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NUMERICAL REALIZATION OF A FICTITIOUS DOMAIN APPROACH USED IN SHAPE OPTIMIZATION PART I: DISTRIBUTED CONTROLS

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Summary. We deal with practical aspects of an approach to the numerical realization of optimal shape design problems, which is based on a combination of the fictitious domain method with the optimal control approach. Introducing a new control variable in the right-hand side of the state problem, the original problem is transformed into a new one, where all the calculations are performed on a fixed domain. Some model examples are presented.

Keywords: shape optimization, fictitious domain approach

AMS classification: 49A22, 49D30

1. INTRODUCTION

A fictitious domain approach is now very popular when solving large systems arising from the discretization of partial differential equations. Recently an optimal control based approach has been used in [1], [3]. Fictitious domain methods applied in shape optimization problems are also very attractive because of the specific character of the problem. The classical boundary variation technique does not seem to be very efficient: one has to create a new partition of the current domain, then to recompute the stiffness matrix, the load vector and to solve the corresponding system of algebraic equations. All this has to be done at each iteration. The use of a fictitious domain approach makes the numerical solution of the state problem more efficient: the problem is solved on a fixed domain $\widehat{\Omega}$ with a simple shape, which does not change during the computation. Moreover, the state problem is still given by the same differential operator. The information on the geometry (related to our original optimal shape design problem) is encoded into the process through an aditional control variable appearing on the right-hand side of the state problem solved on $\widehat{\Omega}$.

Thus the stiffness matrix can be computed just at the beginning and factorized for ever. As usual, we have to pay for that. The external level (i.e., the minimization process) is now more involved:

— the number of control variables increases,

— the problem, originally smooth (differentiable), is in general converted into a non-differentiable one, and consequently special minimization methods have to be used.

Mathematical analysis of the fictitious domain approach in shape optimization can be found in [5], where *distributed* controls are used, and in [6], where the technique of Lagrange multipliers is presented. The aim of the present paper is to study practical aspects of the approach based on the distributed controls. Recently, in [2] the method of Lagrange multipliers has been utilized also for shape optimization. However, there is one fundamental difference between our approach and that from [2]: in [2] the fictitious domain method serves as a "black box" solving the state problem, the output of which is then substituted into the optimal shape design problem. In our approach, the resulting optimal control problem involves two types of control variables which appear simultaneously: *the design variable* and *an auxiliary variable,* physical meaning of which is a force, by means of which the new formulation is linked to the original optimal shape design problem.

The paper is organized as follows: Section 2 contains the formulation of an optimal shape design problem, the state equation of which is given by the homogeneous Dirichlet boundary value problem. In Section 3 we recall the main idea of the fictitious domain approach applied in shape optimization and based on the distributed controls. Section 4 deals with the discretization of the method using finite elements for the approximation of the state problem. The mathematical justification of the results presented in Section 3 and 4 can be found in [5]. The most important results of this paper are concentrated in Section 5, where practical aspects of the method are discussed. This part includes also the sensitivity analysis and the analysis of the smoothness of the resulting optimal control problem. Finally, in Section 6 some model examples are presented.

2. Setting of the problem

Let $\Omega(\alpha)$ be a domain defined by

$$\Omega(\alpha) = \{[x_1, x_2] \in \mathbb{R}^2 |\ 0 < x_1 < \alpha(x_2),\ x_2 \in (0,1)\}$$

where α is a positive, Lipschitz-continuous function, which describes the variable part $\Gamma(\alpha)$ of $\partial\Omega(\alpha)$ with

$$\Gamma(\alpha) = \{[x_1, x_2] \in \mathbb{R}^2 |\ x_1 = \alpha(x_2),\ x_2 \in (0,1)\}.$$

(see Fig. 2.1).

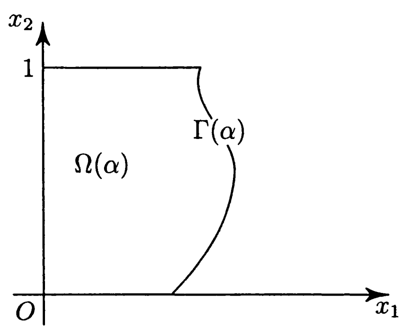


Fig. 2.1

Next we shall suppose that the function α belongs to the set U_{ad}, where

$$\begin{aligned} U_{ad} = \{\alpha \in C^{0,1}([0,1])|\ 0 < C_0 \leqslant \alpha(x_2) \leqslant C_1\ \forall x_2 \in (0,1), \\ |\alpha'(x_2)| \leqslant C_2 \text{ a.e. in } (0,1),\ \operatorname{meas} \Omega(\alpha) = C_3\}, \end{aligned}$$

i.e., U_{ad} contains all functions which are uniformly bounded and uniformly Lipschitz-continuous in $[0,1]$ and preserve the area of $\Omega(\alpha)$. The constants $C_0, \ldots, C_3$ are chosen in such a way that $U_{ad} \neq \emptyset$.

On each $\Omega(\alpha)$, $\alpha \in U_{ad}$, we shall solve the homogeneous Dirichlet boundary value problem

$$(\mathcal{P}(\alpha)) \qquad \begin{cases} -\Delta u(\alpha) = f \text{ in } \Omega(\alpha), \\ \quad u(\alpha) = 0 \text{ on } \partial\Omega(\alpha), \end{cases}$$

or in the weak form:

$$(\mathcal{P}(\alpha)) \qquad \begin{cases} \text{find } u(\alpha) \in V_1(\alpha) \text{ such that} \\ (\nabla u(\alpha), \nabla\varphi)_{\Omega(\alpha)} = (f, \varphi)_{\Omega(\alpha)} \ \forall \varphi \in V_1(\alpha), \end{cases}$$

where $V_1(\alpha) \equiv H_0^1(\Omega(\alpha))$ and the symbol $(.,.)_{\Omega(\alpha)}$ stands for the $L^2(\Omega(\alpha))$ scalar product. Let $\widehat{\Omega}$ be a domain containing $\Omega(\alpha)$ for all $\alpha \in U_{ad}$, $\widehat{\Omega} = (0, d) \times (0, 1)$, $d > C_1$ and $f \in L^2(\widehat{\Omega})$.

Finally, let $I\colon (\alpha, y) \to \mathbb{R}^1$, $\alpha \in U_{ad}$ a $y \in V_1(\alpha)$ be an objective functional and let us define the following problem:

$$(\mathbf{P}) \qquad \begin{cases} \text{find } \alpha^* \in U_{ad} \text{ such that} \\ I(\alpha^*, u(\alpha^*)) \leqslant I(\alpha, u(\alpha)) \ \forall \alpha \in U_{ad}, \end{cases}$$

with $u(\alpha) \in V_1(\alpha)$ being the solution of $(\mathcal{P}(\alpha))$.

The existence of at least one solution α^* of $(\mathbf{P})$ will be guaranteed provided the objective functional I is lower semicontinuous in the following sense (see [7]):

$$((\mathcal{A})) \qquad \left.\begin{cases} \alpha_n \rightrightarrows \alpha \text{ (uniformly) in } [0,1], \ \alpha_n, \alpha \in U_{ad} \\ \hat{y}_n \rightharpoonup \hat{y} \text{ in } H_0^1(\widehat{\Omega}), \ \hat{y}_n, \hat{y} \in H_0^1(\widehat{\Omega}) \end{cases}\right\} \Longrightarrow$$

$$\Longrightarrow \liminf_{n\to\infty} I(\alpha_n, \hat{y}_n|_{\Omega(\alpha_n)}) \geqslant I(\alpha, \hat{y}|_{\Omega(\alpha)}).$$

3. Fictitious domain approach: distributed controls

Let $\widehat{\Omega} = (0, d) \times (0, 1)$, where $d > C_1$. Set $\Xi(\alpha) = \widehat{\Omega} \setminus \overline{\Omega(\alpha)}$ (see Fig. 3.1).

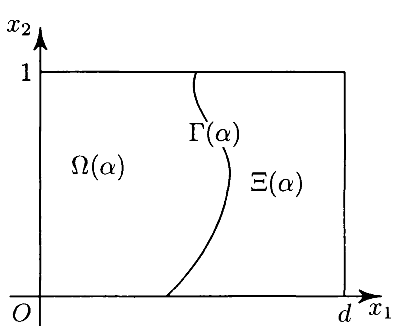


Fig. 3.1

Instead of the problem $(\mathcal{P}(\alpha))$, which is defined on $\Omega(\alpha)$, we consider a new homogeneous Dirichlet boundary value problem which is formulated and solved on $\widehat{\Omega}$:

$$(\widehat{\mathcal{P}}(\alpha, v)) \qquad \begin{cases} -\Delta \hat{u} = v \text{ in } \widehat{\Omega}, \\ \\ \quad \hat{u} = 0 \text{ on } \partial\widehat{\Omega}. \end{cases}$$

The right-hand side v plays the role of *the control variable.* Our goal is to find v in such a way that the solution $\hat{u}$ of $(\widehat{\mathcal{P}}(\alpha, v))$ restricted onto $\Omega(\alpha)$ coincides with the unique solution $u(\alpha)$ of $(\mathcal{P}(\alpha))$. This means that we have to find v such that

$$\text{(i)} \qquad v = f \text{ in } \Omega(\alpha),$$

$$\text{(ii)} \qquad \hat{u}|_{\Gamma(\alpha)} = 0.$$

The condition (i) represents the constraint imposed on the control v, while (ii) is the state constraint.

In order to specify the class of controls v guaranteeing that (ii) is satisfied, we introduce the following notation:

$$V_2(\alpha) \equiv \{v \in H^1(\Xi(\alpha)) |\ v = 0 \text{ on } \ \partial\Xi(\alpha) \setminus \overline{\Gamma(\alpha)}\}.$$

By $V_2'(\alpha)$ we denote the dual space to $V_2(\alpha)$ and by $\langle .,. \rangle_\alpha$ the corresponding duality. We set

$$Q^f(\alpha) = \{v = (v_1, v_2) |\ v_1 = f|_{\Omega(\alpha)},\ v_2 \in V_2'(\alpha)\},\ \alpha \in U_{ad}.$$

Then one can prove

Lemma 3.1. *For any $\alpha \in U_{ad}$ there is an element $\overline{v} \in Q^f(\alpha)$ such that the corresponding solution $\hat{u}$ of $(\mathcal{P}(\alpha, \overline{v}))$ satisfies (ii), i.e., the restriction $\hat{u}|_{\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$. Moreover,*

$$\|\overline{v}|_{\Xi(\alpha)}\|_{*,\alpha} \leqslant K,$$

where the symbol $\|\ \|_{*,\alpha}$ stands for the dual norm in $V_2'(\alpha)$ and the positive constant K depends on $\|f\|_{L^2(\widehat{\Omega})}$, only.

For the proof we refer to [5].

Remark 3.1. If $(\mathcal{P}(\alpha))$ is regular, i.e., its solution $u(\alpha) \in H^2(\Omega(\alpha)) \cap H_0^1(\Omega(\alpha))$, then Lemma 3.1 holds with $\overline{v} \in L^2(\widehat{\Omega})$.

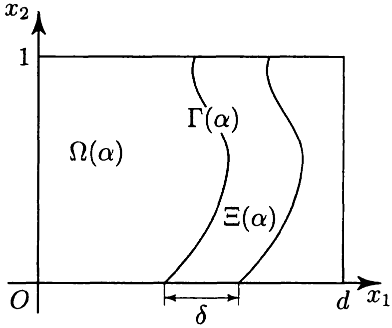


Fig. 3.2

Remark 3.2. Let

$$\Xi_\delta(\alpha) = \{x = [x_1, x_2] \in \mathbb{R}^2,\ \alpha(x_2) < x_1 < \alpha(x_2) + \delta,\ x_2 \in (0,1)\}, \delta > 0.$$

Then Lemma 3.1 still holds provided $\overline{v} = (v_1, v_2) \in Q^f(\alpha)$, $v_2 \in V_2'(\alpha)$ such that $\operatorname{supp} v_2 \subset \overline{\Xi_\delta(\alpha)}$ *(see Fig. 3.2). This means that the behaviour of $\overline{v}$ is important only in a vicinity of $\Gamma(\alpha)$.*

Put

$$Q^f_{ad}(\alpha) = \{v = (v_1, v_2) |\ v_1 = f|_{\Omega(\alpha)},\ v_2 \in V_2'(\alpha), \|v_2\|_{*,\alpha} \leqslant K\},\ \alpha \in U_{ad},$$
$$Q^f_{ad} = \bigcup_{\alpha \in U_{ad}} Q^f_{ad}(\alpha),$$

where $K > 0$ is the same constant as in Lemma 3.1.

The weak formulation of $(\widehat{\mathcal{P}}(\alpha, v))$ for $\alpha \in U_{ad}$ and $v \in Q^f_{ad}(\alpha)$ now reads as follows:

$$(\widehat{\mathcal{P}}(\alpha, v)) \qquad \begin{cases} \text{find } \hat{u} \equiv \hat{u}(\alpha, v) \in \widehat{V} \text{ such that} \\ (\nabla \hat{u}, \nabla \hat{\varphi})_{\widehat{\Omega}} = (f, \hat{\varphi})_{\Omega(\alpha)} + \langle v_2, \hat{\varphi} \rangle_\alpha \ \forall \hat{\varphi} \in \widehat{V}, \end{cases}$$

where $\widehat{V} = H^1_0(\widehat{\Omega})$.

Let $\varepsilon > 0$ be a penalty parameter and define a functional $E_\varepsilon \colon U_{ad} \times Q^f_{ad} \to \mathbb{R}^1$ by

$$E_\varepsilon(\alpha, v) = I(\alpha, \hat{u}|_{\Omega(\alpha)}) + \frac{1}{\varepsilon} \int_0^1 \hat{u}^2 \, \mathrm{d}x_2,$$

where I is the original cost functional appearing in the definition of $(\mathbf{P})$, $\hat{u} \equiv \hat{u}(\alpha, v)$ is the solution of $(\widehat{\mathcal{P}}(\alpha, v))$ and

$$\int_0^1 \hat{u}^2 \, \mathrm{d}x_2 \equiv \int_0^1 (\hat{u}(\alpha(x_2), x_2))^2 \, \mathrm{d}x_2.$$

Instead of $(\mathbf{P})$ we shall introduce the family of problems

$$(\widehat{\mathbf{P}})_\varepsilon \qquad \begin{cases} \text{find } (\alpha_\varepsilon^*, v_\varepsilon^*) \in U_{ad} \times Q_{ad}^f(\alpha_\varepsilon^*) \text{ such that} \\ E_\varepsilon(\alpha_\varepsilon^*, v_\varepsilon^*) \leqslant E_\varepsilon(\alpha, v) \;\; \forall (\alpha, v) \in U_{ad} \times Q_{ad}^f(\alpha). \end{cases}$$

In [5], the following existence result is established:

Theorem 3.1. *Let the cost functional I satisfy $(\mathcal{A})$. Then $(\widehat{\mathbf{P}})_\varepsilon$ has at least one solution for any $\varepsilon > 0$.*

Let $\varepsilon_k \searrow 0$, $k \to \infty$, be a sequence of penalty parameters and define the problems $(\widehat{\mathbf{P}})_{\varepsilon_k}$ as above. As far as the mutual relation between $(\widehat{\mathbf{P}})$ and $(\widehat{\mathbf{P}})_{\varepsilon_k}$, $\varepsilon_k \searrow 0$, is concerned, one can prove

Theorem 3.2. *Let $(\alpha_k^*, v_k^*) \in U_{ad} \times Q_{ad}^f(\alpha_k^*)$ be solutions of $(\widehat{\mathbf{P}})_{\varepsilon_k}$. Then there exists a subsequence $\{(\alpha_{k_j}^*, v_{k_j}^*)\}$ and an element $(\alpha^*, v^*) \in U_{ad} \times Q_{ad}^f(\alpha^*)$ such that*

$$\begin{cases} \alpha_{k_j}^* \rightrightarrows \alpha^* \;\; \textit{uniformly in } [0,1], \\ v_{k_j}^* \rightharpoonup v^* \;\; \textit{in } H^{-1}(\widehat{\Omega}), \\ \hat{u}_{k_j}^* \rightharpoonup \hat{u}^* \;\; \textit{in } \widehat{V}, \end{cases}$$

where $\hat{u}_k^* \in \widehat{V}$ solve the corresponding state problems $(\widehat{\mathcal{P}}(\alpha_k^*, v_k^*))$. Moreover, α^* is a solution of $(\mathbf{P})$ and $\hat{u}^*|_{\Omega(\alpha^*)}$ solves $(\mathcal{P}(\alpha^*))$.

For the proof see [5].

Remark 3.3. Theorem 3.2 says that the problems $(\mathbf{P})$ and $(\widehat{\mathbf{P}})_{\varepsilon_k}$ are close on subsequences for $\varepsilon_k \searrow 0+$. The problem $(\widehat{\mathbf{P}})_{\varepsilon_k}$ is nothing else than the penalty approach treating the state constraint (ii).

From the definition of $(\widehat{\mathbf{P}})_{\varepsilon_k}$, the main advantage of the fictitious domain approach is readily seen: the state problem $(\widehat{\mathcal{P}}(\alpha, v))$ is still solved on the fixed domain $\widehat{\Omega}$. This fact is very important when realizing this method.

If $v = (v_1, v_2) \in Q_{ad}^f(\alpha)$, then v_2 belongs to $V_2'(\alpha)$. For practical reasons, however, such a choice of v is not suitable. A natural question arises, what happens if more regular controls are admitted, namely if $v \in \widetilde{Q}_{ad}^f(r, \alpha)$, where

$$\widetilde{Q}_{ad}^f(r, \alpha) = \{v = (v_1, v_2) |\; v_1 = f|_{\Omega(\alpha)}, v_2 \in L^2(\Xi(\alpha)),\; \|v_2\|_{\Xi(\alpha)} \leqslant r\},$$

and $r > 0$ is a given constant which does not depend on $\alpha \in U_{ad}$. If the state problem $(\mathcal{P}(\alpha))$ is regular for any $\alpha \in U_{ad}$ in the sense of Remark 3.1, then the previous approach can be applied with $\widetilde{Q}^f_{ad}(r,\alpha)$ replacing $Q^f_{ad}(\alpha)$ taking $r > 0$ sufficiently large. Unfortunately this is not our case. On the other hand using the fact that $L^2(\Xi(\alpha))$ is dense in $V_2'(\alpha)$, one can define the problem $(\widetilde{\mathbf{P}})_\varepsilon$, using controls $v \in \widetilde{Q}^f_{ad}(r,\alpha)$:

$$(\widetilde{\mathbf{P}})_\varepsilon \qquad \begin{cases} \text{find } (\tilde{\alpha}_\varepsilon, \tilde{v}_\varepsilon) \in U_{ad} \times \widetilde{Q}^f_{ad}(r,\tilde{\alpha}_\varepsilon) \text{ such that} \\ E_\varepsilon(\tilde{\alpha}_\varepsilon, \tilde{v}_\varepsilon) \leqslant E_\varepsilon(\alpha, v) \ \ \forall (\alpha, v) \in U_{ad} \times \widetilde{Q}^f_{ad}(r,\alpha), \end{cases}$$

where $E_\varepsilon(\alpha, v)$ is the same as before, i.e.,

$$E_\varepsilon(\alpha, v) = I(\alpha, \hat{u}|_{\Omega(\alpha)}) + \frac{1}{\varepsilon}\int_0^1 \hat{u}^2 \,\mathrm{d}x_2,$$

and $\hat{u} \equiv \hat{u}(\alpha, v) \in \widehat{V}$ solves the problem

$$(\widehat{\mathcal{P}}(\alpha, v)) \qquad (\nabla \hat{u}, \nabla \hat{\varphi})_{\widehat{\Omega}} = (f, \hat{\varphi})_{\Omega(\alpha)} + (v, \hat{\varphi})_{\Xi(\alpha)} \ \forall \hat{\varphi} \in \widehat{V}.$$

The relation between $(\mathbf{P})$ and $(\widetilde{\mathbf{P}})_\varepsilon$ is analyzed in [4], where the following result is established:

Theorem 3.3. *Let the cost functional I be uniformly continuous on $U_{ad} \times \widehat{V}$ in the following sense:*

$$\left.\begin{aligned} &\forall \varkappa > 0 \ \exists \delta > 0 \ \forall \hat{u}, \hat{v} \in \widehat{V} \ \|\hat{u} - \hat{v}\|_{\widehat{\Omega}} \leqslant \delta \\ &\forall \alpha, \beta \in U_{ad} \ \|\alpha - \beta\|_{C([0,1])} \leqslant \delta \end{aligned}\right\} \Longrightarrow$$
$$\Longrightarrow |I(\alpha, \hat{u}|_{\Omega(\alpha)}) - I(\beta, \hat{v}|_{\Omega(\beta)})| \leqslant \varkappa$$

and let $\varepsilon > 0$ be fixed and $\eta > 0$ be an arbitrary number. Then there exists $\bar{r} = \bar{r}(\eta, \varepsilon)$ such that for any solution $(\tilde{\alpha}_\varepsilon, \tilde{v}_\varepsilon) \in U_{ad} \times \widetilde{Q}^f_{ad}(\bar{r}, \tilde{\alpha}_\varepsilon)$ of $(\widetilde{\mathbf{P}})_\varepsilon$ we have

$$|E_\varepsilon(\tilde{\alpha}_\varepsilon, \tilde{v}_\varepsilon) - q| \leqslant \mathcal{F}(\eta) + c\frac{\eta}{\varepsilon} + g(\varepsilon),$$

where $q = \inf\limits_{\alpha \in U_{ad}} I(\alpha, u(\alpha))$, $\mathcal{F}(\eta) \to 0+$, $g(\varepsilon) \to 0+$ if $\varepsilon, \eta \to 0+$ and c is a positive constant.

In the subsequent sections we shall deal only with the regular controls v, i.e., $v \in L^2(\widehat{\Omega})$.

4. Finite element approximation of $(\widetilde{\mathbf{P}})_\varepsilon$

The aim of the present section is to describe briefly an approximation of the problem $(\widetilde{\mathbf{P}})_\varepsilon$ for a fixed value of the penalty parameter $\varepsilon > 0$. The approximation will be based on a suitable finite-element discretization of the state problem, solved on the fixed domain $\widehat{\Omega}$.

First, we start with the approximation of the space $\widehat{V}$. Let $\{\mathcal{T}_h\}$, $h \to 0+$, be a regular family of partitions of $\widehat{\Omega}$ in the standard sense. With any $\mathcal{T}_h$ we associate a finite-dimensional space $\widehat{V}_h \subseteq \widehat{V}$ containing *piecewise polynomial* functions over $\mathcal{T}_h$.

Instead of U_{ad} its approximation U_{ad}^H by *piecewise linear* functions will be used:

$$U_{ad}^H = \{\alpha_H \in C([0,1]) |\ \alpha_H \text{ is piecewise linear in } [0,1]\} \cap U_{ad}.$$

Remark 4.1. The reason why we restrict ourselves to the case of piecewise linear functions is that the constraints imposed on $\alpha \in U_{ad}$ can be easily realized by them. The mesh sizes h and H related to $\widehat{V}_h$ and U_{ad}^H are independent of each other, in general.

Let $\alpha_H \in U_{ad}^H$ be given. Then $\overline{\widehat{\Omega}} = \overline{\Omega(\alpha_H)} \cup \overline{\Xi(\alpha_H)}$. Now, it remains to construct an approximation of the set of controls $\widetilde{Q}_{ad}^f(r, \alpha_H)$, defined on $\Xi(\alpha_H)$. Let $\{\mathcal{R}_K(\alpha_H)\}$ be a regular family of partitions of $\Xi(\alpha_H)$. With any $\mathcal{R}_K(\alpha_H)$ we associate a finite-dimensional subspace $L_K(\alpha_H)$ of $L^2(\Xi(\alpha_H))$ containing piecewise polynomial functions on $\mathcal{R}_K(\alpha_H)$, and define

$$\begin{aligned}\widetilde{Q}_{K,ad}^f(r, \alpha_H) = \{v_K \in L^2(\widehat{\Omega}) |\ & v_K = f \text{ in } \Omega(\alpha_H),\\ & v_K \in L_K(\alpha_H) \text{ on } \Xi(\alpha_H),\ \|v_K\|_{\Xi(\alpha_H)} \leqslant r\}.\end{aligned}$$

Remark 4.2. Again, the partition $\mathcal{R}_K(\alpha_H)$ can be independent of $\widehat{\mathcal{T}}_h$ and consequently the mesh sizes h, H, K are independent of each other. In practical realization, however, the situation is less complicated. Indeed, the partitions defining U_{ad}^H and $\widetilde{Q}_{K,ad}^f(r, \alpha_H)$ are usually defined by means of $\widehat{\mathcal{T}_h}$ (see Fig. 4.1).

In the situation depicted in Fig. 4.1 all nodes determining the piecewise linear function $\alpha_H \in U_{ad}^H$ lie on horizontal sides of rectangular elements R belonging to $\widehat{\mathcal{T}}_h$, while the partition $\mathcal{R}_K(\alpha_H)$ of $\Xi(\alpha_H)$ consists of all elements of the form $R \cap \Xi(\alpha_H)$, $R \in \widehat{\mathcal{T}}_h$. Consequently, instead of three independent mesh parameters h, H, K only one, namely h describes the corresponding partitions.

First, we start with the approximation of the state problem $(\widehat{\mathcal{P}}(\alpha, v))$. Let $\alpha_H \in U_{ad}^H$ and $v_K \in \widetilde{Q}_{K,ad}^f(r, \alpha_H)$ be given. We define

$$(\widehat{\mathcal{P}}(\alpha_H, v_K))_h \qquad \begin{cases} \text{find } \hat{u}_h \equiv \hat{u}_h(\alpha_H, v_K) \in \widehat{V}_h \text{ such that} \\ (\nabla \hat{u}_h, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = (f, \hat{\varphi}_h)_{\Omega(\alpha_H)} + (v_K, \hat{\varphi}_h)_{\Xi(\alpha_H)} \forall \hat{\varphi}_h \in \widehat{V}_h. \end{cases}$$

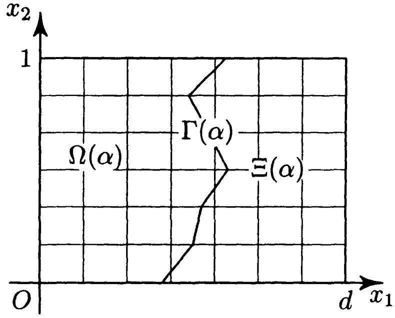


Fig. 4.1

The approximation of the whole problem $(\widetilde{\mathbf{P}})_\varepsilon$ ($\varepsilon > 0$ being fixed) now reads as follows:

$$(\widetilde{\mathbf{P}})_{\varepsilon,\text{approx}} \quad \begin{cases} \text{find } \alpha^*_{\varepsilon H} \in U^H_{ad},\ v^*_{\varepsilon K} \in \widetilde{Q}^f_{K,ad}(r,\alpha^*_{\varepsilon H}) \text{ such that} \\ E_\varepsilon(\alpha^*_{\varepsilon H}, v^*_{\varepsilon K}) \leqslant E_\varepsilon(\alpha_H, v_K)\ \forall \alpha_H \in U^H_{ad},\ \forall v_K \in \widetilde{Q}^f_{K,ad}(r,\alpha_H). \end{cases}$$

Here we will not deal with the mutual relation between $(\widetilde{\mathbf{P}})_\varepsilon$ and $(\widetilde{\mathbf{P}})_{\varepsilon,\text{approx}}$ when $h, H, K \to 0+$. The detailed analysis of this problem can be found in [5], in the case when $\widehat{V}_h$ consists of piecewise linear triangular elements and the controls v are approximated by piecewise constant functions. Again it is possible to show that $(\widetilde{\mathbf{P}})_\varepsilon$ and $(\widetilde{\mathbf{P}})_{\varepsilon,\text{approx}}$ are close on subsequences (in the sense of Theorem 3.2).

5. On practical aspects of the fictitious domain approach used in shape optimization

In this section we will discuss the main features of the approach presented above. To this end we shall present the matrix form of the state problem $(\widehat{\mathcal{P}}(\alpha_H, v_K))_h$. Let us suppose that the mesh sizes h, H, K are given. Then the problem $(\widehat{\mathcal{P}}(\alpha_H, v_K))_h$ reduces to a system of linear algebraic equations

$$\widehat{\mathbf{A}}\mathbf{u}(\boldsymbol{\alpha}, \mathbf{v}) = \mathbf{F}_1(\boldsymbol{\alpha}) + \mathbf{F}_2(\boldsymbol{\alpha}, \mathbf{v}), \tag{5.1}$$

where

$$\widehat{\mathbf{A}} = \{\hat{a}_{ij}\}^n_{i,j=1},\ \hat{a}_{ij} = (\nabla\hat{\varphi}_i, \nabla\hat{\varphi}_j)_{\widehat{\Omega}},$$
$$\mathbf{F}_1(\boldsymbol{\alpha}) = \{\mathbf{F}_1{}^{(i)}(\boldsymbol{\alpha})\}^n_{i=1},\ \mathbf{F}_1{}^{(i)} = \int_{\Omega(\alpha_H)} f\hat{\varphi}_i d\hat{x},$$
$$\mathbf{F}_2(\boldsymbol{\alpha}, \mathbf{v}) = \{\mathbf{F}_2{}^{(i)}(\boldsymbol{\alpha}, \mathbf{v})\}^n_{i=1},\ \mathbf{F}_2{}^{(i)} = \int_{\Xi(\alpha_H)} v_K\hat{\varphi}_i d\hat{x}$$

are the stiffness matrix and the discretization of the right-hand sides of $(\widehat{\mathcal{P}}(\alpha_H, v_K))_h$, respectively. The symbols $\boldsymbol{\alpha} \in \mathbb{R}^m$, $\mathbf{v} \in \mathbb{R}^p$ stand for the discrete variables related to $\alpha_H \in U^H_{ad}$, $v_K \in L_K(\alpha_H)$, respectively. Finally, $\{\hat{\varphi}_i\}$ are the basis functions of $\widehat{V}_h$.

From (5.1) the main advantage of the fictitious domain approach is readily seen: the stiffness matrix $\widehat{\mathbf{A}}$ does not change during computations. It is computed at the beginning once for ever. Thus the linear system (5.1) can be solved very efficiently for various right-hand sides.

Next, we shall analyze in more details the properties of the mappings

$$\boldsymbol{\alpha} \to \mathbf{F}_1(\boldsymbol{\alpha}),\ (\boldsymbol{\alpha}, \mathbf{v}) \to \mathbf{F}_2(\boldsymbol{\alpha}, \mathbf{v})$$

and consequently also of

$$(\boldsymbol{\alpha}, \mathbf{v}) \to \mathbf{u}(\boldsymbol{\alpha}, \mathbf{v})$$

by virtue of (5.1).

Clearly, the mapping

$$\mathbf{v} \to \mathbf{F}_2(\boldsymbol{\alpha}, \mathbf{v})$$

is linear and continuous, but the mappings

$$\begin{aligned} &\boldsymbol{\alpha} \to \mathbf{F}_1(\boldsymbol{\alpha}),\\ &\boldsymbol{\alpha} \to \mathbf{F}_2(\boldsymbol{\alpha}, \mathbf{v}) \end{aligned}$$

are *Lipschitz-continuous* only, in general. For the detailed analysis we pass from the matrix formulation of the state problem to its finite element formulation.

Let $\alpha \in U_{ad}$ and $g \in L^2(\widehat{\Omega})$ be given. Then the functional

$$\Phi(\alpha) = \int_{\Omega(\alpha)} g \,\mathrm{d}x$$

is only Lipschitz-continuous on U_{ad}, in general. If the function g is more regular, namely if $g \in H^1(\widehat{\Omega})$, then Φ is already continuously differentiable on U_{ad}. Indeed, let $\mathcal{F}_t\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping of the form

$$\mathcal{F}_t = \mathrm{id} + t(\Theta, 0),\ t \geqslant 0,$$

where the function $\Theta \in W^{1,\infty}(\widehat{\Omega})$ is such that $\mathcal{F}_t(\widehat{\Omega}) = \widehat{\Omega}$ and $\Theta(\alpha(x_2), x_2) = \theta(x_2)$, i.e. $\mathcal{F}_t$ maps $\widehat{\Omega}$ onto $\widehat{\Omega}$ and the boundary $\Gamma(\alpha)$ onto $\Gamma(\alpha + t\theta)$. Denote $\Omega_t = \mathcal{F}_t(\Omega(\alpha))$ and $\Phi_t(\alpha) = \int_{\Omega_t} g \,\mathrm{d}x$. Using Fubini's theorem, one can easily prove that

$$\text{(5.2)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(\alpha)\bigg|_{t=0} = \int_{\Omega(\alpha)} \frac{\partial}{\partial x_1}(g\Theta)\,\mathrm{d}x + \int_{\Gamma(\alpha)} g\theta n_1 \,\mathrm{d}s,$$

where $n = (n_1, n_2)$ is the unit outer normal vector to $\Omega(\alpha)$. This yields that the smoothness of $\Phi(\alpha)$ depends on the regularity of the integrands. Applying this result to the mapping

$$\alpha_H \to \int_{\Xi(\alpha_H)} v_K \hat{\varphi}_i \,\mathrm{d}x, \; v_K \in L_K(\alpha_H), \; \alpha_H \in U^H_{ad},$$

we arrive at the following conclusion: *smoothness of the state problem* (5.1) *depends on the type of finite elements used for the approximation of the control variable* v*: if the elements are of the class* C^0*, i.e.,* $L_K(\alpha_H) \subset H^1(\Xi(\alpha_H))$*, and if we assume that the right-hand side* $f \in H^1(\widehat{\Omega})$*, then the state problem is continuously differentiable, if not, the solution of* (5.1) *is only a Lipschitz-continuous function of* $\boldsymbol{\alpha}$*, in general.*

Next we will analyse the smooth case only, i.e., $L_K(\alpha_H) \subset H^1(\Xi(\alpha_H))$. Moreover, we will assume that $\{\widehat{\mathcal{T}}_h\}$ is a regular family of rectangulations of $\overline{\widehat{\Omega}}$ for $h \to 0+$ and any $\widehat{V}_h$ is the space of *piecewise bilinear functions* over $\widehat{\mathcal{T}}_h$. Let $\theta_H \in C^{0,1}([0,1])$ be a piecewise linear function having the same nodes as the functions from U^H_{ad}. The *directional derivative* of $\hat{u}_h$ at α_H in the direction θ_H is defined as follows:

$$(\hat{u}_h)'_{\alpha_H}(\theta_H) \equiv \lim_{t \to 0+} \frac{\hat{u}_h(\alpha_H + t\theta_H, v_K) - \hat{u}_h(\alpha_H, v_K)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_h(\alpha_H + t\theta_H, v_K)\Big|_{t=0+}$$

Using (5.2) one can easily show that $(\hat{u}_h)'_{\alpha_H} \equiv (\hat{u}_h)'_{\alpha_H}(\theta_H) \in \widehat{V}_h$ solves the problem

$$(5.3) \qquad (\nabla(\hat{u}_h)'_{\alpha_H}, \nabla\hat{\varphi}_h)_{\widehat{\Omega}} = \int_{\Gamma(\alpha_H)} (f - v_K)\theta_H n_1 \hat{\varphi}_h \,\mathrm{d}s \;\; \forall \hat{\varphi}_h \in \widehat{V}_h.$$

Next we shall consider the cost functionals

$$\begin{cases} J_1(\alpha_H, \hat{u}_h) = \|\hat{u}_h - z\|^2_{L^2(\Omega(\alpha_H))}, \\ J_2(\alpha_H, \hat{u}_h) = \|\hat{u}_h - z\|^2_{L^2(\widetilde{\Omega})}, \\ J_3(\alpha_H, \hat{u}_h) = \|\nabla(\hat{u}_h - z)\|^2_{L^2(\widetilde{\Omega})}, \\ J_4(\alpha_H, \hat{u}_h) = -(\hat{u}_h, f)_{\Omega(\alpha_H)}, \end{cases}$$

where $\hat{u}_h \equiv \hat{u}_h(\alpha_H, v_K)$ solves $(\widehat{\mathcal{P}}(\alpha_H, v_K))_h$, $\widetilde{\Omega} \subset \Omega(\alpha_H) \; \forall \alpha_H \in U^H_{ad}$ is a given domain which does not depend on α_H and z is a given function. If $\hat{u}_h$ is continuously differentiable with respect to $\alpha_H \in U^H_{ad}$ and $v_K \in L_K(\alpha_H)$, then the same holds for the mappings $(\alpha_H, v_K) \to J_i(\alpha_H, \hat{u}_h(\alpha_H, v_K))$, $i = 1, \dots, 4$. Indeed, using the *adjoint state technique,* one can easily derive formulae for the gradients of the cost functionals $J_i(\alpha_H, \hat{u}_h(\alpha_H, v_K))$, $i = 1, \dots, 4$, with respect to the variable v_K:

$$\begin{cases} \nabla_{v_K} J_1(\alpha_H, \hat{u}_h|_{\Omega(\alpha_H)}) = p_{1h}|_{\Xi(\alpha_H)}, \text{ where } p_{1h} \in \widehat{V}_h \text{ solves} \\ (\nabla p_{1h}, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = 2(\hat{u}_h - z, \hat{\varphi}_h)_{\Omega(\alpha_H)} \ \forall \hat{\varphi}_h \in \widehat{V}_h; \\ \\ \nabla_{v_K} J_2(\alpha_H, \hat{u}_h|_{\Omega(\alpha_H)}) = p_{2h}|_{\Xi(\alpha_H)}, \text{ where } p_{2h} \in \widehat{V}_h \text{ solves} \\ (\nabla p_{2h}, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = 2(\hat{u}_h - z, \hat{\varphi}_h)_{\widehat{\Omega}} \ \forall \hat{\varphi}_h \in \widehat{V}_h; \\ \\ \nabla_{v_K} J_3(\alpha_H, \hat{u}_h|_{\Omega(\alpha_H)}) = p_{3h}|_{\Xi(\alpha_H)}, \text{ where } p_{3h} \in \widehat{V}_h \text{ solves} \\ (\nabla p_{3h}, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = 2(\nabla \hat{u}_h - \nabla z, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} \ \forall \hat{\varphi}_h \in \widehat{V}_h; \\ \\ \nabla_{v_K} J_4(\alpha_H, \hat{u}_h|_{\Omega(\alpha_H)}) = p_{4h}|_{\Xi(\alpha_H)}, \text{ where } p_{4h} \in \widehat{V}_h \text{ solves} \\ (\nabla p_{4h}, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = -(f, \hat{\varphi}_h)_{\Omega(\alpha_H)} \ \forall \hat{\varphi}_h \in \widehat{V}_h. \end{cases}$$

The corresponding derivatives with respect to the design variable $\alpha_H \in U_{ad}^H$ are given by

$$\begin{cases} (J_1)'_{\alpha_H}(\theta_H) = \int_{\Gamma(\alpha_H)} ((\hat{u}_h - z)^2 + (f - v_K)p_{1h})\theta_H n_1 \, \mathrm{d}s, \\ (J_2)'_{\alpha_H}(\theta_H) = \int_{\Gamma(\alpha_H)} (f - v_K)p_{2h}\theta_H n_1 \, \mathrm{d}s, \\ (J_3)'_{\alpha_H}(\theta_H) = \int_{\Gamma(\alpha_H)} (f - v_K)p_{3h}\theta_H n_1 \, \mathrm{d}s, \\ (J_4)'_{\alpha_H}(\theta_H) = \int_{\Gamma(\alpha_H)} (-f\hat{u}_h + (f - v_K)p_{4h})\theta_H n_1 \, \mathrm{d}s, \end{cases}$$

where $p_{1h}, \dots, p_{4h}$ are the adjoint states introduced above.

Now let us examine in more detail the penalty term[1]

$$J_0(\alpha_H, \hat{u}_h) = \int_{\Gamma(\alpha_H)} (\hat{u}_h)^2 \, \mathrm{d}s.$$

[1] Instead of the integral $\int_0^1 \hat{u}^2 \, \mathrm{d}x_2$, used for theoretical purposes, the integral along $\Gamma(\alpha_H)$ is now considered.

Differentiating J_0 with respect to v_K, we obtain

$$\nabla_{v_K} J_0(\alpha_H, \hat{u}_h) = p_{0h}|_{\Xi(\alpha_H)}, \text{ where } p_{0h} \text{ solves}$$
$$(\nabla p_{0h}, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = 2 \int_{\Gamma(\alpha_H)} \hat{u}_h \hat{\varphi}_h \, \mathrm{d}s \ \forall \hat{\varphi}_h \in \widehat{V}_h.$$

The derivative of J_0 with respect to α_H is more involved. The *formal differentiation* leads to the following expression:

$$(5.4) \begin{cases} (J_{0h})'_{\alpha_H}(\theta_H) = \displaystyle\int_{\Gamma(\alpha_H)} (f - v_K) p_{0h} \theta_H n_1 \, \mathrm{d}s + 2 \int_{\Gamma(\alpha_H)} \hat{u}_h \frac{\partial \hat{u}_h}{\partial x_1} \theta_H \, \mathrm{d}s \\ \qquad + \displaystyle\int_0^1 \hat{u}_h^2(\alpha_H(x_2), x_2) \frac{\theta'_H(x_2)\alpha'_H(x_2)}{\sqrt{1 + (\alpha'_H(x_2))^2}} \, \mathrm{d}x_2. \end{cases}$$

As the solution $\hat{u}_h$ is a piecewise bilinear function, its derivative $\partial \hat{u}_h / \partial x_1$ is discontinuous on vertical interelement boundaries M, the union of which will be denoted by $\mathcal{M}$. If the intersection of $\Gamma(\alpha_H)$ with all $M \in \mathcal{M}$ is a set with a positive measure (see Fig. 5.1b)) then only the directional derivative of J_0 with respect to α_H exists. On the other hand, if such an intersection contains a finite number of points, then J_0 is continuously differentiable at α_H (cf. Fig. 5.1a)).

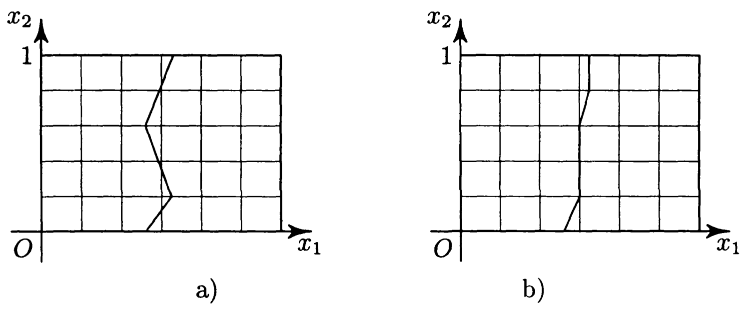


Fig. 5.1

We use a more convenient expression of the second term on the right-hand side of (5.4), namely

$$2 \int_{\Gamma(\alpha_H)} \hat{u}_h \frac{\partial^+ \hat{u}_h}{\partial x_1} (\theta_H)^+ \, \mathrm{d}s + 2 \int_{\Gamma(\alpha_H)} \hat{u}_h \frac{\partial^- \hat{u}_h}{\partial x_1} (\theta_H)^- \, \mathrm{d}s,$$

where $\partial^{\pm} \hat{u}_h / \partial x_1$ stands for the partial derivatives of $\hat{u}_h$ from the right and the left, respectively, $\theta_H^+ = \max(\theta_H, 0)$, $\theta_H^- = \min(\theta_H, 0)$. From the analysis presented above we see that the whole optimal control problem $(\widetilde{\mathbf{P}})_{\varepsilon,\mathrm{approx}}$ is *non-smooth*, in general. The possible non-differentiability is due to the penalty term $\int_{\Gamma(\alpha_H)} (\hat{u}_h)^2\, ds$. A natural idea arises, namely to replace this term by a "smooth" one. Proposition A.1 in [5] implies that there is an element $\bar{v} \in Q^f_{ad}(\alpha)$ such that the corresponding solution $\hat{u}(\alpha, \bar{v})$ of $(\widehat{\mathcal{P}}(\alpha, \bar{v}))$ is identically equal to zero not only on $\Gamma(\alpha)$ but also in $\Xi_\delta(\alpha)$ (defined in Remark 3.2). Thus as the penalty term we can take

$$J_\delta(\alpha_H, \hat{u}_h) = \|\hat{u}_h\|^2_{L^2(\Xi_\delta(\alpha_H))}.$$

An easy computation shows that J_δ is continuously differentiable with respect to $v_K \in L_K(\alpha_H)$ and $\alpha_H \in U^H_{ad}$. Indeed,

$$\begin{aligned} &\nabla_{v_K} J_\delta(\alpha_H, \hat{u}_h) = p^\delta_h|_{\Xi(\alpha_H)}, \text{ where } p^\delta_h \in \widehat{V}_h \text{ solves} \\ &(\nabla p^\delta_h, \nabla \hat{\varphi}_h)_{\widehat{\Omega}} = 2(\hat{u}_h, \hat{\varphi}_h)_{\Xi_\delta(\alpha_H)}\ \forall \hat{\varphi}_h \in \widehat{V}_h \end{aligned}$$

and

$$(J_\delta)'_{\alpha_H}(\theta_H) = \int_{\Gamma(\alpha_H)} ((f - v_K)p^\delta_h - \hat{u}^2_h)\theta_H n_1\, ds + \int_{\Gamma(\alpha_H+\delta)} \hat{u}^2_h \theta_H n_1\, ds,$$

where $\Gamma(\alpha_H + \delta)$ is parallel to $\Gamma(\alpha_H)$ and $\mathrm{dist}(\Gamma(\alpha_H + \delta), \Gamma(\alpha_H)) = \delta$. Replacing J_0 by J_δ, the whole problem $(\widetilde{\mathbf{P}})_{\varepsilon,\mathrm{approx}}$ is converted to a smooth one.

The main difficulty, however, is related to the so called *locking effect*. Let $\alpha_H \in U^H_{ad}$ be given. If there is a $\bar{v}_K \in L_K(\alpha_H)$ such that the corresponding solution $\hat{u}_h(\alpha_H, \bar{v}_K)$ of $(\widehat{\mathcal{P}}(\alpha_H, \bar{v}_K))_h$ vanishes on $\Gamma(\alpha_H)$ and if $\Gamma(\alpha_H)$ does not contain vertical segments (i.e., parallel to x_2-axis), then $\hat{u}_h$ is identically equal to zero in all elements from $\widehat{\mathcal{T}}_h$, the interior of which has a non-empty intersection with $\Gamma(\alpha_H)$. Such phenomenon is undesirable in practical realization because it makes minimization procedures *non sensitive* with respect to the change of the design variable $\alpha_H \in U^H_{ad}$.

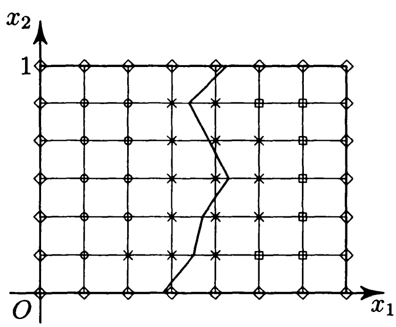
Now we show that under certain circumstances the locking effect may arise. To this end we again pass to the algebraic formulation of the state problem. Let $\alpha_H \in U^H_{ad}$ be fixed and let us consider the situation depicted in Fig. 4.1 and mentioned in Remark 4.1: all vertices of the polygonal domain $\Omega(\alpha_H)$ lie on the horizontal interelement boundaries of elements $R \in \widehat{\mathcal{T}}_h$. The partition $\mathcal{R}(\alpha_H)$ used for the construction of $L_K(\alpha_H)$ is made of elements $R \cap \overline{\Xi}(\alpha_H)$, $R \in \widehat{\mathcal{T}}_h$ and the space $L_K(\alpha_H)$ itself contains all *continuous piecewise bilinear functions on* $\mathcal{R}(\alpha_H)$. Let $\mathcal{I}$ be the set of the indices of all nodes $\{a_i\}, i \in \mathcal{I}$, belonging to $\widehat{\mathcal{T}}_h$. The set $\mathcal{I}$ will be split as follows (see Fig. 5.2):

$$\mathcal{I} = \mathcal{J} \cup \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2,$$

where

$$\begin{aligned}
\mathcal{J} &= \{i \in \mathcal{I} \mid a_i \in \partial\widehat{\Omega}\},\\
\mathcal{I}_0 &= \{i \in \mathcal{I} \setminus \mathcal{J} \mid a_i \text{ is a vertex of a rectangle } R \in \widehat{\mathcal{T}_h} \text{ such that}\\
&\quad \operatorname{int}(R) \cap \Gamma(\alpha_H) \neq \emptyset\} \cup \{i \in \mathcal{I} \setminus \mathcal{J} \mid a_i \in \Gamma(\alpha_H)\},\\
\mathcal{I}_1 &= \{i \in \mathcal{I} \mid a_i \in \Omega(\alpha_H),\ i \notin \mathcal{I}_0\},\\
\mathcal{I}_2 &= \{i \in \mathcal{I} \mid a_i \in \Xi(\alpha_H), i \notin \mathcal{I}_0\}.
\end{aligned}$$

Set n_i =card $\mathcal{I}_i$, $i = 0, 1, 2$.



$\times$ the nodes a_i, $i \in \mathcal{I}_0$
$\circ$ the nodes a_i, $l \in \mathcal{I}_1$
$\square$ the nodes a_i, $l \in \mathcal{I}_2$
$\diamond$ the nodes a_i, $l \in \mathcal{J}$

Fig. 5.2

Now we use the matrix formulation (5.1) of the state problem $(\widehat{\mathcal{P}}(\alpha_H, v_K))_h$. As $\alpha_H \in U_{ad}^H$ is assumed to be fixed, we omit the symbol $\boldsymbol{\alpha}$ in what follows. The resulting system of linear algebraic equations, which takes into account the definition of $L_K(\alpha_H)$, reads as follows:

$$\widehat{\mathbf{A}}\mathbf{u} = \mathbf{F}_1\mathbf{f} + \mathbf{F}_2\mathbf{v}, \tag{5.5}$$

where

$$\begin{aligned}
\mathbf{F}_1 &= \{f_1^{kl}\},\ f_1^{kl} = (\hat{\varphi}_k, \hat{\varphi}_l)_{\Omega(\alpha_H)},\\
\mathbf{F}_2 &= \{f_2^{kl}\},\ f_2^{kl} = (\hat{\varphi}_k, \hat{\varphi}_l)_{\Xi(\alpha_H)},
\end{aligned}$$

$k \in \mathcal{I} \setminus \mathcal{J}$, $l \in \mathcal{I}$. Here $\{\hat{\varphi}_j\}$, $j \in \mathcal{I}$ are piecewise bilinear functions over $\widehat{\mathcal{T}_h}$ such that

$$\hat{\varphi}_j(a_i) = \delta_{ij},\ i, j \in \mathcal{I}$$

and $\mathbf{f}$, $\mathbf{v}$ are the vectors of the nodal values of f (here we assume $f \in C(\overline{\widehat{\Omega}})$, otherwise we pass to a suitable approximation of f) and $\widetilde{v_K}$ (here $\widetilde{v_K}$ is a continuous piecewise bilinear extension of v_K on $\widehat{\Omega}$) at a_i, $i \in \mathcal{I}$. We shall show that for any choice of $z_l \in \mathbb{R}^1$, $l \in \mathcal{I}_0 \cup \mathcal{I}_2$, there is a right-hand side $\mathbf{v}$ for which the corresponding solution $\mathbf{u}$ of (5.5) satisfies $u_l = z_l$ for all $l \in \mathcal{I}_0 \cup \mathcal{I}_2$.

First of all it is easy to see that if u_l, $l \in \mathcal{I}_0$, are given a-priori, the components u_l, $l \in \mathcal{I}_1$, of the vector $\mathbf{u}$ are already uniquely determined. Indeed, let $\tilde{\mathbf{u}}$ be the vector the components of which are u_l, $l \in \mathcal{I}_1$. Using the fact that $(\mathbf{F}_2\mathbf{v})_i = 0$ for all $i \in \mathcal{I}_1$ and for all $\mathbf{v}$, we can see that $\tilde{\mathbf{u}}$ solves the system of linear equations

$$\tilde{\widehat{\mathbf{A}}}\tilde{\mathbf{u}} = \widetilde{\mathbf{F}_1\mathbf{f}} - \tilde{\mathbf{d}}, \tag{5.6}$$

where $\tilde{\widehat{\mathbf{A}}} = (\hat{a}_{ij})$, $i,j \in \mathcal{I}_1$, $\widetilde{\mathbf{F}_1\mathbf{f}}$ and $\tilde{\mathbf{d}}$ are vectors the components of which are $(\mathbf{F}_1\mathbf{f})_i$, $i \in \mathcal{I}_1$ and $\sum\limits_{j\in\mathcal{I}_0} \hat{a}_{ij}z_j$, $i \in \mathcal{I}_1$(here we use the fact that $\hat{a}_{ij} = 0$ for $i \in \mathcal{I}_1$, $j \in \mathcal{I}_2$), respectively.

Such $\tilde{\mathbf{u}}$ solving (5.6) is unique and independent of $\mathbf{v}$. Substituting $\tilde{\mathbf{u}}$ together with z_l, $l \in \mathcal{I}_0 \cup \mathcal{I}_2$, into (5.5), we get equations from which $\mathbf{v}$ can be determined. Its components v_i, $i \in \mathcal{J} \cup \mathcal{I}_1$, can be chosen arbitrarily. The remaining $(n_0 + n_2)$ components v_i, $i \in \mathcal{I}_0 \cup \mathcal{I}_2$, form a vector denoted as $\mathbf{v}^*$, which can be found by solving the system

$$\overline{\mathbf{F}_2}\mathbf{v}^* = \overline{\widehat{\mathbf{A}}\mathbf{u}} - \bar{\mathbf{g}} - \overline{\mathbf{F}_1\mathbf{f}}, \tag{5.7}$$

where $\overline{\mathbf{F}}_2 = \{f_2^{kl}\}$, $k,l \in \mathcal{I}_0 \cup \mathcal{I}_2$, $\overline{\widehat{\mathbf{A}}\mathbf{u}}$, $\overline{\mathbf{F}_1\mathbf{f}}$ and $\bar{\mathbf{g}}$ are the vectors the components of which are $\{\widehat{\mathbf{A}}\mathbf{u}\}_i, \{\mathbf{F}_1\mathbf{f}\}_i$, $i \in \mathcal{I}_0 \cup \mathcal{I}_2$ and $\sum\limits_{l\in\mathcal{J}} f_2^{kl}v_l$, $k \in \mathcal{I}_0 \cup \mathcal{I}_2$. As $\overline{\mathbf{F}_2}$ is regular, there is a unique $\mathbf{v}^*$ solving (5.7). Applying this result to the special choice of z_l, namely $z_l = 0$ for all $l \in \mathcal{I}_0$ we arrive at the assertion on the existence of the locking effect at the nodes a_i, $i \in \mathcal{I}_0$.

To avoid the locking effect, we need to modify the definition of $L_K(\alpha_H)$. Instead of the partition $\mathcal{R}(\alpha_H)$ created directly from $\widehat{\mathcal{T}}_h$, a corser partition of $\Xi(\alpha_H)$ has to be used. Nevertheless such a partition still can be related to $\widehat{\mathcal{T}}_h$ (see Fig. 5.3, where each element from $\mathcal{R}(\alpha_H)$ is the intersection of $\Xi(\alpha_H)$ with the union of four elements from $\widehat{\mathcal{T}}_h$).

As in all state constrained optimal control problems, special attention has to be paid to the mutual relation between the penalty parameter ε and the mesh size $h > 0$. The general strategy is to choose $\varepsilon > 0$ fixed and let $h \to 0+$.

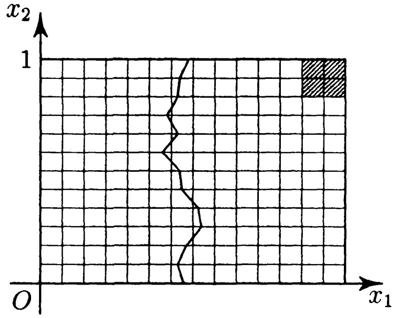


Fig. 5.3

6. Examples

In examples presented below, the piecewise bilinear approximation of the solution u as well as of the control variable v will be used and $\widehat{\Omega} = (0, 1.5) \times (0, 1)$. The fictitious domain $\widehat{\Omega}$ is cut into squares with the size h.

Example 1. Here

$$
\begin{aligned}
&f(x_1, x_2) = 1, \\
&C_0 = 0.7,\ C_1 = 1.3,\ C_2 = 1,\ C_3 = 1
\end{aligned}
$$

(for the meaning of C_0, C_1, C_2, C_3 see the definition of U_{ad}) and the cost functional $J = J_4$ (see Section 5).

Example 2. Here

$$
\begin{aligned}
&z(x_1, x_2) = x_1 x_2 (1 - x_2) \left(\frac{\sin(\pi x_2)}{2} - x_1 + 1 - \frac{1}{\pi} \right), \\
&f(x_1, x_2) = -\Delta z, \\
&C_0 = 0.6,\ C_1 = 1.3,\ C_2 = 1.6,\ C_3 = 1, \\
&\tilde{\Omega} = (0, 0.5) \times (0, 1),
\end{aligned}
$$

and the cost functional $J = J_i$, $i = 1, 2, 3$.

In both the examples $h = 1/16$ and the sequential quadratic programming method NLPQL (see [8]) has been used. The initial shape $\Omega_{in} = (0,1) \times (0,1)$ and the initial value v_{in} of v is equal to 0. Two choices of the functional $E_{\varepsilon h}$ have been tested, namely

$$
E_\varepsilon(\alpha, v) = J_i(\alpha, \hat{u}(\alpha, v)|_{\Omega(\alpha)}) + \frac{1}{\varepsilon} \int_{\Gamma(\alpha)} \hat{u}^2 \, \mathrm{d}s, \ i = 1, \ldots, 4, \tag{6.1}
$$

$$
E_\varepsilon(\alpha, v) = J_i(\alpha, \hat{u}(\alpha, v)|_{\Omega(\alpha)}) + \frac{1}{\varepsilon} J_\delta(\alpha, \hat{u}), \ i = 1, \ldots, 4, \tag{6.2}
$$

where

$$J_\delta(\alpha, \hat{u}) = \|\hat{u}\|^2_{L^2(\Xi_\delta(\alpha))}$$

(see Section 5). Results of the first example for different values of the penalty parametr ε are depicted in Figs. 6.1, 6.2 when (6.1) was used and in Fig. 6.3 for (6.2) with $\delta = 0.05$ (and $i = 4$). Figures are amended by tables, the columns of which have the following meaning (TABLE 1):

- IT = the number of iterations needed;
- $E(0)$ = the initial value of E_ε: $E_\varepsilon(\alpha_{in}, \hat{u}(\alpha_{in}, v_{in}))$;
- $E(IT)$ = the final value of E_ε: $E_\varepsilon(\alpha_{opt}, \hat{u}_{opt}))$;
- $J_0(IT)$ = the value of $\int_{\Gamma(\alpha_{opt})}(\hat{u}(\alpha_{opt}, v_{opt}))^2\,\mathrm{d}s$ (the information how the homogeneous Dirichlet boundary condition is satisfied);
- $J_4(IT)$ = the value of $J_4(\alpha_{opt}, \hat{u}(\alpha_{opt}, v_{opt}))$;
- $J_4^p(IT)$ = the value of the cost functional J_4 recomputed by using the fictitious domain approach (see [1]) with $h = 1/64$ over the domain Ω_{opt}.

The last line contains the following information:

- $J_0(0)$ = the value of $\int_{\Gamma(\alpha_{in})}(\hat{u}(\alpha_{in}, v_{in}))^2\,\mathrm{d}s$;
- $J_4(0)$ = the value of $J_4(\alpha_{in}, \hat{u}(\alpha_{in}, v_{in}))$;
- $J_4^p(0)$ = the value of J_4 computed by using the fictitious domain approach with $h = 1/64$ over the domain $\Omega_{in} = (0,1) \times (0,1)$.

Table 2 shows the results when (6.2) was used. This example was recomputed by R. Mäkinen (University of Jyväskylä) by using the classical boundary variation approach. His result corresponds to ours for $\varepsilon = 0.1$. In Fig. 6.2 the influence of the locking effect is readily seen. The designs only slightly change $\Omega_{in} = (0,1) \times (0,1)$. In Fig. 6.4, 6.5 and 6.6 the final shapes for Example 2 and the cost functional (6.1) with $i = 1, 2$ and 3, respectively, are shown. Results for E_ε given by (6.2) are almost the same, therefore the variant (6.1) is presented, only.

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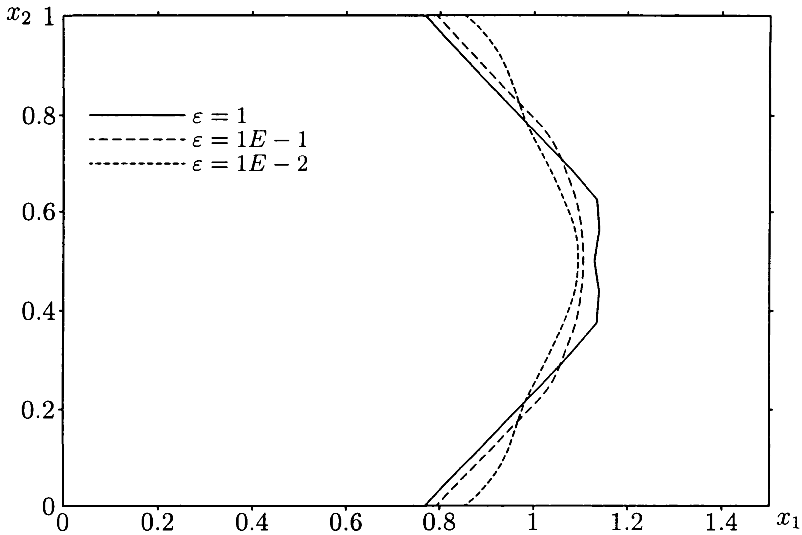

Fig. 6.1

ε	IT	$E(0)$	$E(IT)$	$J_0(IT)$	$J_4(IT)$	$J_4^p(IT)$
$.1E+01$	50	$-.439E-01$	$-.568E-01$	$.139E-01$	$-.707E-01$	$-.366E-01$
$.1E+00$	44	$-.296E-01$	$-.389E-01$	$.249E-03$	$-.414E-01$	$-.369E-01$
$.1E-01$	61	$.114E+00$	$-.366E-01$	$.377E-05$	$-.369E-01$	$-.367E-01$
$.1E-02$	69	$.155E+01$	$-.360E-01$	$.883E-07$	$-.361E-01$	$-.367E-01$
$.1E-03$	63	$.159E+02$	$-.359E-01$	$.180E-08$	$-.359E-01$	$-.361E-01$
$.1E-04$	200	$.159E+03$	$-.357E-01$	$.227E-08$	$-.359E-01$	$-.361E-01$

$J^c(0) = .15931E-02,\ J_4(0) = -.455E-01,\ J_4^p(0) = .351E-01$

Table 1

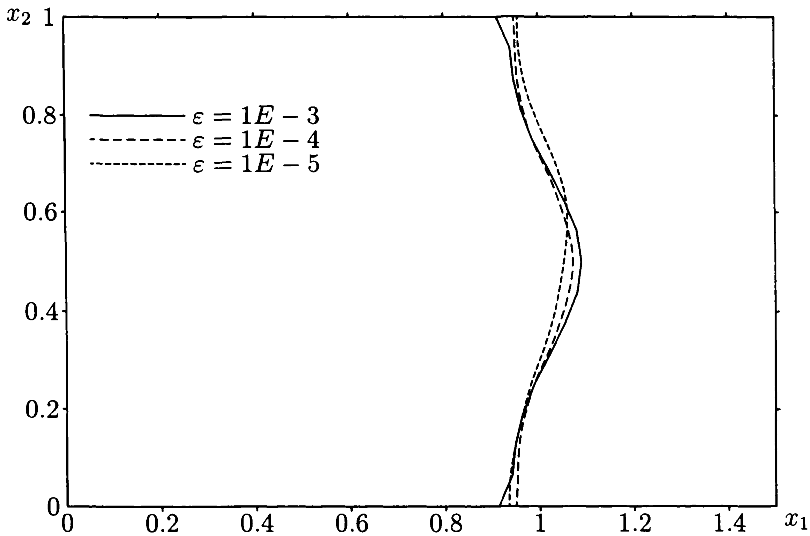

Fig. 6.2

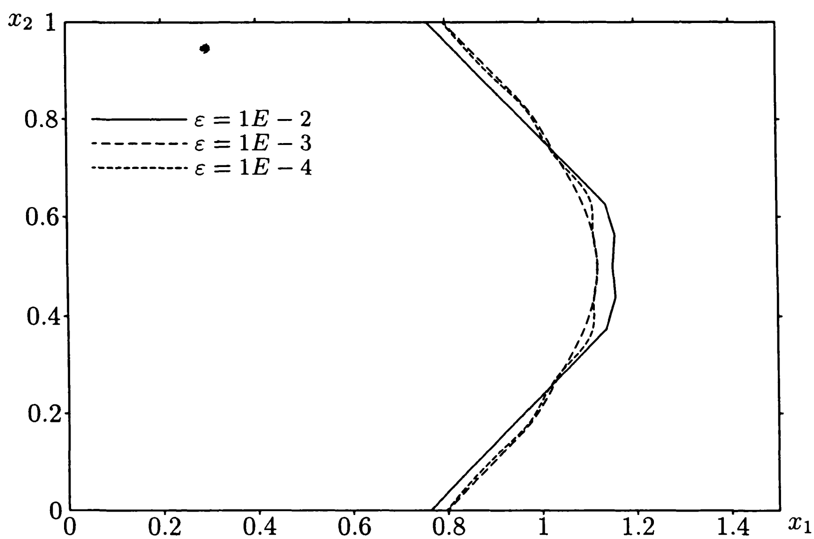

Fig. 6.3

ε	IT	$E(0)$	$E(IT)$	$J^c(IT)$	$J_4(IT)$	$J_4^p(IT)$
$.1E-01$	59	$-.387E-01$	$-.437E-01$	$.592E-04$	$-.496E-01$	$-.365E-02$
$.1E-02$	50	$.223E-01$	$-.383E-01$	$.107E-05$	$-.394E-01$	$-.366E-02$
$.1E-03$	124	$.632E+00$	$-.362E-01$	$.161E-06$	$-.378E-01$	$-.368E-02$

$J^c(0) = .67764E-04,\ J_4(0) = -.455E-01,\ J_4^p(0) = .351E-01$

Table 2

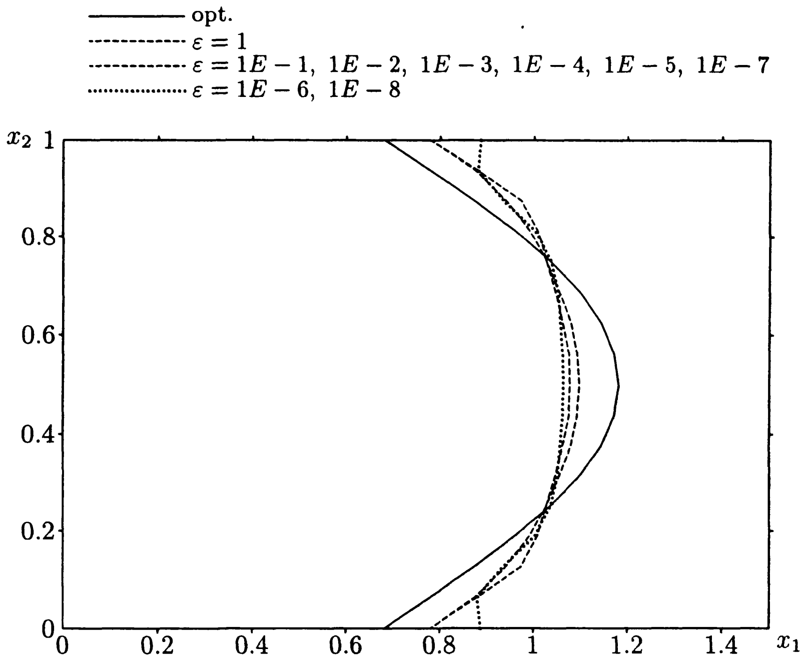

Fig. 6.4

ε	IT	$E(0)$	$E(IT)$	$J_0(IT)$	$J_1(IT)$	$J_1^p(IT)$
$.1E+01$	147	$.118E-02$	$.154E-04$	$.324E-05$	$.121E-04$	$.186E-04$
$.1E+00$	84	$.116E-01$	$.201E-04$	$.107E-05$	$.190E-04$	$.186E-04$
$.1E-01$	74	$.115E+00$	$.243E-04$	$.371E-07$	$.206E-04$	$.197E-04$
$.1E-02$	59	$.115E+00$	$.567E-04$	$.365E-07$	$.202E-04$	$.193E-04$
$.1E-03$	60	$.115E+02$	$.269E-03$	$.256E-07$	$.136E-04$	$.126E-04$
$.1E-04$	76	$.115E+03$	$.248E-02$	$.246E-07$	$.141E-04$	$.132E-04$
$.1E-05$	200	$.115E+04$	$.106E-02$	$.103E-08$	$.268E-04$	$.269E-04$
$.1E-06$	53	$.115E+05$	$.249E+00$	$.249E-07$	$.139E-04$	$.129E-04$
$.1E-07$	200	$.115E+06$	$.428E+00$	$.428E-08$	$.243E-04$	$.239E-04$

$J_0(0) = .11532E-02,\ J_1(0) = -.307E-04,\ J_1^p(0) = .682E-04$

Table 3

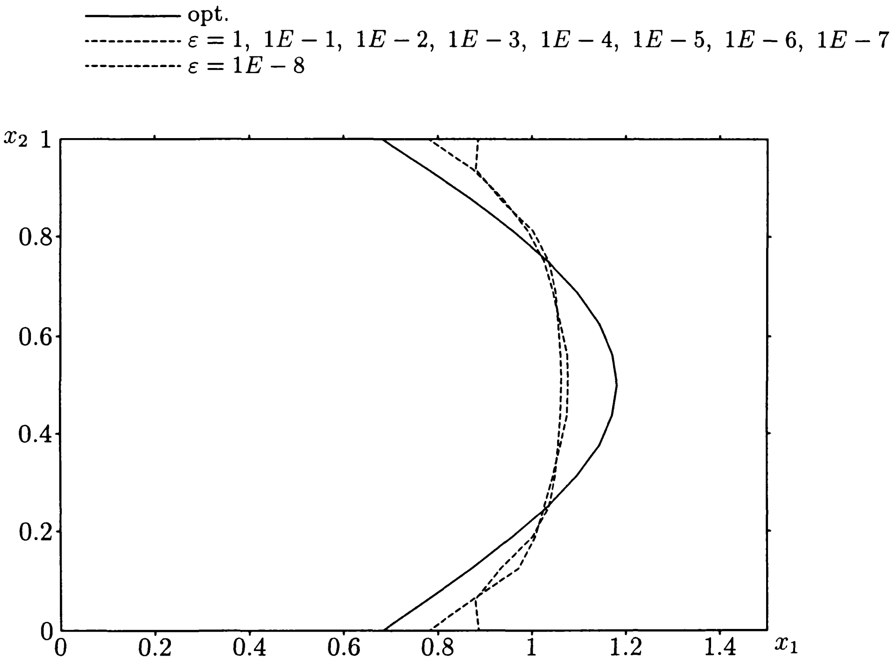

Fig. 6.5

ε	IT	$E(0)$	$E(IT)$	$J_0(IT)$	$J_2(IT)$	$J_2^p(IT)$
$.1E+01$	154	$.115E-02$	$.359E-06$	$.561E-07$	$.303E-06$	$.152E-06$
$.1E+00$	100	$.115E-01$	$.689E-06$	$.369E-07$	$.319E-06$	$.157E-06$
$.1E-01$	77	$.115E+00$	$.387E-05$	$.354E-07$	$.327E-06$	$.163E-06$
$.1E-02$	59	$.115E+01$	$.368E-04$	$.365E-07$	$.310E-06$	$.150E-06$
$.1E-03$	60	$.115E+02$	$.254E-03$	$.254E-07$	$.195E-06$	$.730E-07$
$.1E-04$	76	$.115E+03$	$.246E-02$	$.246E-07$	$.199E-06$	$.754E-07$
$.1E-05$	200	$.115E+04$	$.567E-02$	$.567E-08$	$.388E-06$	$.231E-06$
$.1E-06$	53	$.115E+05$	$.249E+00$	$.249E-07$	$.196E-06$	$.723E-07$
$.1E-07$	200	$.115E+06$	$.254E+00$	$.254E-08$	$.432E-06$	$.278E-06$

$J_0(0) = .11532E-02,\ J_2(0) = -.775E-06,\ J_2^p(0) = .179E-05$

Table 4

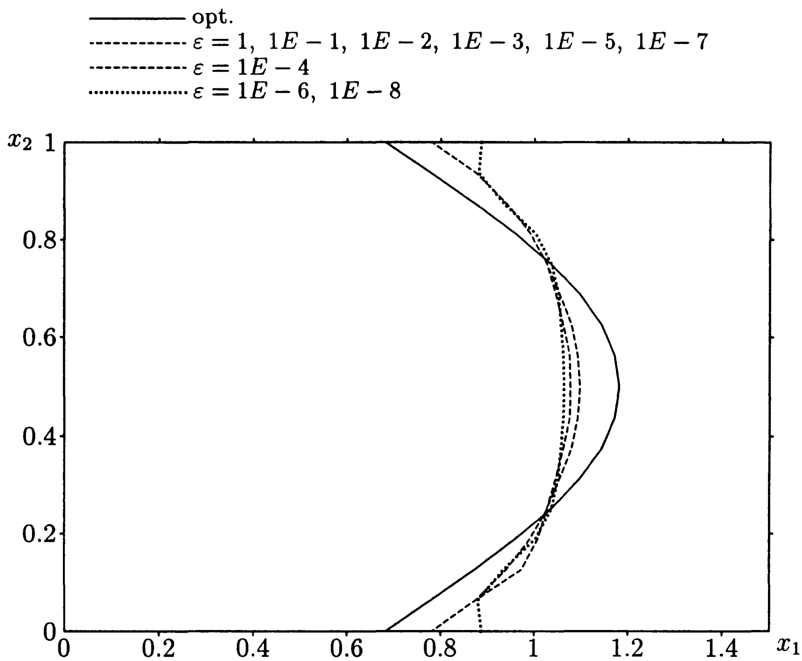

Fig. 6.6

ε	IT	$E(0)$	$E(IT)$	$J_0(IT)$	$J_3(IT)$	$J_3^p(IT)$
$.1E+01$	154	$.118E-02$	$.498E-05$	$.171E-05$	$.327E-05$	$.197E-05$
$.1E+00$	66	$.116E-01$	$.774E-05$	$.796E-07$	$.694E-05$	$.371E-05$
$.1E-01$	64	$.115E+00$	$.116E-04$	$.363E-07$	$.798E-05$	$.432E-05$
$.1E-02$	57	$.115E+01$	$.443E-04$	$.365E-07$	$.776E-05$	$.413E-05$
$.1E-03$	57	$.115E+02$	$.260E-03$	$.255E-07$	$.476E-05$	$.195E-05$
$.1E-04$	80	$.115E+03$	$.247E-02$	$.246E-07$	$.485E-05$	$.210E-05$
$.1E-05$	61	$.115E+04$	$.253E-01$	$.253E-07$	$.493E-05$	$.130E-05$
$.1E-06$	48	$.115E+05$	$.249E+00$	$.249E-07$	$.481E-05$	$.210E-05$
$.1E-07$	200	$.115E+06$	$.408E+00$	$.408E-08$	$.174E-04$	$.124E-04$

$J_0(0) = .11532E-02,\ J_3(0) = -.225E-04,\ J_3^p(0) = .486E-04$

Table 5

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