\partial c_l^N(t)}{\partial t} w_i^l(\mathbf{x}) \right) w_i^\alpha(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&+ \int_{\mathbb{R}^n} \tau_{ij}(\mathbf{e}(\mathbf{u}^N(\mathbf{x},t))) e_{ij}(\mathbf{w}^\alpha(\mathbf{x}))\, \mathrm{d}\mathbf{x} \\
&+ 2\mu_1 \int_{\mathbb{R}^n} \frac{\partial e_{ij}(\mathbf{u}^N(\mathbf{x},t))}{\partial x_k} \frac{\partial e_{ij}(\mathbf{w}^\alpha(\mathbf{x}))}{\partial x_k}\, \mathrm{d}\mathbf{x} \\
&+ \int_{\mathbb{R}^n} \left(\sum_{l=1}^N c_l^N(t) w_j^l(\mathbf{x}) \right) \left(\sum_{k=1}^N c_k^N(t) \frac{\partial w_i^k(\mathbf{x})}{\partial x_j} \right) w_i^\alpha(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&- \int_{\mathbb{R}^n} f_i(\mathbf{x},t) w_i^\alpha(\mathbf{x}) = 0 \qquad \forall \mathbf{w}^\alpha \quad \alpha = 1,2,\dots,N.
\end{aligned} \tag{3.21}$$

Using the Carathéodory theorem (see [7]) we get the existence of the Galerkin approximation locally in time. From the apriori estimates in $L^\infty(I;H)$ we have the existence on each time interval $(0,T)$, $T < \infty$.

3.22. Remark. For the Carathéodory theorem we need that the matrix with the elements $a^{l\alpha} = \int_{\mathbb{R}^n} w_i^l w_i^\alpha\, \mathrm{d}\mathbf{x}$ be regular. It is the so-called Gram matrix, and it is known that the Gram matrix is regular provided $\{\mathbf{w}^\alpha\}_{\alpha=1}^N$ are linearly independent.

3.23. Lemma. *Let $\mathbf{u}_0 \in H$, $\mathbf{f} \in L^2(I; L^2(\mathbb{R}^n)^n)$. Then the sequence of Galerkin approximations satisfies the following uniform estimates:*

$$\|\mathbf{u}^N\|_{L^\infty(I;L^2(\mathbb{R}^n)^n)} \leqslant c_5, \tag{3.24}$$

$$\|D\mathbf{u}^N\|_{L^p(I;L^p(\mathbb{R}^n)^{n^2})} \leqslant c_6, \tag{3.25}$$

$$\|\mathbf{u}^N\|_{L^2(I;W^{2,2}(\mathbb{R}^n)^n)} \leqslant c_7. \tag{3.26}$$

Proof. Multiplying (3.21) by $c_\alpha^N(t)$ and summing up the equations we get (using the fact that $\int_{\mathbb{R}^n} u_j^N \frac{\partial u_i^N}{\partial x_j} u_i^N \,\mathrm{d}\mathbf{x} = 0$ for divergence free functions)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^n} |\mathbf{u}^N|^2 \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \tau_{ij}(\mathbf{e}(\mathbf{u}^N))e_{ij}(\mathbf{u}^N)\,\mathrm{d}\mathbf{x} + 2\mu_1 \int_{\mathbb{R}^n} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} f_i u_i^N \,\mathrm{d}\mathbf{x}.$$

Integrating over $(0,t)$ and using the coercivity condition (1.7) and the Korn inequality (2.7) we obtain

$$\frac{1}{2}\|\mathbf{u}^N(t)\|_2^2 \,\mathrm{d}t + c_p \int_0^t \|D\mathbf{u}^N\|_p^p \,\mathrm{d}t + \tilde{c}_2\mu_1 \int_0^t \|D^{(2)}\mathbf{u}^N\|_2^2 \,\mathrm{d}t \leqslant \left|\int_0^t \int_{\mathbb{R}^n} f_i u_i^N \,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right| + \frac{1}{2}\|\mathbf{u}_0\|_2^2. \tag{3.27}$$

Taking the first term on the left hand side we get

$$\|\mathbf{u}^N(t)\|_2^2 \leqslant \int_0^t \|\mathbf{f}\|_2(1 + \|\mathbf{u}^N\|_2^2)\,\mathrm{d}t + \|\mathbf{u}_0\|_2^2, \tag{3.28}$$

which after employing the Gronwall inequality (see e.g. [7]), proves (3.24). The other two estimates we get from (3.27) and Lemma 3.16. □

3.29. Remark. By means of (1.6) and (1,7) it is possible to show that there exist constants c_8 and c_9 such that $\forall \mathbf{u} \in W^{1,2}(\mathbb{R}^n)^n \cap W^{1,p}(\mathbb{R}^n)^n$,

$$c_8\|\mathbf{e}(\mathbf{u})\|_p^p \leqslant \|\vartheta(\mathbf{e}(\mathbf{u}))\|_1 \leqslant c_9(\|\mathbf{e}(\mathbf{u})\|_2^2 + \|\mathbf{e}(\mathbf{u})\|_p^p).$$

3.30. Lemma. *Let $n \leqslant 3$, $\mathbf{f} \in L^2(I; L^2(\mathbb{R}^n)^n)$, $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^n)^n \cap V_p$, $p > 1$. Then*

$$\|\mathbf{u}^N\|_{L^\infty(I;W^{2,2}(\mathbb{R}^n)^n)} \leqslant c_{10}, \tag{3.31}$$

$$\left\|\frac{\partial \mathbf{u}^N}{\partial t}\right\|_{L^2(I;L^2(\mathbb{R}^n)^n)} \leqslant c_{11}. \tag{3.32}$$

P r o o f. Multiplying (3.21) by $\frac{\partial c_\alpha^N(t)}{\partial t}$, summing up from 1 to N and integrating over $(0,t)$, $t \in (0,T]$ we have

$$
\begin{aligned}
(3.33) \qquad & \int_{Q_t} \Big|\frac{\partial \mathbf{u}^N}{\partial t}\Big|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^n} \vartheta(\mathbf{e}(\mathbf{u}^N(t)))\,\mathrm{d}\mathbf{x} - \int_{\mathbb{R}^n} \vartheta(\mathbf{e}(\mathbf{u}^N(0)))\,\mathrm{d}\mathbf{x} \\
&+ \mu_1 \int_{\mathbb{R}^n} \Big|\frac{\partial e_{ij}(\mathbf{u}^N(t))}{\partial x_k}\Big|^2 \,\mathrm{d}\mathbf{x} - \mu_1 \int_{\mathbb{R}^n} \Big|\frac{\partial e_{ij}(\mathbf{u}^N(0))}{\partial x_k}\Big|^2 \,\mathrm{d}\mathbf{x} \\
&+ \int_{Q_t} u_j^N \frac{\partial u_i^N}{\partial x_j}\frac{\partial u_i^N}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_t} f_i \frac{\partial u_i^N}{\partial t}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
$$

The assumptions on $\mathbf{u}_0$, $\mathbf{f}$ and the scalar potential ϑ (non-negativity), the Korn inequality and in the case of the last two terms in (3.32) also the Hölder and Young inequalities yield

$$
(3.34) \qquad \frac{1}{2}\Big\|\frac{\partial \mathbf{u}^N}{\partial t}\Big\|^2_{L^2(Q_t)} + \mu_1\tilde{c}_2\|D^{(2)}\mathbf{u}^N(t)\|_2^2 \leqslant c(\mathbf{u}_0,\mathbf{f}) + \int_{Q_t} |\mathbf{u}^N|^2|D\mathbf{u}^N|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
$$

The convective term on the right hand side of (3.34) can be estimated by means of Lemmas 3.17 and 3.23:

$$
\begin{aligned}
\int_0^t \int_{\mathbb{R}^n} |\mathbf{u}^N|^2|D\mathbf{u}^N|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t &\leqslant c_4 \int_0^t \|\mathbf{u}^N\|_{2,2}^2\|D\mathbf{u}^N\|_2^2\,\mathrm{d}t \\
&\leqslant c_3c_4\int_0^t \|\mathbf{u}^N\|_{2,2}^2(\varepsilon\|D^{(2)}\mathbf{u}^N\|_2^2 + \lambda(\varepsilon)\|\mathbf{u}^N\|_2^2)\,\mathrm{d}t.
\end{aligned}
$$

In the first term we take in $\|D^{(2)}\mathbf{u}^N\|_2$ the supremum over $(0,t)$ and transfer it with a small coefficient ε to the left hand side of (3.34). The other term is finite thanks to the apriori estimates in $L^\infty(I;H)$ and $L^2(I;W^{2,2}(\mathbb{R}^n)^n)$. The estimates (3.31) and (3.32) follow from (3.34) and Lemma 3.16. $\square$

3.35. R e m a r k. Multiplying (3.9) by $\xi^2(t)\frac{\partial c_\alpha^N(t)}{\partial t}$, where $\xi(t) = 0$ on $[0,\frac{\delta}{2}]$, $\xi(t) = 1$ on $[\delta,T]$ and $\xi(t) \in C^\infty([0,T])$ we can get the same estimates as (3.31)–(3.32) with $\mathbf{u}_0 \in H$ only, but on $[\delta,T]$ with $\delta > 0$ arbitrary.

3.36. Theorem. *Let* $n \leqslant 3$ *and let all the assumptions of Lemmas* 3.23 *and* 3.29 *be satisfied. Then there exists a unique weak solution of the problem* (CBN) *in the sense of Definition* 3.8. *Moreover,* $\mathbf{u} \in C^{\frac{1}{2}}(I;H)$.

P r o o f. *Existence.* Denote by $\mathbf{u}^N/B_R$ the restriction of the Galerkin approximation to the ball in $\mathbb{R}^n$ with diameter R. First, we take B_1 and denote $A_1 = \{\alpha \in \mathbb{N};\ \operatorname{supp}\mathbf{w}^\alpha \subset B_1\}$, where $\{\mathbf{w}^i\}_{i=1}^\infty$ is our dense countable subset in U.

As $\mathbf{u}^N/B_1$ is bounded in both $L^2(I;W^{2,2}(B_1)^n)$ and $L^p(I;W^{1,p}(B_1)^n)$ and $\frac{\partial \mathbf{u}^N}{\partial t}/B_1$ in $L^2(I;L^2(B_1)^n)$ we can derive by means of Lions-Aubin Lemma (see e.g. [10, Theorem 5.1]) that there exists a subsequence $\mathbf{u}_1^N$ such that

$$\mathbf{u}_1^N/B_1 \to \mathbf{u}_1 \text{ strongly in } L^2(I;W^{1,\tilde{p}}(B_1)^n)$$

with $\tilde{p} \in (1,\infty)$ for $n=2$ and $\tilde{p} \in (1,6)$ for $n=3$. Now we are able to carry out the limiting passage in (3.21) for fixed $\mathbf{w}^\alpha$ with $\alpha \in A_1$. (In the nonlinear term thanks to the above mentioned strong convergence in $L^2(I;W^{1,\tilde{p}}(B_1)^n)$, i.e. $D\mathbf{u}^{N'} \to D\mathbf{u}$ a.e. in $B_1 \times I$, and thanks to Lemma 2.9.)

Now we take B_2 and denote again $A_2 = \{\alpha \in \mathbb{N};\ \operatorname{supp} \mathbf{w}^\alpha \subset B_2\}$. Evidently $A_1 \subset A_2$ and we can deduce the existence of a subsequence $\mathbf{u}_2^N$ (chosen from $\mathbf{u}_1^N$) such that

$$\mathbf{u}_2^N/B_2 \to \mathbf{u}_2 \text{ strongly in } L^2(I;W^{1,\tilde{p}}(B_2)^n).$$

Evidently $\mathbf{u}_2/B_1 = \mathbf{u}_1$. So we can construct a "diagonal" sequence $\{\mathbf{u}_N^N\}_{N=1}^\infty$ such that

$$\mathbf{u}_N^N/B_R \to \mathbf{u}/B_R \text{ strongly in } L^2(I;W^{1,\tilde{p}}(B_R)^n)$$

for an arbitrary $R > 0$.

Now we use the fact that the system $\{\mathbf{w}^\alpha\}_{\alpha=1}^\infty$ is dense in U. We can close the test functions in U and thanks to the apriori estimates of the solution we get that the equality (3.9) is satisfied for every $\varphi \in U$ a.e. in I.

As $\mathbf{u} \in L^2(I;W^{2,2}(\mathbb{R}^n)^n) \cap L^2(I;H)$ and $\frac{\partial \mathbf{u}}{\partial t} \in L^2(I;H)$, it follows from Theorem 1.17, Chapter IV in [6] that $\mathbf{u} \in C(I;H)$. Moreover, we can show that $\mathbf{u} \in C^{\frac{1}{2}}(I;H)$. Put

$$\mathbf{u}(t) = \int_{t_1}^t \dot{\mathbf{u}}(s)\,\mathrm{d}s + \mathbf{u}(t_1).$$

From the Hölder inequality and the apriori estimate of the time derivative we conclude:

$$\|\mathbf{u}(t) - \mathbf{u}(t_1)\|_2^2 \leqslant |t - t_1| \int_{t_1}^t \|\dot{\mathbf{u}}(s)\|_2^2\,\mathrm{d}s$$

and therefore $\mathbf{u} \in C^{\frac{1}{2}}(I;H)$.

Uniqueness. Let $\mathbf{u}$, $\mathbf{v}$ be two weak solutions of the problem (CBN). Taking $\mathbf{w} = \mathbf{u} - \mathbf{v}$ as a test function for both equations for $\mathbf{u}$ and $\mathbf{v}$ we get

$$(3.37)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_2^2 + \int_{\mathbb{R}^n} (\tau_{ij}(\mathbf{e}(\mathbf{u})) - \tau_{ij}(\mathbf{e}(\mathbf{v})))e_{ij}(\mathbf{w}))\,\mathrm{d}\mathbf{x} + 2\mu_1 \int_{\mathbb{R}^n} \frac{\partial e_{ij}(\mathbf{w})}{\partial x_k}\frac{\partial e_{ij}(\mathbf{w})}{\partial x_k}\,\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^n} w_j \frac{\partial u_i}{\partial x_j} w_i\,\mathrm{d}\mathbf{x}.$$

It follows from the condition (1.10) that the second term in (3.37) is non-negative:

$$
\begin{aligned}
(3.38)\qquad & (\tau_{ij}(\mathbf{e}(\mathbf{u})) - \tau_{ij}(\mathbf{e}(\mathbf{v})))e_{ij}(\mathbf{u}-\mathbf{v})) \\
&= \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\alpha}\tau_{ij}(\mathbf{e}(\mathbf{v}+\alpha(\mathbf{u}-\mathbf{v})))\,\mathrm{d}\alpha\right) e_{ij}(\mathbf{u}-\mathbf{v}) \\
&= \left(\int_0^1 \frac{\partial^2\vartheta}{\partial e_{ij}\partial e_{kl}}(\mathbf{e}(\mathbf{v}+\alpha(\mathbf{u}-\mathbf{v})))\,\mathrm{d}\alpha\right) e_{ij}(\mathbf{u}-\mathbf{v}))e_{kl}(\mathbf{u}-\mathbf{v})) \\
&\geqslant c_3|\mathbf{e}(\mathbf{v}+\xi(\mathbf{u}-\mathbf{v}))|^{p-2}e_{ij}(\mathbf{u}-\mathbf{v})e_{ij}(\mathbf{u}-\mathbf{v}) \geqslant 0
\end{aligned}
$$

with $\xi \in [0,1]$. We obtain from (3.37) by means of the Korn inequality that

$$(3.39)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_2^2 + \tilde{c}_2|\mathbf{w}|_{2,2}^2 \leqslant \|\mathbf{w}\|_4^2|\mathbf{u}|_{1,2}.$$

The interpolation inequality, Lemmas 3.18, 3.16 and the Young inequality yield

$$
\begin{aligned}
(3.40)\qquad \|\mathbf{w}\|_4^2 \leqslant \|\mathbf{w}\|_2\|\mathbf{w}\|_\infty &\leqslant \tilde{c}\|\mathbf{w}\|_2\|\mathbf{w}\|_{2,2} \\
&\leqslant \tilde{c}\|\mathbf{w}\|_2(\|\mathbf{w}\|_2^2 + |\mathbf{w}|_{1,2}^2 + |\mathbf{w}|_{2,2}^2)^{\frac{1}{2}} \\
&\leqslant \varepsilon|\mathbf{w}|_{2,2}^2 + \lambda(\varepsilon)\|\mathbf{w}\|_2^2.
\end{aligned}
$$

Integrating (3.39) over $(0,t)$ we get along with (3.40) and the apriori estimate of the solution in $L^\infty(I;W^{2,2}(\mathbb{R}^n)^n)$ $(\mathbf{w}(0)=0)$

$$\frac{1}{2}\|\mathbf{w}(t)\|_2^2 \leqslant \tilde{c}_8\int_0^t \|\mathbf{w}(\tau)\|_2^2\,\mathrm{d}\tau.$$

The Gronwall inequality implies

$$\|\mathbf{w}\|_2 = 0 \qquad \text{a.e. in I}$$

and therefore $\mathbf{u}=\mathbf{v}$ a.e. in Q_T. □

At the end of this part we prove regularity of the weak solution, i.e. that $\mathbf{u} \in L^2(I;W^{3,2}(\mathbb{R}^n)^n)$.

Let $\mathbf{e}_k$ be a unit vector in the direction of $\mathbf{x}_k$. Then

$$(3.41)\qquad \Delta_k^h u(t) = \frac{u(\mathbf{x}+h\mathbf{e}_k,t)-u(\mathbf{x},t)}{h},$$

$$(3.42)\qquad \Delta_k^{2,h}u(t) = \frac{u(\mathbf{x}+h\mathbf{e}_k,t)-2u(\mathbf{x},t)+u(\mathbf{x}-h\mathbf{e}_k,t)}{h^2}.$$

The proofs of the following two lemmas can be found for example in [13].

3.43. Lemma. *Let $u \in W^{1,p}(\mathbb{R}^n)$. Then $\|\Delta_k^h u\|_p \leqslant c_{12} \|\frac{\partial u}{\partial x_k}\|_p$, where $c_{12} > 0$ does not depend on h.*

3.44. Lemma. *Let $\|\Delta_k^h u\|_p \leqslant c_{13} \quad \forall h > 0$, $c_{13} > 0$. Then $\|\frac{\partial u}{\partial x_k}\|_p \leqslant c_{13}$.*

3.45. Theorem. *Let all assumptions of Theorem 3.36 be satisfied. Then the weak solution of the problem* (CBN) $\mathbf{u} \in L^2(I; W^{3,2}(\mathbb{R}^n)^n)$.

Proof. We take $\frac{\mathbf{u}(\mathbf{x}+h\mathbf{e}_k,t)}{h^2}$, $\frac{-2\mathbf{u}(\mathbf{x},t)}{h^2}$ and $\frac{\mathbf{u}(\mathbf{x}-h\mathbf{e}_k,t)}{h^2}$ ($h > 0$ arbitrary but fixed) as test functions and integrate over $(0, T)$:

$$
\begin{aligned}
(3.46) \qquad & \int_0^T \int_{\mathbb{R}^n} \frac{\partial u_i(\mathbf{x},t)}{\partial t} \Delta_k^{2,h} u_i(t) \\
& + \int_0^T \int_{\mathbb{R}^n} u_j(\mathbf{x},t) \frac{\partial u_i(\mathbf{x},t)}{\partial x_j} \Delta_k^{2,h} u_i(t) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& + \int_0^T \int_{\mathbb{R}^n} \tau_{ij}(\mathbf{e}(\mathbf{u}(\mathbf{x},t))) e_{ij}\left(\Delta_k^{2,h}\mathbf{u}(t)\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& + 2\mu_1 \int_0^T \int_{\mathbb{R}^n} \frac{\partial e_{ij}(\mathbf{u}(\mathbf{x},t))}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k}\left(\Delta_k^{2,h}\mathbf{u}(t)\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& - \int_0^T \int_{\mathbb{R}^n} f_i(\mathbf{x}) \Delta_k^{2,h} u_i(t) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0.
\end{aligned}
$$

We use the substitution $\mathbf{x} - h\mathbf{e}_k = \tilde{\mathbf{x}}$, sum up the equations for $k = 1 \dots n$ and get

$$
\begin{aligned}
(3.47) \quad & \frac{1}{2} \int_{\mathbb{R}^n} \left(\Delta_k^h u_i(T) \Delta_k^h u_i(T)\right) \mathrm{d}\mathbf{x} \\
& + \int_0^T \int_{\mathbb{R}^n} \frac{\tau_{ij}(\mathbf{e}[\mathbf{u}(\mathbf{x}+h\mathbf{e}_k,t)]) - \tau_{ij}(\mathbf{e}[\mathbf{u}(\mathbf{x},t)])}{h} e_{ij}\left(\Delta_k^h \mathbf{u}(t)\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& + 2\mu_1 \int_0^T \int_{\mathbb{R}^n} \frac{\partial e_{ij}}{\partial x_l}\left(\Delta_k^h \mathbf{u}(t)\right) \frac{\partial e_{ij}}{\partial x_l}\left(\Delta_k^h \mathbf{u}(t)\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
\leqslant{} & \frac{1}{2} \int_{\mathbb{R}^n} \left(\Delta_k^h u_{0i} \Delta_k^h u_{0i}\right) \mathrm{d}\mathbf{x} + \left| \int_0^T \int_{\mathbb{R}^n} f_i(\mathbf{x},t) \Delta_k^{2,h} u_i(t) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \right| \\
& + \left| \int_0^T \int_{\mathbb{R}^n} \frac{u_j(\mathbf{x}+h\mathbf{e}_k,t) \frac{\partial u_i(\mathbf{x}+h\mathbf{e}_k,t)}{\partial x_j} - u_j(\mathbf{x},t) \frac{\partial u_i(\mathbf{x},t)}{\partial x_j}}{h} \Delta_k^h u_i(t) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \right|.
\end{aligned}
$$

The first term on the left hand side of (3.47) is evidently non-negative. Similarly as in (3.38) we can show that also the second term is non-negative.

As $\mathbf{u} \in L^2(I; W^{2,2}(\mathbb{R}^n)^n)$ and $\mathbf{u}_0 \in W^{2,2}(\mathbb{R}^n)^n$, the first two terms on the right hand side of (3.47) can be estimated by means of the Hölder inequality and

Lemma 3.43. The estimate of the convective term is a bit more complicated. It follows from the Hölder inequality and Lemma 3.42 that

$$\int_0^T \int_{\mathbb{R}^n} \frac{u_j(\mathbf{x}+h\mathbf{e}_k,t)\frac{\partial u_i(\mathbf{x}+h\mathbf{e}_k,t)}{\partial x_j} - u_j(\mathbf{x},t)\frac{\partial u_i(\mathbf{x},t)}{\partial x_j}}{h} \Delta_k^h u_i(t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$\leqslant c\int_0^T |\mathbf{u}(t)|_{1,2}\left(\int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_k}\left(u_j(\mathbf{x},t)\frac{\partial u_i(\mathbf{x},t)}{\partial x_j}\right)\right)^2 \mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \mathrm{d}t \leqslant \text{using}(3.31)$$

$$\leqslant c_8\int_0^T \left(\int_{\mathbb{R}^n} |\mathbf{u}(t)|^2|D^{(2)}\mathbf{u}(t)|^2\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |D\mathbf{u}(t)|^4\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \mathrm{d}t$$

$$\leqslant \bar{c}_8\int_0^T \left(\|\mathbf{u}(t)\|_\infty|\mathbf{u}(t)|_{2,2} + |\mathbf{u}(t)|_{1,4}^2\right)\,\mathrm{d}t.$$

The first term is estimated by means of Lemma 3.18 and the apriori estimate in $L^2(I;W^{2,2}(\mathbb{R}^n)^n)$. The other term can be estimated by the following interpolation inequalities and imbeddings:

a) $n=3$

$$|\mathbf{u}|_{1,4} \leqslant |\mathbf{u}|_{1,2}^{\frac{1}{4}}|\mathbf{u}|_{1,6}^{\frac{3}{4}} \leqslant |\mathbf{u}|_{1,2}^{\frac{1}{4}}|\mathbf{u}|_{2,2}^{\frac{3}{4}}$$

and the boundedness follows from the estimate in $L^2(I;W^{2,2}(\mathbb{R}^n)^n)$;

b) $n=2$

$$|\mathbf{u}|_{1,4} \leqslant |\mathbf{u}|_{1,2}^{\frac{1}{2}}|\mathbf{u}|_{2,2}^{\frac{1}{2}}$$

(see e.g. [18]) and we use again the same apriori estimate as above. Thanks to the Korn inequality we have that $\int_0^T\int_{\mathbb{R}^n}\Delta_k^h(D^{(2)}(\mathbf{u}(t))\Delta_k^h(D^{(2)}(\mathbf{u}(t)) \leqslant \tilde{c}_{14}$ which together with Lemma 3.44 gives $\|\mathbf{u}\|_{L^2(I;W^{3,2}(\mathbb{R}^n)^n)} \leqslant c_{14}$. Moreover, the term $|\int_0^T\int_{\mathbb{R}^n}\tau_{ij}(\mathbf{e}(\mathbf{u}(\mathbf{x},t)))e_{ij}(\Delta_k^{2,h}\mathbf{u}(t))\,\mathrm{d}\mathbf{x}\,\mathrm{d}t|$ is uniformly bounded for arbitrary $h>0$. □

4. Weak solution for the monopolar fluid

Hereafter we will study the problem (3.1)–(3.3), assuming $\mu_1^N \to 0^+$. We denote by $\mathbf{u}^N$ the solution of the problem (CBN) (i.e. with μ_1^N), by $\mathbf{u}$ the solution of the problem (CMN) (i.e. with $\mu_1 = 0$). Our system of equations (for the monopolar fluid) takes the form

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{4.1}$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial \pi}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} + f_i, \tag{4.2}$$

$$u_i(\mathbf{x},0) = u_{0i}(\mathbf{x}), \tag{4.3}$$

where the tensor function $\tau_{ij}(\mathrm{e})$ satisfies the conditions (1.6)–(1.10).

Using the apriori estimates in $L^\infty(I;H)$ and $L^p(I;V_p)$ (only these do not depend on μ_1^N) we can get thanks to the estimate of the time derivative in $L^{p'}(I;(X(\tilde\Omega)')$ $(X(\tilde\Omega) = \left\{\varphi \in W_0^{4,2}(\tilde\Omega) \cap W^{1,p}(\tilde\Omega) \cap W^{1,p'}(\tilde\Omega);\ \operatorname{div}\varphi = 0\right\}$, $\tilde\Omega$ a bounded open subset of $\mathbb{R}^n$) the existence of the measure-valued solution of the problem (CMN) for $p > \frac{2n}{n+2}$. It means that there exists a couple $(\mathbf{u},\nu)$,

$$\mathbf{u} \in L^p(I;V_p) \cap L^\infty(I;H),$$
$$\nu \in L_w^\infty(Q_T; M(\mathbb{R}^{n^2}))$$

($M(\mathbb{R}^{n^2})$ is the space of the Radon measures on $\mathbb{R}^{n^2}$) such that

$$\int_{Q_T}\left(-u_i\frac{\partial\varphi_i}{\partial t} - u_ju_i\frac{\partial\varphi_i}{\partial x_j} + e_{ij}(\varphi)\int_{\mathbb{R}^{n^2}}\tau_{ij}(\mathbf{e}(\lambda))\,\mathrm{d}\nu(\lambda) - f_i\varphi_i\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{\mathbb{R}^n} u_{0i}\varphi_i\,\mathrm{d}\mathbf{x} \tag{4.4}$$

for every $\varphi \in C^1(I;\mathscr{D}_0(\mathbb{R}^n)^n)$, $\varphi(T) = 0$ and

$$D\mathbf{u}(\mathbf{x},t) = \int_{\mathbb{R}^{n^2}}\lambda\,\mathrm{d}\nu(\lambda) \text{ a.e. in } Q_T. \tag{4.5}$$

In the case of the pertubated linear model (i.e. $\tau(\mathbf{e}) = (\nu_0 + \nu_1|\mathbf{e}|^{p-2})\mathbf{e}$ for $p < 2$) we get the same result as above for arbitrary $p > 1$ in both the two- and the three-dimensional case. This is connected with the fact that we have also an independent estimate in $L^2(I;V_2)$. For more detailed description see [16] or for the Dirichlet problem [2] or [11].

In the next part we will try to find new estimates of solution of the problem (CBN) which will make the limiting passage in the nonlinear term possible. So we will get a weak solution of the problem (CMN). In fact the estimates will guarantee that $D\mathbf{u}^N \to D\mathbf{u}$ in $L^{\tilde p}(I;L^{\tilde p}(\mathbb{R}^n)^{n^2})$, i.e. $\nabla\mathbf{u}^N \to \nabla\mathbf{u}$ a.e. in Q_T. Then, using Lemma 2.9, we will get the desired limiting passage. About the data we will assume the following:

$$\mathbf{u}_0 \in W^{1,2}(\mathbb{R}^n)^n \cap H,$$
$$\mathbf{f} \in \begin{cases} L^2(I;L^2(\mathbb{R}^n)^n), & p \geqslant 2 \\ L^{p'}(I;L^{p'}(\mathbb{R}^n)^n), & p < 2,\ p' = \frac{p}{p-1}. \end{cases} \tag{4.6}$$

The weak solution of the problem (CMN) is defined as follows:

4.7. Definition. Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (4.6), and let $p \geqslant 1 + \frac{2n}{n+2}$. Then a function $\mathbf{u}$, where

$$\mathbf{u} \in L^p(I; V_p) \cap C(I; H) \cap L^2(I; W^{1,2}(\mathbb{R}^n)^n), \tag{4.8}$$

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(I; H), \tag{4.9}$$

is called a weak solution of the problem (CMN) if

$$\int_{\mathbb{R}^n} \frac{\partial u_i}{\partial t} \varphi_i \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\varphi) \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} f_i \varphi_i \,\mathrm{d}\mathbf{x} \tag{4.10}$$

is satisfied a.e. in I for every $\varphi \in V_p \cap W^{1,2}(\mathbb{R}^n)^n$.

4.11. Definition. Let $\mathbf{u}_0$, $\mathbf{f}$ satisfy (4.6), and let $p \geqslant \frac{3n}{n+2}$. Then a function $\mathbf{u}$, where

$$\mathbf{u} \in L^p(I; V_p) \cap L^\infty(I; H), \tag{4.12}$$

is called a weak solution of the problem (CMN) if

$$\begin{aligned} -\int_{Q_T} u_i \frac{\partial \varphi_i}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\varphi) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ = \int_{Q_T} f_i \varphi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^n} u_{0i} \varphi_i(0) \,\mathrm{d}\mathbf{x} \end{aligned} \tag{4.13}$$

is satisfied for every $\varphi \in C^1(I; \mathscr{D}_0(\mathbb{R}^n)^n)$ with $\varphi(T) = 0$.

4.14. Remark. The existence of a weak solution means that the Young measure ν from (4.4) is the Dirac measure a.e. in Q_T, i.e. $\nu_{\mathbf{x},t} = \delta(\lambda - D\mathbf{u}(\mathbf{x},t))$ for a.e. $(\mathbf{x},t) \in Q_T$.

Let $\mathbf{u}^N$ be a solution of the problem (CBN), i.e. with $\mu_1^N > 0$. Let $\mu_1^N \to 0^+$ for $N \to \infty$. From Chapter 3 we have the following apriori estimates, which do not depend on μ_1:

$$\|\mathbf{u}^N\|_{L^\infty(I;H)} \leqslant c_1, \tag{4.15}$$

$$\|D\mathbf{u}^N\|_{L^p(I;L^p(\mathbb{R}^n)^{n^2})} \leqslant c_2. \tag{4.16}$$

From Theorem 3.45 we know that $\mathbf{u}^N$ is bounded in $L^2(I; W^{3,2}(\mathbb{R}^n)^n)$ and therefore also in $L^2(I; W^{1,\infty}(\mathbb{R}^n)^n)$ (of course, the estimate tends to ∞ when $\mu_1^N \to 0^+$). We want to use $\Delta \mathbf{u}^N$ as a test function in (3.9). However it is not possible to use it directly. Let us assume that the test function $\mathbf{w} \in \mathscr{D}_0(\mathbb{R}^n)^n$ and $\mathbf{w} = \Delta\mathbf{v}$. Integrating by parts we get

$$
\begin{aligned}
(4.17)\qquad & \int_{\mathbb{R}^n} \frac{\partial^2 u_i^N}{\partial t \partial x_k} \frac{\partial v_i}{\partial x_k}\, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \frac{\partial}{\partial x_k} \left(u_j^N \frac{\partial u_i^N}{\partial x_j} \right) \frac{\partial v_i}{\partial x_k}\, \mathrm{d}\mathbf{x} \\
&+ \int_{\mathbb{R}^n} \frac{\partial \tau_{ij}(\mathbf{e}(\mathbf{u}^N))}{\partial x_k} e_{ij}\Big(\frac{\partial \mathbf{v}}{\partial x_k} \Big)\, \mathrm{d}\mathbf{x} \\
&+ 2\mu_1^N \int_{\mathbb{R}^n} e_{ij}(\Delta \mathbf{u}^N) e_{ij}(\Delta \mathbf{v})\, \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} f_i \Delta v_i\, \mathrm{d}\mathbf{x} = 0.
\end{aligned}
$$

The fourth term in (4.17) is finite thanks to the regularity of the solution. The equality (4.17) is satisfied for arbitrary $\mathbf{v} \in \mathscr{D}_0(\mathbb{R}^n)^n$. Thanks to the density property (Lemma 3.10) and regularity result (Theorem 3.45) we take a sequence $\mathbf{v}_N^n$ such that $\mathbf{v}_N^n \to \mathbf{u}^N$ in $W^{3,2}(\mathbb{R}^n)^n \cap V_p$ for a.e. $t \in I$, N fixed. For $p \geqslant 2$ all the limiting passages can be done very easily. For $p < 2$ we should get similar results, but the limiting passage in the nonlinear term is not completely clear to the author. After multiplying by $(1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda}$, $\lambda \geqslant 0$ and integrating over $(0, T)$ we get

$$
\begin{aligned}
(4.18)\quad & \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_{\mathbb{R}^n} |\nabla \mathbf{u}^N|^2\, \mathrm{d}\mathbf{x} \right) \mathrm{d}t \\
&+ \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \left(\int_{\mathbb{R}^n} \frac{\partial u_j^N}{\partial x_k} \frac{\partial u_i^N}{\partial x_j} \frac{\partial u_i^N}{\partial x_k}\, \mathrm{d}\mathbf{x} \right) \mathrm{d}t \\
&+ \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \left(\frac{\partial \tau_{ij}(\mathbf{e}(\mathbf{u}^N))}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\, \mathrm{d}\mathbf{x} \right) \mathrm{d}t \\
&+ \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} 2\mu_1^N \left(\int_{\mathbb{R}^n} e_{ij}(\Delta \mathbf{u}^N) e_{ij}(\Delta \mathbf{u}^N)\, \mathrm{d}\mathbf{x} \right) \mathrm{d}t \\
&+ \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \left(\int_{\mathbb{R}^n} f_i \Delta u_i^N\, \mathrm{d}\mathbf{x} \right) \mathrm{d}t = 0.
\end{aligned}
$$

We can calculate:

$$
(4.19)\qquad (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |\mathbf{u}^N|_{1,2}^2 = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2(1-\lambda)} (1 + |\mathbf{u}^N|_{1,2}^2)^{1-\lambda}, & \lambda \neq 1 \\ \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \log(1 + |\mathbf{u}^N|_{1,2}^2), & \lambda = 1. \end{cases}
$$

By means of (1.9) and (1.10) we get from the nonlinear term

$$
\begin{aligned}
(4.20)\quad & \int_{\mathbb{R}^n} \frac{\partial \tau_{ij}(\mathbf{e}(\mathbf{u}^N))}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \frac{\partial^2 \vartheta(\mathbf{e}(\mathbf{u}^N))}{\partial e_{lm} \partial e_{ij}} \frac{\partial e_{lm}(\mathbf{u}^N)}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k} \\
&\geqslant \begin{cases} c_3 \int_{\mathbb{R}^n} (1 + |\mathbf{e}(\mathbf{u}^N)|^{p-2}) \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}, & p \geqslant 2 \\ c_3 \int_{\mathbb{R}^n} |\mathbf{e}(\mathbf{u}^N)|^{p-2} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k}, & p < 2. \end{cases}
\end{aligned}
$$

4.21. Remark. For the perturbated linear problem we get on the right hand side of (4.20)

$$c_3 \int_{\mathbb{R}^n} (1 + |\mathbf{e}(\mathbf{u}^N)|^{p-2}) \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k} \frac{\partial e_{ij}(\mathbf{u}^N)}{\partial x_k} \, \mathrm{d}\mathbf{x} \quad p > 1.$$

We denote the term on the right hand side of (4.20) by $\mathscr{J}$ and

$$(1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \mathscr{J} = \mathscr{K}. \tag{4.22}$$

As the term with μ_1^N is obviously non-negative, we can rewrite (4.18) as follows:

$$\begin{aligned} \frac{1}{2(1-\lambda)}(1 + |\mathbf{u}^N(T)|_{1,2}^2)^{1-\lambda} + c_3 \int_0^T \mathscr{K} \, \mathrm{d}t \\ \leqslant + c(\mathbf{u}_0) + \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} |\mathbf{u}^N|_{1,3}^3 \, \mathrm{d}t \\ + \int_0^T (1 + |\mathbf{u}^N|_{1,2}^2)^{-\lambda} \left| \int_{\mathbb{R}^n} f_i \Delta u_i^N \, \mathrm{d}\mathbf{x} \right| \mathrm{d}t \end{aligned} \tag{4.23}$$

(for $\lambda = 1$ the first terms on the left hand side is replaced by $\frac{1}{2} \log(1 + |\mathbf{u}^N|_{1,2}^2)$).

4.24. Remark. Using a similar technique as in Remark 3.34 we would obtain the same results for $\mathbf{u}_0 \in H$ but only on $[\delta, T]$, $\delta > 0$ arbitrary. This remark holds for everything which will be proved in this chapter.

4.25. Lemma. *For* $\mathbf{u}$ *smooth enough we have*

$$\|D^{(2)}\mathbf{u}\|_2 \leqslant c_4 \mathscr{J}^{\frac{1}{2}} \quad \text{for } p \geqslant 2, \tag{4.26}$$

$$\|D^{(2)}\mathbf{u}\|_p \leqslant c_5 |\mathbf{u}|_{1,p}^{\frac{2-p}{2}} \mathscr{J}^{\frac{1}{2}} \quad \text{for } 1 < p \leqslant 2. \tag{4.27}$$

Proof. The inequality (4.26) is a direct consequence of the Korn inequality and the definition of $\mathscr{J}$. The other one follows from the Hölder and Korn inequalities:

$$\begin{aligned} \|D^{(2)}\mathbf{u}\|_p^p \leqslant c \int_{\mathbb{R}^n} \left| \frac{\partial e_{ij}(\mathbf{u})}{\partial x_k} \right|^p \mathrm{d}\mathbf{x} = c \int_{\mathbb{R}^n} \left(\frac{\partial e_{ij}}{\partial x_k} \frac{\partial e_{ij}}{\partial x_k} |e|^{p-2} \right)^{\frac{p}{2}} |\mathbf{e}|^{\frac{(2-p)p}{2}} \mathrm{d}\mathbf{x} \\ \leqslant c \mathscr{J}^{\frac{p}{2}} \left(\int_{\mathbb{R}^n} |\mathbf{e}|^p \, \mathrm{d}\mathbf{x} \right)^{\frac{2-p}{2}} \leqslant \tilde{c} \mathscr{J}^{\frac{p}{2}} |\mathbf{u}|_{1,p}^{\frac{(2-p)p}{2}}. \end{aligned}$$

□

4.28. Remark. For the perturbated linear model we get that the inequality (4.26) is satisfied for $p > 1$.

The forcing term can be now estimated by means of (4.26) (for $p \geqslant 2$) or (4.27) (for $p < 2$). Let us demonstrate this in the latter case.

From (4.21) and the Young inequality it follows that

$$
\begin{aligned}
(4.29)\qquad & \int_0^T (1+|\mathbf{u}^N|_{1,2}^2)^{-\lambda} \Big| \int_{\mathbb{R}^n} f_i \Delta u_i^N \,\mathrm{d}\mathbf{x} \Big| \,\mathrm{d}t \\
& \leqslant \int_0^T (1+|\mathbf{u}^N|_{1,2}^2)^{-\lambda} \|\mathbf{f}\|_{p'} \|D^{(2)}\mathbf{u}^N\|_p \,\mathrm{d}t \\
& \leqslant c_5 \int_0^T \mathscr{I}^{\frac{1}{2}} |\mathbf{u}^N|_{1,p}^{\frac{2-p}{2}} (1+|\mathbf{u}^N|_{1,2})^{-\frac{\lambda}{2}} \|\mathbf{f}\|_{p'} (1+|\mathbf{u}^N|_{1,2})^{-\frac{\lambda}{2}} \\
& \leqslant \varepsilon \int_0^T \mathscr{K} \,\mathrm{d}t + c(\varepsilon) \int_0^T (1+|\mathbf{u}^N|_{1,2}^2)^{-\lambda} \|\mathbf{f}\|_{p'}^2 |\mathbf{u}^N|_{1,p}^{2-p} \,\mathrm{d}t.
\end{aligned}
$$

The first term is transferred to the left hand side of (4.23) with a small coefficient while the second term can be estimated by means of the Hölder inequality:

$$
\begin{aligned}
(4.30)\qquad \int_0^T \|\mathbf{f}\|_{p'}^2 |\mathbf{u}|_{1,p}^{2-p} \,\mathrm{d}t & \leqslant \Big(\int_0^T \|\mathbf{f}\|_{p'}^{p'} \,\mathrm{d}t \Big)^{\frac{2(p-1)}{p}} \Big(\int_0^T |\mathbf{u}^N|_{1,p}^p \,\mathrm{d}t \Big)^{\frac{2-p}{p}} \\
& = \|\mathbf{f}\|_{L^{p'}(I;L^{p'}(\mathbb{R}^n)^n)}^2 \|\mathbf{u}^N\|_{L^p(I;V_p)}^{2-p} \leqslant c(\mathbf{f}).
\end{aligned}
$$

We obtain

$$
\begin{aligned}
(4.31)\qquad & \frac{1}{2(1-\lambda)} (1+|\mathbf{u}^N(T)|_{1,2}^2)^{1-\lambda} + \frac{c_3}{2} \int_0^T \mathscr{K} \,\mathrm{d}t \\
& \leqslant c(\mathbf{u}_0,\mathbf{f}) + \int_0^T (1+|\mathbf{u}^N|_{1,2}^2)^{-\lambda} |\mathbf{u}^N|_{1,3}^3 \,\mathrm{d}t.
\end{aligned}
$$

Now it remains to estimate the convective term on the right hand side of (4.31). If we get such an estimate with $\lambda \leqslant 1$ then we have from the first term on the left hand side of (4.31) that $\mathbf{u} \in L^\infty(I; W^{1,2}(\mathbb{R}^n)^n)$ and consequently from the other term (if $p \geqslant 2$) we obtain our desired estimate in $L^2(I; W^{2,2}(\mathbb{R}^n)^n)$. For $\lambda > 1$ we have only $\int_0^T \mathscr{K} \,\mathrm{d}t \leqslant \text{const.}$

Hereafter, we will write only $\mathbf{u}$ instead of $\mathbf{u}^N$. We will deal with the problem for $n = 3$ and at the end we will only give a sketch of the proof for $n = 2$.

a. Cauchy problem in 3 space dimensions.

4.32. Lemma. *For* $\mathbf{u}$ *smooth enough we have*

$$
(4.33)\qquad |\mathbf{u}|_{1,3p} \leqslant c_6 \mathscr{I}^{\frac{1}{p}}.
$$

Proof.

$$\||\mathbf{e}|^{\frac{p}{2}}|_{1,2}^2 = \int_{\mathbb{R}^n} \left(\nabla(|\mathbf{e}|^{\frac{p}{2}})\right)^2 \,\mathrm{dx} \leqslant \int_{\mathbb{R}^n} |\mathbf{e}|^{p-2} \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} \,\mathrm{dx} \leqslant \mathscr{I},$$

the inequality (4.33) follows from the imbedding $W^{1,2}(\mathbb{R}^3)^3 \hookrightarrow L^6(\mathbb{R}^3)^3$ and the Korn inequality. □

We will solve separately two cases:

(i) $p \geqslant 3$

(ii) $1 < p < 3$

ad i) $p \geqslant 3$

Put $\lambda = 0$. From Lemmas 3.16 and 4.25 we see that $|\mathbf{u}|_{1,2}^2 \leqslant c\|\mathbf{u}\|_2 \mathscr{I}^{\frac{1}{2}}$. The interpolation inequality

$$|\mathbf{u}|_{1,3}^3 \leqslant |\mathbf{u}|_{1,2}^{2\frac{p-3}{p-2}} |\mathbf{u}|_{1,p}^{\frac{p}{p-2}} \tag{4.34}$$

and the Young inequality yield

$$\begin{aligned}\int_0^T |\mathbf{u}|_{1,3}^3 \,\mathrm{d}t &\leqslant c\int_0^T \|\mathbf{u}\|_2^{\frac{p-3}{p-2}} \mathscr{I}^{\frac{p-3}{2(p-2)}} |\mathbf{u}|_{1,p}^{\frac{p}{p-2}} \\ &\leqslant \varepsilon \int_0^T \mathscr{I}\,\mathrm{d}t + c(\varepsilon)\int_0^T |\mathbf{u}|_{1,p}^{\frac{2p}{p-1}} \,\mathrm{d}t.\end{aligned}$$

The first term is transferred to the left hand side of (4.31), the other is finite because $\frac{2}{p-1} \leqslant 1$ for $p \geqslant 3$.

ad ii) $1 < p < 3$

Considering the fact that $2 < 3 < 3p$ and $p < 3 < 3p$ for $p \in (1,3)$ we can use the following interpolation inequalities:

$$|\mathbf{u}|_{1,3} \leqslant |\mathbf{u}|_{1,2}^{2\frac{p-1}{3p-2}} |\mathbf{u}|_{1,3p}^{\frac{p}{3p-2}}, \tag{4.35}$$

$$|\mathbf{u}|_{1,3} \leqslant |\mathbf{u}|_{1,p}^{\frac{p-1}{2}} |\mathbf{u}|_{1,3p}^{\frac{3-p}{2}}. \tag{4.36}$$

From (4.33) we get

$$\begin{aligned}|\mathbf{u}|_{1,3}^3(1+|\mathbf{u}|_{1,2}^2)^{-\lambda} &= |\mathbf{u}|_{1,3}^{3(\alpha+1-\alpha)}(1+|\mathbf{u}|_{1,2}^2)^{-\lambda} \\ &\leqslant \tilde{c}_7 \mathscr{I}^{Q_1} |\mathbf{u}|_{1,p}^{Q_2}(1+|\mathbf{u}|_{1,2}^2)^{-\lambda+3\frac{(p-1)(1-\alpha)}{3p-2}},\end{aligned} \tag{4.37}$$

where $Q_1 = \frac{3(1-\alpha)}{3p-2} + 3\alpha\frac{3-p}{2p}$, $Q_2 = 3\alpha\frac{p-1}{2}$. Integrating (4.37) over $(0,T)$ we obtain

$$\int_0^T |\mathbf{u}|_{1,3}^3(1+|\mathbf{u}|_{1,2}^2)^{-\lambda}\,\mathrm{d}t \leqslant \tilde{c}_7 \int_0^T \mathscr{K}^{Q_1} |\mathbf{u}|_{1,p}^{Q_2}(1+|\mathbf{u}|_{1,2}^2)^{-\lambda+3\frac{(p-1)(1-\alpha)}{3p-2}+\lambda Q_1}\,\mathrm{d}t. \tag{4.38}$$

We claim

$$-\lambda + 3\frac{(p-1)(1-\alpha)}{3p-2} + \lambda Q_1 = 0\,. \tag{4.39}$$

Using the Hölder and Young inequalities under the assumptions $Q_1\delta = 1$, $Q_2\delta' = p$ and $\frac{1}{\delta} + \frac{1}{\delta'} = 1$ we get

$$\int_0^T |\mathbf{u}|_{1,3}^3 (1 + |\mathbf{u}|_{1,2}^2)^{-\lambda}\,\mathrm{d}t \leqslant c\Big(\int_0^T \mathscr{K}\,\mathrm{d}t\Big)^{\frac{1}{\delta}} \Big(\int_0^T |\mathbf{u}|_{1,p}^p\,\mathrm{d}t\Big)^{\frac{1}{\delta'}}. \tag{4.40}$$

From (4.16) and (4.40) it follows by means of the Young inequality

$$\frac{1}{2(1-\lambda)}(1 + |\mathbf{u}(T)|_{1,2}^2)^{1-\lambda} + \tilde{c}_4 \int_0^T \mathscr{K}\,\mathrm{d}t \leqslant c(\mathbf{u}_0, \mathbf{f}). \tag{4.41}$$

It remains to find the values of α, λ, δ, δ' and verify whether their values are in the required intervals. Solving the above mentioned system of equations we find

$$\alpha = \frac{p(3p-5)}{6(p-1)}, \tag{4.42}$$

$$\lambda = 2\frac{3-p}{3p-5}. \tag{4.43}$$

Hence $\alpha \in [0,1] \iff p \in [\frac{5}{3}, 3]$ and $\lambda \geqslant 0 \iff p \in (\frac{5}{3}, 3]$. Moreover, $\lambda \leqslant 1 \iff p \geqslant \frac{11}{5}$. We can also verify that δ and $\delta' > 1$.

4.44. Remark. In the case of the perturbated linear model we can use the interpolation of $W^{1,3}(\mathbb{R}^n)$ between $W^{1,2}(\mathbb{R}^n)$ and $W^{1,6}(\mathbb{R}^n)$ and then, thanks to the imbedding of $W^{2,2}(\mathbb{R}^n) \hookrightarrow W^{1,6}(\mathbb{R}^n)$, we can estimate the convective term by means of a similar technique as above for $\lambda = 2(3-p)$, i.e. $\lambda \geqslant 0\ \forall p > 1$.

As $\lambda \leqslant 1$ for $p \geqslant \frac{11}{5}$ we get the following lemma:

4.45. Lemma. *Let $p \geqslant \frac{11}{5}$ and let $\mathbf{u}_0$, $\mathbf{f}$ satisfy* (4.6). *Then the sequence of solutions of the problem* (CBN) *for $\mu_1^N \to 0^+$ is uniformly bounded in the following norms:*

$$\|\mathbf{u}^N\|_{L^\infty(I;W^{1,2}(\mathbb{R}^3)^3)} \leqslant c_5, \tag{4.46}$$

$$\|\mathbf{u}^N\|_{L^2(I;W^{2,2}(\mathbb{R}^3)^3)} \leqslant c_6, \tag{4.47}$$

$$\|\mathbf{u}^N\|_{L^p(I;W^{1,3p}(\mathbb{R}^3)^3)} \leqslant c_7. \tag{4.48}$$

Proof. From the inequalities (4.15) and (4.41) we immediately get (4.46). As $p > 2$, the estimate (4.47) is a consequence of (4.46) and (4.26). The last inequality follows from (4.46), (4.33) and the following considerations:

$$W^{1,q}(\mathbb{R}^3) \hookrightarrow L^{3p}(\mathbb{R}^3) \iff q = \frac{3p}{p+1}.$$

As $2 < q < 3p$, we get

$$|\mathbf{u}^N|_{1,q} \leqslant |\mathbf{u}|_{1,2}^{\frac{2p}{3p-2}} |\mathbf{u}|_{1,3p}^{\frac{p-2}{3p-2}}.$$

Evidently $\frac{p-2}{3p-2} \leqslant 1$. Then

$$\begin{aligned}(4.49)\qquad \int_0^T \|\mathbf{u}^N\|_{3p}^p \, dt &\leqslant \int_0^T |\mathbf{u}^N|_{1,q}^p \leqslant c \int_0^T |\mathbf{u}|_{1,2}^{\frac{2p^2}{3p-2}} |\mathbf{u}|_{1,3p}^{p\frac{p-2}{3p-2}} \, \mathrm{d}t \\ &\leqslant \bar{c} \|\mathbf{u}^N\|_{L^\infty(I;W^{1,2}(\mathbb{R}^3)^3)}^{\frac{2p^2}{3p-2}} \|D\mathbf{u}^N\|_{L^p(I;L^{3p}(\mathbb{R}^3)^9)}.\end{aligned}$$

□

4.50. Lemma. *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy* (4.6) *and let* $\mathbf{u}_0 \in V_p$. *Let* $p \geqslant \frac{11}{5}$. *Then* $\frac{\partial \mathbf{u}^N}{\partial t}$ *is uniformly bounded in* $L^2(I; L^2(\mathbb{R}^3)^3)$.

Proof. We revert to the Galerkin approximation of the (CBN) problem and get a new estimate of the time derivative, which does not depend on μ_1. Using $\mathbf{w}^i(\mathbf{x})$ as test functions, multiplying by $\frac{dc_i(t)}{\mathrm{d}t}$ and summing up we obtain

$$\begin{aligned}(4.51)\qquad \left\|\frac{\partial \mathbf{u}^n}{\partial t}\right\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \vartheta(\mathbf{e}(\mathbf{u}^n)) \, \mathrm{d}\mathbf{x} + 2\mu_1 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{\partial e_{ij}(\mathbf{u}^n)}{\partial x_k}\frac{\partial e_{ij}(\mathbf{u}^n)}{\partial x_k} \, \mathrm{d}\mathbf{x} \\ = \int_{\mathbb{R}^3} f_i \frac{\partial u_i^n}{\partial t} \, \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^3} u_j^n \frac{\partial u_i^n}{\partial x_j}\frac{\partial u_i^n}{\partial t} \, \mathrm{d}\mathbf{x}.\end{aligned}$$

As $\mathbf{u}_0 \in W^{1,2}(\mathbb{R}^3)^3 \cap H$ we can construct such a sequence of initial conditions that $\mu_1^N \|D^{(2)}\mathbf{u}_0^N\|$ remains bounded. Integrating over $(0,T)$ and using the Hölder and Young inequalities we obtain

$$(4.52)\qquad \frac{1}{2}\int_0^T \left\|\frac{\partial \mathbf{u}^n}{\partial t}\right\|_2^2 \mathrm{d}t + \int_{\mathbb{R}^3} \vartheta(\mathbf{e}(\mathbf{u}^n(T))) \, \mathrm{d}\mathbf{x} \leqslant c(\mathbf{u}_0, \mathbf{f}) + \int_{Q_T} |\mathbf{u}^n|^2 |D\mathbf{u}^n|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

Using (4.46), (4.47) and Lemma 3.16 we obtain an indepedent estimate of the convective term

$$\begin{aligned}(4.53)\qquad \int_{Q_T} |\mathbf{u}^n|^2 |D\mathbf{u}^n|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t &\leqslant c \int_0^T \|\mathbf{u}^n\|_{2,2}^2 |D\mathbf{u}^n|_{1,2}^2 \, \mathrm{d}t \\ &\leqslant c \|\mathbf{u}^n\|_{L^2(I;W^{2,2}(\mathbb{R}^3)^3)}^2 \|\mathbf{u}^n\|_{L^\infty(I;W^{1,2}(\mathbb{R}^3)^3)}^2,\end{aligned}$$

which, taking $\vartheta(\mathbf{e}) \geqslant 0$ into account, gives the desired estimate. □

4.54. Theorem. *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy* (4.6) *and let* $\mathbf{u}_0 \in V_p$. *Let* $p \geqslant \frac{11}{5}$. *Then there exists a unique weak solution of the problem* (CMN) *in the sense of Definition* 4.7. *Moreover, the solution is regular, i.e.* $\mathbf{u} \in L^\infty(I; W^{1,2}(\mathbb{R}^3)^3) \cap L^2(I; W^{2,2}(\mathbb{R}^3)^3) \cap L^p(I; W^{1,3p}(\mathbb{R}^3)^3)$.

Proof. *Existence.* The same method as in Lemma 3.10 gives that there exists a countable subset of $\mathscr{D}_0(\mathbb{R}^3)^3$ which is dense in $W^{1,2}(\mathbb{R}^3)^3 \cap V_p$. Similarly as in Theorem 3.36 we get a "diagonal" subsequence such that for arbitrary R positive $D\mathbf{u}^{N'} \to D\mathbf{u}$ in $L^2(I; L^2(B_R)^9)$, i.e. $D\mathbf{u}^{N'} \to D\mathbf{u}^N$ a.e. in $I \times B_R$. Using Lemma 2.9 we get

$$(4.55)\quad \int_{Q_T} \frac{\partial u_i}{\partial t} w_i \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} u_j \frac{\partial u_i}{\partial x_j} w_i \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\mathbf{w}) \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T} f_i w_i \psi \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \quad \forall \psi \in C^\infty(I), \forall \mathbf{w} \in \mathscr{D}_0(\mathbb{R}^3)^3.$$

Similarly as in Theorem 3.36 we will prove that (4.55) is satisfied for all $\mathbf{w} \in W^{1,2}(\mathbb{R}^3)^3 \cap V_p$ and a.e. in $(0, T)$.

Let $\mathbf{w}$ be an arbitrary function from $W^{1,2}(\mathbb{R}^3)^3 \cap V_p$, $\mathbf{w}^n \in \mathscr{D}_0(\mathbb{R}^3)^3$, $\mathbf{w}^n \to \mathbf{w}$ (i.e. $\mathbf{w}^n \to \mathbf{w}$ in $W^{1,2}(\mathbb{R}^3)^3$ and $\nabla \mathbf{w}^n \to \nabla \mathbf{w}$ in $L^p(\mathbb{R}^3)^9$). Then, thanks to the estimates from Lemmas 4.45 and 4.50 we get that (4.55) is satisfied for all $\mathbf{w} \in W^{1,2}(\mathbb{R}^3)^3 \cap V_p$.

To complete the existence part of the proof we must verify that $\mathbf{u} \in C(I; H)$. This can be done in the same way as in Theorem 3.36. We get again that $\mathbf{u} \in C^{\frac{1}{2}}(I; H)$.

Uniqueness. It will be proved similarly as for the (CBN) problem. Let $\mathbf{u}$, $\mathbf{v}$ be two different solutions, $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Using $\mathbf{w}$ as a test function in (4.10) we obtain

$$(4.56)\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{w}\|_2^2 + \int_{\mathbb{R}^3} (\tau_{ij}(\mathbf{e}(\mathbf{u})) - \tau_{ij}(\mathbf{e}(\mathbf{v}))) e_{ij}(\mathbf{w}) \,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} w_j \frac{\partial u_i}{\partial x_j} w_i \,\mathrm{d}\mathbf{x}.$$

In the same way as in the uniqueness part of Theorem 3.36 we can prove

$$\int_{\mathbb{R}^3} (\tau_{ij}(\mathbf{e}(\mathbf{u})) - \tau_{ij}(\mathbf{e}(\mathbf{v}))) e_{ij}(\mathbf{w}) \,\mathrm{d}\mathbf{x} \geqslant \int_{\mathbb{R}^3} \int_0^1 \frac{\partial^2 \vartheta^\alpha}{\partial e_{ij} \partial e_{kl}} e_{ij}(\mathbf{w}) e_{kl}(\mathbf{w}) \,\mathrm{d}\alpha \,\mathrm{d}\mathbf{x}$$

with $\vartheta^\alpha = \vartheta(\mathbf{e}[v + \alpha(u - v)])$.

However, (1.10), the Korn inequality and the fact that $p \geqslant 2$ imply

$$(4.57)\qquad \int_{\mathbb{R}^3} (\tau_{ij}(\mathbf{e}(\mathbf{u})) - \tau_{ij}(\mathbf{e}(\mathbf{v}))) e_{ij}(\mathbf{w}) \,\mathrm{d}\mathbf{x} \geqslant c|\mathbf{w}|_{1,2}^2.$$

Integrating (4.56) over $(0, t)$, using (4.57) and $\mathbf{w}(0) = 0$ we obtain

$$(4.58)\qquad \frac{1}{2}\|\mathbf{w}(t)\|_2^2 + c\int_0^t |\mathbf{w}|_{1,2}^2 \,\mathrm{d}t = \int_0^t \|\mathbf{w}\|_4^2 |\mathbf{u}|_{1,2} \,\mathrm{d}t.$$

From the interpolation inequality $\|\mathbf{w}\|_4 \leqslant \|\mathbf{w}\|_2^{\frac{1}{4}} \|\mathbf{w}\|_6^{\frac{3}{4}}$, the imbedding $W^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and the apriori estimate of $\mathbf{u}$ in $L^\infty(I; W^{1,2}(\mathbb{R}^3)^3)$ we get

$$\|\mathbf{w}(t)\|_2^2 \leqslant c \int_0^t \|\mathbf{w}(\tau)\|_2^2 \, d\tau. \tag{4.59}$$

Then the Gronwall inequality gives $\|\mathbf{w}(t)\|_2 = 0$ a.e. in I, i.e. $\mathbf{u} = \mathbf{v}$ a.e. in Q_T. □

4.60. Remark. When $\mathbf{u}_0 \notin V_p$, we do not have the information about the time derivative from Lemma 4.50. Nevertheless we can get the estimate of the time derivative in $L^2(I; (W^{2,2}(\mathbb{R}^n)^n \cap V_p)')$ which implies (see [5 pp. 147–149]) that $\mathbf{u}$ belongs to $C(I; H)$. So we get the existence (and uniqueness) of the weak solution of the problem (CMN). However we have to assume (4.13) instead of (4.10) with test functions $\varphi \in L^2(I; V_p \cap W^{1,2}(\mathbb{R}^n)^n)$ with $\frac{\partial \varphi}{\partial t} \in L^2(I; L^2(\mathbb{R}^n)^n)$.

Now we will deal with the case $p < \frac{11}{5}$, separately for $p \geqslant 2$ and $p < 2$.

ad a) $\mathbf{p} \geqslant 2$

We can dispose only with

$$\int_0^T (1 + |\mathbf{u}|_{1,2}^2)^{-2\frac{3-p}{3p-5}} \mathscr{I} \, dt \leqslant \text{ const.} \tag{4.61}$$

4.62. Lemma. *Let* (4.61), (4.15) *and* (4.16) *hold. Then* $\int_0^T \|D^{(2)}\mathbf{u}\|_2^{2\beta} \leqslant c_7$ *with* $\beta = \frac{4p-8}{3p-5}$ *for* $p > 2$, $\beta = \frac{1}{3}$ *for* $p = 2$.

Proof. For some $\beta < 1$ (which will be specified later) we have

$$\begin{aligned} \int_0^T \|D^{(2)}\mathbf{u}\|_2^{2\beta} \, dt &\leqslant c_4 \int_0^T \mathscr{I}^\beta (1 + |\mathbf{u}|_{1,2}^2)^{-2\frac{3-p}{3p-5}\beta} (1 + |\mathbf{u}|_{1,2}^2)^{2\frac{3-p}{3p-5}\beta} \, dt \\ &\leqslant c\Big(\int_0^T \mathscr{K} \, dt\Big)^\beta \Big(\int_0^T (1 + \|\mathbf{u}\|_2 \|D^2(\mathbf{u})\|)^{2\frac{3-p}{3p-5}\frac{\beta}{1-\beta}}\Big)^{1-\beta}, \end{aligned} \tag{4.62}$$

where Lemma 3.16 was used. As we know that $\mathbf{u}$ is bounded in $L^\infty(I; H)$ we can put

$$2\beta = 2\frac{3-p}{3p-5}\frac{\beta}{1-\beta}$$

and get $\beta = \frac{4p-8}{3p-5}$. Now we use the Young inequality and transfer the term $\varepsilon \int_0^T \|D^{(2)}\mathbf{u}\|_2^{2\beta} \, dt$ with ε sufficiently small to the left hand side. For $p = 2$ we can use directly the estimate in $L^2(I; V_2)$ and get $2\frac{3-2}{6-5}\frac{\beta}{1-\beta} = 1$, i.e. $\beta = \frac{1}{3}$. □

Let us note that $\beta \in (0, \frac{1}{2})$ for $p \in (2, \frac{11}{5})$.

Thanks to the apriori estimates we have

$$\int_0^T \|\mathbf{u}\|_{2,2}^{2\beta} \leqslant c_8. \tag{4.64}$$

Using the imbedding $W^{2,2}(\mathbb{R}^3) \hookrightarrow W^{1+s,p}(\mathbb{R}^3)$ which holds for $s = \frac{6-p}{2p}$ (i.e. $s \in (\frac{19}{22}, 1]$ for $p \in [2, \frac{11}{5})$) we see that $\int_0^T \|\mathbf{u}\|_{1+s,p}^{2\beta} \leqslant c_9$. We choose $q \in (1, p)$. Let $\sigma \in (0, s)$ (which will be specified later). The interpolation inequality (2.4) implies

$$\|\mathbf{u}\|_{1+\sigma,p} \leqslant c\|\mathbf{u}\|_{1,p}^{1-\frac{\sigma}{s}} \|u\|_{1+s,p}^{\frac{\sigma}{s}} \tag{4.65}$$

and therefore

$$\begin{aligned} \int_0^T \|\mathbf{u}\|_{1+\sigma,p}^q &\leqslant c \int_0^T \|\mathbf{u}\|_{1,p}^{q(1-\frac{\sigma}{s})} \|\mathbf{u}\|_{1+s,p}^{q\frac{\sigma}{s}} \\ &\leqslant \left(\int_0^T \|\mathbf{u}\|_{1,p}^{q(1-\frac{\sigma}{s})\delta} \right)^{\frac{1}{\delta}} \left(\int_0^T \|\mathbf{u}\|_{1+s,p}^{q\frac{\sigma}{s}\delta'} \right)^{\frac{1}{\delta'}}, \end{aligned} \tag{4.66}$$

where $\frac{1}{\delta} + \frac{1}{\delta'} = 1$. Both terms on the right hand side are bounded when

$$\begin{aligned} q\left(1 - \frac{\sigma}{s}\right)\delta &= p, \\ q\frac{\sigma}{s}\delta' &= 2\beta. \end{aligned} \tag{4.67}$$

($\int_0^T \|\mathbf{u}\|_p^p \,\mathrm{d}t < \infty$ because of the imbedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ and the interpolation between $L^2(\mathbb{R}^3)$ and $L^{\frac{3p}{3-p}}(\mathbb{R}^3)$.)

Solving the system (4.67) we get

$$\sigma = \frac{s}{q}\frac{(p-q)2\beta}{p-2\beta}. \tag{4.68}$$

We can also verify that δ, $\delta' > 1$. So we have

4.69. Theorem. *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy* (4.6) *and let* $p \in [2, \frac{11}{5})$. *Then there exists a weak solution* $\mathbf{u}$ *of the problem* (CMN) *in the sense of Definition* 4.11. *Moreover,* $\mathbf{u} \in L^q(I; W^{1+\sigma,p}(\mathbb{R}^3))$, *where* $q \in (1,p)$ *and* σ *satisfy* (4.68).

Proof. Let $\mathbf{u}^N$ be our bounded sequence in $L^q(I; W^{1+\sigma,p}(\mathbb{R}^3)^3)$. Because of the estimate of the time derivative mentioned at the beginning of this chapter we get from the Lions-Aubin Lemma that $\mathbf{u}^N \to \mathbf{u}$ in $L^q(I; W^{1,q}(\tilde{\Omega})^3)$, where $\tilde{\Omega}$ is an arbitrary bounded open subset of $\mathbb{R}^3$. We use again the technique of the "diagonal" subsequence. From Theorem 2.9 we get for $\varphi \in C^1(I; \mathscr{D}_0(\mathbb{R}^3)^3)$, $\varphi(T) = 0$

$$-\int_{Q_T} u_i \frac{\partial \varphi_i}{\partial t} \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T} \tau_{ij}(\mathbf{e}(\mathbf{u})) e_{ij}(\varphi) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = \int_{Q_T} f_i \varphi_i \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\mathbb{R}^n} u_{0i} \varphi_i(0) \,\mathrm{d}\mathbf{x}. \tag{4.70}$$

□

4.71. Remark. We can try to close the test functions in $C^1(I; V_p \cap W^{1,2}(\mathbb{R}^3)^3)$. (We know that $\mathscr{D}_0(\mathbb{R}^3)^3$ is dense in $V_p \cap W^{1,2}(\mathbb{R}^3)^3$.) We would have to assume that $|\tau| \leqslant c|\mathbf{e}|^{p-1}$ in order to control the nonlinear term.

ad b) $p < 2$

Now let $p < 2$. We can make use only of (4.61) and the apriori estimates (4.15) and (4.16). Using (4.27) (Lemma 4.25) we see that

$$\int_0^T |\mathbf{u}|_{2,p}^2 |\mathbf{u}|_{1,p}^{p-2} (1 + |\mathbf{u}|_{1,2}^2)^{-2\frac{3-p}{3p-5}} \,\mathrm{d}t \leqslant \text{ const.} \tag{4.72}$$

For some $\beta < 1$ we calculate

$$\begin{aligned} &\int_0^T \|D^{(2)}\mathbf{u}\|_p^{2\beta} \,\mathrm{d}t \\ &\leqslant \int_0^T \left(|\mathbf{u}|_{2,p}^2 |\mathbf{u}|_{1,p}^{p-2} (1 + |\mathbf{u}|_{1,2}^2)^{-2\frac{3-p}{3p-5}} \right)^{\beta} |\mathbf{u}|_{1,p}^{\beta(2-p)} (1 + |\mathbf{u}|_{1,2}^2)^{2\beta\frac{3-p}{3p-5}} \,\mathrm{d}t \\ &\leqslant c(\beta) \int_0^T |\mathbf{u}|_{2,p}^2 |\mathbf{u}|_{1,p}^{p-2} (1 + |\mathbf{u}|_{1,2}^2)^{-2\frac{3-p}{3p-5}} \,\mathrm{d}t \\ &+ \bar{c}(\beta) \int_0^T |\mathbf{u}|_{1,p}^{\frac{\beta(2-p)}{1-\beta}} (1 + |\mathbf{u}|_{1,2}^2)^{2\frac{\beta}{1-\beta}\frac{3-p}{3p-5}} \,\mathrm{d}t. \end{aligned} \tag{4.73}$$

Using the interpolation inequality

$$|\mathbf{u}|_{1,2} \leqslant |\mathbf{u}|_{1,p}^{\frac{5p-6}{2p}} |\mathbf{u}|_{1,\frac{3p}{3-p}}^{\frac{3(2-p)}{2p}} \tag{4.74}$$

and the imbedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^{\frac{3p}{3-p}}(\mathbb{R}^3)$ we see that the second term in (4.73) is bounded by

$$\int_0^T |\mathbf{u}|_{1,p}^{Q_1} |\mathbf{u}|_{2,p}^{Q_2} \,\mathrm{d}t + \int_0^T |\mathbf{u}|_{1,p}^{(2-p)\frac{\beta}{1-\beta}} \,\mathrm{d}t \tag{4.75}$$

with $Q_1 = \left(2 - p + 2\frac{(3-p)(5p-6)}{(3p-5)p}\right)\frac{\beta}{1-\beta}$ and $Q_2 = 6\frac{(2-p)(3-p)}{p(3p-5)}\frac{\beta}{1-\beta}$. The second term in (4.75) is finite if $\beta \leqslant \frac{p}{2}$. The first term can be estimated by means of the Young inequality

$$\int_0^T |\mathbf{u}|_{1,p}^{Q_1} |\mathbf{u}|_{2,p}^{Q_2}\,\mathrm{d}t \leqslant \varepsilon \int_0^T |\mathbf{u}|_{2,p}^{Q_2\delta'}\,\mathrm{d}t + c(\varepsilon)\int_0^T |\mathbf{u}|_{1,p}^{Q_2\delta}. \tag{4.76}$$

The first integral is transferred to the left hand side of (4.73) and the other is finite when the following holds ($\frac{1}{\delta} + \frac{1}{\delta'} = 1$):

$$\begin{aligned} Q_2\delta' &= 2\beta \\ Q_1\delta &= p. \end{aligned} \tag{4.77}$$

Solving (4.77) we get

$$\beta = \frac{p(5p-9)}{2(-p^2+8p-9)} \tag{4.78}$$

and therefore $\beta \in (0, \frac{1}{3})$ for $p \in (\frac{9}{5}, 2)$. We get that p must be greater than $\frac{9}{5}$ instead of $\frac{5}{3}$, which was the bound from the estimate of the convective term. The case $p = \frac{9}{5}$ must be excluded. Evidently, the condition $\beta < \frac{p}{2}$ is satisfied as well as δ, $\delta' > 1$. From (4.16) and the above proved estimate we see that

$$\int_0^T \|D\mathbf{u}\|_{1,p}^{2\beta} \leqslant c_{10} \tag{4.79}$$

with β satisfying (4.78).

Let us choose $\sigma \in (0,1)$. More precisely, σ must satisfy (4.83) as will be seen later. Thanks to the interpolation inequality (2.5) we get $\|D\mathbf{u}\|_{\sigma,p} \leqslant c\|D\mathbf{u}\|_{1,p}^{\sigma}\|D\mathbf{u}\|_p^{1-\sigma}$.

Let $q > 1$. Then

$$\begin{aligned} \int_0^T \|D\mathbf{u}\|_{\sigma,p}^q &\leqslant c\int_0^T \|D\mathbf{u}\|_{1,p}^{\sigma q}\|D\mathbf{u}\|_p^{(1-\sigma)q}\,\mathrm{d}t \\ &\leqslant \left(\int_0^T \|D\mathbf{u}\|_{1,p}^{\sigma q\delta'}\,\mathrm{d}t\right)^{\frac{1}{\delta'}} \left(\int_0^T \|D\mathbf{u}\|_p^{(1-\sigma)q\delta}\,\mathrm{d}t\right)^{\frac{1}{\delta}}. \end{aligned} \tag{4.80}$$

Solving the system ($\frac{1}{\delta} + \frac{1}{\delta'} = 1$)

$$\begin{aligned} \sigma q\delta' &= 2\beta \\ (1-\sigma)q\delta &= p \end{aligned} \tag{4.81}$$

we obtain

$$q = \frac{2\beta p}{\sigma p + 2\beta(1-\sigma)}, \tag{4.82}$$

where σ must satisfy

$$\sigma < \frac{(p-1)(5p-9)}{p(3-p)}. \tag{4.83}$$

Therefore we have

4.84. Theorem. *Let $p > \frac{9}{5}$, let $\mathbf{u}_0$, $\mathbf{f}$ satisfy* (4.6). *Then there exists a weak solution of the problem* (CMN) *in the sense of Definition* 4.11.

Proof. It is analogous to the proof of Theorem 4.69. □

4.85. Remark. It is possible to close the test function in $C^1(I; V_p)$ but only for $p \geqslant \frac{3+\sqrt{39}}{5}$. This bound follows from the estimate of the convective term.

When we assume the perturbated linear model, i.e. with $\tau(\mathbf{e}) = (\nu_0 + \nu_1|\mathbf{e}|^{p-2})\mathbf{e}$ for $p < 2$, we can get the existence of a weak solution for all $p > 1$. As we have the estimate of the convective term for $p > 1$ we can get by means of a similar technique as above that $\int_0^T \|D\mathbf{u}\|_{1,p}^{2\beta} \leqslant c$ with $\beta = \frac{1}{7-2p}$ and therefore we have

4.86. Theorem. *Let $p > 1$, let $\mathbf{u}_0$, $\mathbf{f}$ satisfy* (4.6) *and $\tau(\mathbf{e}) = (\nu_0 + \nu_1|\mathbf{e}|^{p-2})\mathbf{e}$. Then there exists at least one weak solution of the problem* (CMN) *in the sense of Definition 4.11.*

b. Sketch of the proof in 2 space dimensions.

First we will estimate the convective term in (4.31). As for $p \geqslant 3$ the proof is completely analogous to the three-dimensional case, we will deal separately with two cases:

(i) $2 \leqslant p < 3$

(ii) $p < 2$

ad i) $2 < p < 3$

4.87. Lemma. *Let $p \in [2,3)$. Then for $\mathbf{u}$ smooth enough we have*

$$|\mathbf{u}|_{1,3}^3 \leqslant c|\mathbf{u}|_{1,2}^2 \left(\mathscr{I}^{\frac{1}{2}} + \|\mathbf{u}\|_2^{\frac{1}{2}} \mathscr{I}^{\frac{1}{4}}\right), \tag{4.88}$$

$$|\mathbf{u}|_{1,3}^3 \leqslant c|\mathbf{u}|_{1,p}^p \left(\mathscr{I}^{\frac{3-p}{2}} + \|\mathbf{u}\|_2^{\frac{3-p}{2}} \mathscr{I}^{\frac{3-p}{4}}\right). \tag{4.89}$$

Proof. $W^{\frac{1}{3},2}(\mathbb{R}^2) \hookrightarrow L^3(\mathbb{R}^2)$, which together the interpolation inequality $\|\nabla \mathbf{u}\|_{\frac{1}{3},2} \leqslant c\|\mathbf{u}\|_2^{\frac{2}{3}}\|\mathbf{u}\|_{1,2}^{\frac{1}{3}}$ and Lemma 3.16 gives the first inequality.

The other one follows from the imbedding $W^{2\frac{3-p}{3p},p}(\mathbb{R}^2) \hookrightarrow L^3(\mathbb{R}^2)$, the interpolation inequality $\|\nabla \mathbf{u}\|_{2\frac{3-p}{3p},p} \leqslant c\|\nabla \mathbf{u}\|_p^{\frac{p}{3}}\|\nabla \mathbf{u}\|_{\frac{2}{p},p}^{\frac{3-p}{3}}$, the imbedding $W^{1,2}(\mathbb{R}^2) \hookrightarrow W^{\frac{2}{p},p}$, (4.26) and Lemma 3.16. □

Now we can apply Lemma 4.87 to the convective term and we get, similarly as in the three-dimensional case, the following system of equations ($\frac{1}{\delta} + \frac{1}{\delta'} = 1$):

$$
\begin{aligned}
&-\lambda + \alpha + \lambda\frac{\alpha + (1-\alpha)(3-p)}{2} = 0 \\
&\frac{\alpha + (1-\alpha)(3-p)}{2}\delta = 1 \\
&p(1-\alpha)\delta' = p.
\end{aligned} \tag{4.90}
$$

Solving the system (4.90) we obtain

$$\alpha = \frac{3-p}{4-p}, \tag{4.91}$$

$$\lambda = 3 - p, \tag{4.92}$$

i.e. $\alpha \in (0, \frac{1}{2}]$ and $\lambda \leqslant 1$ for $p \in [2, 3)$. We can also verify that $\delta, \delta' > 1$.

ad ii) $1 < \mathbf{p} < 2$

4.93. Lemma. *Let $p \in [\frac{6}{5}, 2)$. Then we have for $\mathbf{u}$ smooth enough*

$$|\mathbf{u}|_{1,3}^3 \leqslant c|\mathbf{u}|_{1,2}^{\frac{5p-6}{2(p-1)}}|\mathbf{u}|_{1,p}^{\frac{p(2-p)}{4(p-1)}}\mathcal{J}^{\frac{p}{4(p-1)}}, \tag{4.94}$$

$$|\mathbf{u}|_{1,3}^3 \leqslant c|\mathbf{u}|_{1,p}^p\mathcal{J}^{\frac{3-p}{p}}. \tag{4.95}$$

Proof. As $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ and (4.21) hold, the first inequality is a consequence of the interpolation inequality $|\mathbf{u}|_{1,3}^3 \leqslant |\mathbf{u}|_{1,2}^{\frac{5p-6}{2(p-1)}}|\mathbf{u}|_{1,\frac{2p}{2-p}}^{\frac{p}{2(p-1)}}$.

The other one follows from by same argument and the interpolation inequality $|\mathbf{u}|_{1,3}^3 \leqslant |\mathbf{u}|_{1,p}^{\frac{5p-6}{p}}|\mathbf{u}|_{1,\frac{2p}{2-p}}^{\frac{2(3-p)}{p}}$. □

Now we will apply the previous lemma to the convective term and get the following system of equations ($\frac{1}{\delta} + \frac{1}{\delta'} = 1$):

$$\frac{\alpha p}{4(p-1)} + (1-\alpha)\frac{3-p}{p} = Q \tag{4.96}$$
$$-\lambda + \alpha\frac{5p-6}{4(p-1)} + \lambda Q = 0$$
$$Q\delta = 1$$
$$\left(\frac{\alpha(2-p)}{4(p-1)} + 1 - \alpha\right)\delta' = 1.$$

Solving the above mentioned system we get

$$\alpha = \frac{2(p-1)(3-p)}{5p-6}, \tag{4.97}$$

$$\lambda = \frac{3-p}{p-1}, \tag{4.98}$$

i.e. $\alpha \in (\frac{1}{2}, 1]$ for $p \in [\frac{3}{2}, 2)$, $\lambda > 1$ for $p < 2$. We can also verify that δ, $\delta' > 1$.

4.99. Remark. For the perturbated linear model we can make use of (4.88) and 4.95. This enables us to estimate the convective term with $\lambda = \frac{2(3-p)}{p}$ for $p > 1$.

Now we revert to the case when $p \geqslant 2$, i.e. $\lambda \leqslant 1$. Similarly as in three space dimensions we get

4.100. Lemma. *Let* $\mathbf{u}^N$ *be solutions of the problem* (CBN) *with* $\mu_1^N > 0$, $\mu_1^N \to 0^+$. *Let* $p \geqslant 2$. *Then* $\mathbf{u}^N$ *are unifromly bounded in the following norms:*

$$\|\mathbf{u}^N\|_{L^\infty(I;W^{1,2}(\mathbb{R}^2)^2)} \leqslant c_{11}, \tag{4.101}$$

$$\|\mathbf{u}^N\|_{L^2(I;W^{2,2}(\mathbb{R}^2)^2)} \leqslant c_{12}, \tag{4.102}$$

$$\|\mathbf{u}^N\|_{L^p(I;W^{1,p}(\mathbb{R}^2)^2)} \leqslant c_{13}. \tag{4.103}$$

4.104. Theorem. *Let* $\mathbf{u}_0$,$\mathbf{f}$ *satisfy* (4.6) *and* $\mathbf{u}_0 \in V_p$. *Let* $p \geqslant 2$. *Then there exists a unique weak solution of the problem* (CMN) *in the sense of Definition* 4.7. *Moreover, the solution is regular, i.e.* $\mathbf{u} \in L^\infty(I; W^{1,2}(\mathbb{R}^2)^2) \cap L^2(I; W^{2,2}(\mathbb{R}^2)^2)$.

Proof. The proof is analogous to the proof of Theorem 4.54 and Lemma 4.50. Only in the uniqueness part we use $\|\mathbf{u}\|_4 \leqslant 2^{\frac{1}{4}}\|\mathbf{u}\|_2^{\frac{1}{2}}|\mathbf{u}|_{1,2}^{\frac{1}{2}}$ (see [17]). □

Now let us solve the case when $p < 2$, i.e. $p \in [\frac{3}{2}, 2)$. We can make use only of

$$\int_0^T (1 + |\mathbf{u}|_{1,2}^2)^{-\frac{3-p}{p-1}} \mathscr{I} \, dt \leqslant \text{ const} \tag{4.105}$$

together with (4.15) and (4.17). From (4.105) and (4.27) we get

$$\int_0^T |\mathbf{u}|_{2,p}^2 |\mathbf{u}|_{1,p}^{p-2} (1 + |\mathbf{u}|_{1,2}^2)^{-\frac{3-p}{p-1}} \, dt \leqslant \text{ const} . \tag{4.106}$$

For $\beta < 1$ (which will be specified later) we get from the interpolation inequality $|\mathbf{u}|_{1,2} \leqslant |\mathbf{u}|_{1,p}^{2\frac{p-1}{p}} |\mathbf{u}|_{1,\frac{2p}{2-p}}^{\frac{2-p}{p}}$ and the imbedding $W^{1,p}(\mathbb{R}^2) \hookrightarrow L^{\frac{2p}{2-p}}(\mathbb{R}^2)$ analogously to the three-dimensional case

$$\int_0^T \|D^{(2)}\mathbf{u}\|_p^{2\beta} \, dt \leqslant c_{14} \tag{4.107}$$

with

$$\beta = \frac{p(2p-3)}{(p-1)(6-p)}. \tag{4.108}$$

The case $p = \frac{3}{2}$ must be excluded again.

4.109. Theorem. *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy* (4.6), $p > \frac{3}{2}$. *Then there exists a weak solution of the problem* (CMN) *in the sense of Definition* 4.11.

Proof. It is completely analogous to the proof of Theorems 4.84 and 4.69 including the part between (4.79) and (4.83). □

4.110. Remark. It is possible to close the test functions in $C^1(I; V_p)$ for $p \geqslant \frac{1+\sqrt{5}}{2}$. This bound follows again from the estimate of the convective term.

For the perturbated linear problem we can get thanks to the estimate of the convective term similarly as above the following theorem:

4.111. Theorem. *Let* $\mathbf{u}_0$, $\mathbf{f}$ *satisfy* (4.6). *Then there exists at least one weak solution of the problem* (CMN) *in the sense of Definition* 4.11.

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