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OPTIMAL DESIGN PROBLEMS FOR A DYNAMIC VISCOELASTIC PLATE I. SHORT MEMORY MATERIAL

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Summary. We deal with an optimal control problem with respect to a variable thickness for a dynamic viscoelastic plate with velocity constraints. The state problem has the form of a pseudohyperbolic variational inequality. The existence and uniqueness theorem for the state problem and the existence of an optimal thickness function are proved.

Keywords: optimal control, viscoelastic plate, variable thickness, pseudohyperbolic variational inequality, penalization

AMS classification: 49J20, 49J40, 35L85, 73F15

Optimal design problems with respect to the thickness of a viscoelastic plate made of a short memory material were considered in papers [2], [3]. State problems were initial-boundary value problems for pseudoparabolic equations and variational inequalities. In contrast to these papers we consider here dynamic problems with velocity constraints. The state problem is then an initial-boundary value problem for a pseudohyperbolic variational inequality. It involves also the hyperbolic case. Unilateral hyperbolic optimal control problems with controls on right-hand sides were studied by D. Tiba ([10], [11]). We consider control parameters in coefficients of the variational inequality as well as on the right-hand side. The first chapter is devoted to the formulation of the state problem. Using the method of penalization we prove the existence and uniqueness theorem for a solution of an initial-boundary value problem for a pseudohyperbolic variational inequality in the second chapter. The existence of an optimal thickness function will be established in the third chapter.

1. Formulation of the state problem

We consider a thin viscoelastic plate made of a short memory material occupying the domain $G \subset \mathbb{R}^3$ of the form

$$G = \Big\{(x,z) \in \mathbb{R}^3 \colon x = (x_1,x_2) \in \Omega, -\frac{1}{2}e(x) < z < \frac{1}{2}e(x)\Big\},$$

where $\Omega \in \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary $\partial\Omega$. We assume the plate to be clamped on the part

$$\gamma_1 = \Big\{(s,z) \in \mathbb{R}^3 \colon s \in \Gamma_1 \subset \partial\Omega, -\frac{1}{2}e(s) < z < \frac{1}{2}e(s)\Big\}$$

of the boundary surface ∂G. Further, it is subjected to surface tractions $g(s)$ acting on the part

$$\gamma_2 = \Big\{(s,z) \in \mathbb{R}^3 \colon s \in \Gamma_2 \subset \partial\Omega, -\frac{1}{2}e(s) < z < \frac{1}{2}e(s)\Big\},$$
$$\Gamma_1 \cap \Gamma_2 = \emptyset, \quad \overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega;$$

and to forces $f(x)$ acting perpendicularly on the part

$$\gamma_3 = \Big\{(x,z) \in \mathbb{R}^3 \colon x \in \Omega, z = \frac{1}{2}e(x)\Big\}.$$

The displacement vector-valued function $\mathbf{u} \colon [0,T] \times G \to \mathbb{R}^3$ and the symmetric tensor-valued function $\sigma \colon [0,T] \times G \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ fulfill the relations

$$\varrho \mathbf{u}_t'' - \operatorname{div} \sigma = \mathbf{O} \quad \text{on } [0,T] \times G, \tag{1.1}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } G, \tag{1.2}$$
$$\mathbf{u}'(0) = \mathbf{v}_0 \quad \text{in } G, \tag{1.3}$$
$$\mathbf{u}(t,s) = \mathbf{O} \quad \text{on } [0,T] \times \gamma_1, \tag{1.4}$$
$$\sigma \cdot \mathbf{n} = \big(0,0,g(t,s)\big) \quad \text{on } [0,T] \times \gamma_2, \tag{1.5}$$
$$\sigma \cdot \mathbf{n} = \big(0,0,f(t,x)\big) \quad \text{on } [0,T] \times \gamma_3, \tag{1.6}$$
$$\sigma = A^{(0)}(t)\varepsilon' + A^{(1)}(t)\varepsilon \quad \text{on } [0,T] \times G, \tag{1.7}$$
$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.8}$$

$\mathbb{R}^{3\times 3}_{\mathrm{sym}}$ is the set of all 3×3 symmetric matrices, $\mathbf{n}$ is the exterior unit normal vector on ∂G, $\varrho \colon G \to \mathbb{R}$ is the density function, $A^{(0)}(.)$, $A^{(1)}(.)$ are fourth-order symmetric tensor-functions fulfilling the assumptions

$$A^{(r)}_{ijkl}(.) \in C^2[0,T], \tag{1.9}$$

$$A^{(r)}_{ijkl}(t) = A^{(r)}_{jikl}(t) = A^{(r)}_{klij}(t), \tag{1.10}$$

$$A^{(0)}_{ijkl}(t)\varepsilon_{ij}\varepsilon_{kl} \geqslant 0, \tag{1.11}$$

$$A^{(1)}_{ijkl}(t)\varepsilon_{ij}\varepsilon_{kl} \geqslant \alpha_1\varepsilon_{ij}\varepsilon_{ij}, \quad \alpha_1 > 0, \tag{1.12}$$

$$\Big[\frac{\mathrm{d}}{\mathrm{d}t}A^{(0)}_{ijkl}(t) + A^{(1)}_{ijkl}(t)\Big]\varepsilon_{ij}\varepsilon_{kl} \geqslant \alpha_2\varepsilon_{ij}\varepsilon_{ij}, \quad \alpha_2 > 0, \tag{1.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{(1)}_{ijkl}(t)\varepsilon_{ij}\varepsilon_{kl} \leqslant 0, \quad \text{for all } \{\varepsilon_{ij}\} \in \mathbb{R}^{3\times 3}_{\text{sym}} \text{ and } t \in [0,T]. \tag{1.14}$$

Multiplying the equilibrium equation (1.1) by a test function $\mathbf{v}$ and integrating by parts through the domain G we obtain the relation

$$\iint_\Omega \int_{-e(x)/2}^{e(x)/2} [\varrho(x)\mathbf{u}''(t,x)\cdot\mathbf{v}(x) + \sigma_{ij}\varepsilon_{ij}(\mathbf{v})]\,\mathrm{d}z\,\mathrm{d}x \tag{1.15}$$
$$= \iint_\Omega f(t,x)v_3(x)\,\mathrm{d}x + \oint_{\Gamma_2}\int_{-e(s)/2}^{e(s)/2} g(t,s)v_3(s)\,\mathrm{d}z\,\mathrm{d}s$$
$$\text{for all } \mathbf{v} \in H^1(\Omega)^3 \text{ such that } \mathbf{v} = \mathbf{O} \text{ on } \gamma_1,$$

where $H^k(\Omega)$ is the Sobolev space of all functions with generalized derivatives up to the order k belonging to the space $L^2(\Omega)$.

Considering the Kirchhoff hypothesis of a plate ([8], 10.4.41) we express the displacement vector $\mathbf{u}$ in the form

$$u_j = -z\frac{\partial w}{\partial x_j}, \ j = 1,2; \quad u_3(x) = w(x).$$

Using the relations

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0,$$
$$v_i = -zv_{'i}, \ v_3 = v, \ \varepsilon_{ij}(\mathbf{v}) = -zv_{'ij}, \quad i,j = 1,2$$

we obtain from (1.15) the integral relation for the deflection function w:

$$\iint_\Omega \Big\{\varrho(X)\Big[e(x)w''(t,x)v(x) + \frac{1}{12}e^3(x)w''_{'i}(t,x)v_{'i}(x)\Big] \tag{1.16}$$
$$+ \frac{1}{12}e^3(x)[A^{(0)}_{ijkl}(t)w_{'kl}(t,x) + A^{(1)}_{ijkl}(t)w_{'kl}(t,x)]v_{'ij}(x)\Big\}\,\mathrm{d}x$$
$$= \iint_\Omega f(t,x)v(x)\,\mathrm{d}x + \oint_{\Gamma_2} e(s)g(t,s)v(s)\,\mathrm{d}s \quad \text{for all } v \in V,$$

where

$$V = \Big\{ v \in H^2(\Omega) \colon v = \frac{\partial v}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_1 \Big\}.$$

In the sequel we shall consider the constraints on velocities of vibrations which will be introduced in the next chapter.

2. Existence and uniqueness theorem for a pseudohyperbolic variational inequality

We assume the dynamic problem for a viscoelastic plate considered in the first chapter with constraints on the velocities of vibrations. The pseudohyperbolic variational inequality then appears instead of the relation (1.16). Admissible deflections of the plate belong to the space V, which is a Hilbert space with the inner product

$$(u, v) = \iint_\Omega u_{,ij} v_{,ij} \, \mathrm{d}x$$

and the norm $\|u\| = (u,u)^{1/2}$. Admissible velocities belong to the closed convex set $K \subset V$, $0 \in K$. We shall deal with an initial-boundary value problem

(2.1) For a.e. $t \in [0,T]$:

$$\begin{aligned} &\iint_\Omega \Big\{ \varrho(x) \Big[e(x) w''(t,x) \big(v(x) - w'(t,x) \big) \\ &\quad + \frac{1}{12} e^3(x) w''_{,i}(t,x) \big(v_{,i}(x) - w'_{,i}(t,x) \big) \Big] \\ &\quad + \frac{1}{12} e^3(x) [A^{(0)}_{ijkl}(t) w'_{,kl}(t,x) + A^{(1)}_{ijkl}(t) w_{,kl}(t,x)][v_{,ij}(x) - w'_{,ij}(t,x)] \Big\} \, \mathrm{d}x \\ &\geqslant \iint_\Omega f(t,x)[v(x) - w'(t,x)] + \oint_{\Gamma_2} e(s) g(t,s) [v(s) - w'(t,s)] \, \mathrm{d}s \end{aligned}$$

for all $v \in K$,

(2.2) $w'(t) \in K$ for a.e. $t \in [0,T]$,

(2.3) $w(0,x) = w_0(e,x)$,

(2.4) $w'(0,x) = w_1(x)$, $x \in \Omega$.

We have $w_0 \equiv 0$ in the case of the Voight model ([6]). In the case of the Zener model ([9]) $w_0(e,.) \in V$ is a solution of the corresponding elastic problem

$$(2.5)\qquad \iint_\Omega \frac{1}{12} e^3(x) A^{(1)}_{ijkl}(0) w_{0'ij}(x) v_{'kl}(x)\,\mathrm{d}x = \iint_\Omega f_0(x) v(x)\,\mathrm{d}x + \oint_{\Gamma_2} e(s) g_0(s) v(s)\,\mathrm{d}s \quad \text{for all } v \in V.$$

Further, we assume

$$w_1 \in K, \quad f \in W^{1,2}(0,T;L^2(\Omega)), \quad g \in W^{1,2}(0,T;L^2(\Gamma_2)),$$

where $W^{1,2}(0,T;X)$ is the Sobolev space of functions defined on $(0,T)$ with values in a Hilbert space X with first-order generalized derivatives (with respect to t) belonging to $L^2(0,T;X)$ (see [1] or [3] for more details).

The density and thickness functions ϱ and e are continuous on $\overline{\Omega}$ and fulfill the bounds $0 < \varrho_1 \leqslant \varrho(x) \leqslant \varrho_2$, $x \in \overline{\Omega}$ and

$$(2.6)\qquad 0 < e_1 \leqslant e(x) \leqslant e_2 \quad \text{for all } x \in \overline{\Omega}.$$

We denote by E the set of all functions $e \in C(\overline{\Omega})$ fulfilling the estimates (2.6).

Let V^* be the dual space with respect to V with the norm $\|.\|_*$ and the duality pairing $\langle .,. \rangle$. The inequality (2.1) can be expressed in the operator form

$$(2.1')\qquad \langle B(e)w''(t) + A_0(e,t)w'(t) + A_1(e,t)w(t), v - w'(t)\rangle \geqslant \langle F(t) + G(e,t), v - w'(t)\rangle \quad \text{for all } v \in K,$$

where the operators $B(e) \in L(H^1(\Omega), V^*)$, $A_0(e,t) \in L(V,V^*)$, $A_1(e,t) \in L(V,V^*)$ and the functionals $F(t)$, $G(e,t) \in V^*$ are defined by

$$(2.7)\qquad \langle B(e)u, v\rangle = \iint_\Omega \varrho(x)\Big[e(x)u(x)v(x) + \frac{1}{12}e^3(x)u_{'i}(x)v_{'i}(x)\Big]\,\mathrm{d}x,$$

$$(2.8)\qquad \langle A_r(e,t)u, v\rangle = \frac{1}{12}\iint_\Omega e^3(x)A^{(r)}_{ijkl}(t)u_{'ij}(x)v_{'kl}(x)\,\mathrm{d}x, \quad r = 0,1;$$

$$(2.9)\qquad \langle F(t), v\rangle = \iint_\Omega f(t,x)v(x)\,\mathrm{d}x,$$

$$(2.10)\qquad \langle G(e,t), v\rangle = \oint_{\Gamma_2} e(s)g(t,s)v(s)\,\mathrm{d}s, \quad v \in V.$$

We solve the problem (2.1′), (2.2), (2.3), (2.4) using the penalized initial value problem

$$(B(e)+\varepsilon J)w''_\varepsilon(t)+A_0(e,t)w'_\varepsilon(t)+\frac{1}{\varepsilon}\beta(w'_\varepsilon(t)) + A_1(e,t)w_\varepsilon(t)=F(t)+G(e,t), \quad t\in[0,T]; \tag{2.11}$$

$$w_\varepsilon(0)=w_0(e), \tag{2.12}$$

$$w'_\varepsilon(0)=w_1, \tag{2.13}$$

where $J\colon V\to V^*$ is the canonical operator defined by

$$\langle Ju,v\rangle=(u,v), \quad u,v\in V,$$

and $\beta\colon V\to V^*$ is the penalty operator defined by

$$\beta(u)=J(u-P_Ku), \quad u\in V.$$

We recall ([3]) that $P_K\colon V\to K$ is the projection operator and the penalized operator β fulfills the conditions

(2.14)
i) $\beta(v)=0\Leftrightarrow v\in K$,
ii) $\langle\beta(v)-\beta(u),v-u\rangle\geqslant 0$,
iii) $\langle\beta(u),u\rangle\geqslant 0$,
iv) $\|\beta(u)-\beta(v)\|_*\leqslant 2\|u-v\|$ for all $u,v\in V$.

Let $H^1(\Omega)$ be the Sobolev space equipped with the inner product

$$(u,v)_1=\iint_\Omega (uv+u_{'i}v_{'i})\,\mathrm{d}x$$

and the norm $\|u\|_1=(u,u)_1^{1/2}$. We assume that the initial functions $w_0(e)$, v_1 fulfill the condition

$$A_0(e,0)w_1+A_1(e,0)w_0(e)\in H^1(\Omega)^* \quad \text{for every } e\in E. \tag{2.15}$$

Theorem 2.1. *Let $T>0$, $\varepsilon>0$. There exists a unique solution $w_\varepsilon\in C^2([0,T];V)$ of the initial value problem* (2.11), (2.12), (2.13) *fulfilling the estimates*

$$\|w'_\varepsilon(t)\|_1^2+\|w_\varepsilon(t)\|^2\leqslant M_1 \quad \text{for every } t\in[0,T], \tag{2.16}$$

$$\|w''_\varepsilon(t)\|_1^2+\|w'_\varepsilon(t)\|^2\leqslant M_2 \quad \text{for every } t\in[0,T], \tag{2.17}$$

with constants M_1, M_2 not depending on e and ε.

Proof. The problem (2.11), (2.12), (2.13) can be expressed as an initial value problem of the first order in $V \times V$:

$$U_\varepsilon'(t) + C_\varepsilon(t)U_\varepsilon(t) = F_\varepsilon(t), \quad t \in [0,T], \tag{2.18}$$

$$U_\varepsilon(0) = U_0, \tag{2.19}$$

where

$$U_0 = \begin{pmatrix} w_0(e) \\ w_1 \end{pmatrix}, \quad F_\varepsilon(t) = \begin{pmatrix} 0 \\ [B(e)+\varepsilon J]^{-1}[F(t)+G(e,t)] \end{pmatrix}$$

and the operator $C_\varepsilon(t)\colon V \times V \to V \times V$ is defined by

$$C_\varepsilon(t) = \begin{pmatrix} 0 & -I \\ [B(e)+\varepsilon J]^{-1}A_1(e,t) & [B(e)+\varepsilon J]^{-1}\big[A_0(e,t)+\frac{1}{\varepsilon}\beta\big] \end{pmatrix}.$$

The operator $B(e)+\varepsilon J\colon V \to V^*$ is linear, bounded and strongly monotone and hence there exists a linear bounded inverse operator $[B(e)+\varepsilon J]^{-1}\colon V^* \to V$. The operator $C_\varepsilon(.)$ is Lipschitz continuous in the space $C([0,T], V \times V)$ and due to the theory of ordinary differential equations in Banach spaces ([7]) there exists a unique solution

$$U_\varepsilon = (u_\varepsilon^{(1)}, u_\varepsilon^{(2)})^T \in C^1([0,T], V \times V)$$

of (2.18), (2.19). The function $w_\varepsilon = u_\varepsilon^{(1)}$ is then a unique solution of the problem (2.11), (2.12), (2.13).

It remains to verify the estimates (2.16), (2.17). After duality pairing of the equation (2.11) with $w_\varepsilon'(t)$ we obtain due to the symmetry of $B(t)$ and $A_1(e,t)$ the relation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\big[\big\langle (B(e)+\varepsilon J)w_\varepsilon'(t), w_\varepsilon'(t)\big\rangle + \langle A_1(e,t)w_\varepsilon(t), w_\varepsilon(t)\rangle\big] \\ &\quad - \langle A_1'(e,t)w_\varepsilon(t), w_\varepsilon(t)\rangle + 2\langle A_0(e,t)w_\varepsilon'(t), w_\varepsilon'(t)\rangle \\ &\quad + \frac{2}{\varepsilon}\big\langle \beta\big(w_\varepsilon'(t)\big), w_\varepsilon'(t)\big\rangle = 2\langle F(t)+G(e,t), w_\varepsilon'(t)\rangle, \quad t \in [0,T]. \end{aligned} \tag{2.20}$$

Let us introduce the real function $\varphi_\varepsilon(t)$ by

$$\varphi_\varepsilon(t) = \big\langle \big(B(e)+\varepsilon J\big)w_\varepsilon'(t), w_\varepsilon'(t)\big\rangle + \langle A_1(e,t)w_\varepsilon(t), w_\varepsilon(t)\rangle. \tag{2.21}$$

Due to the imbedding theorem and the theorem on traces in the Sobolev space $H^1(\Omega)$, the right-hand side of (2.20) fulfills the inequality

$$\begin{aligned} &|\langle F(t)+G(e,t), w_\varepsilon'(t)\rangle| \\ &\quad \leqslant c_1\left\{\left[\iint_\Omega f(t,x)^2\,\mathrm{d}x\right]^{1/2} + \left[\oint_\Gamma \big(e(s)g(t,s)\big)^2\,\mathrm{d}s\right]^{1/2}\right\}\|w_\varepsilon'(t)\|_1 \end{aligned}$$

with a constant c_1 depending only on Ω and Γ_2. Due to (1.10), (1.12), (2.14 iii) the relations (2.20), (2.21) then imply the inequality

$$\varphi_\varepsilon'(t) \leqslant \iint_\Omega f(t,x)^2\,\mathrm{d}x + \oint_\Gamma \big(e(s)g(t,s)\big)^2\,\mathrm{d}s + c_2\|w_\varepsilon'(t)\|^2 + c_3\|w_\varepsilon(t)\|^2, \quad t\in[0,T]$$

and integrating it we obtain

$$\begin{aligned}\varphi_\varepsilon(t) &\leqslant c_4\|w_1\|_1^2 + \varepsilon\|w_1\|^2 + c_5\|w_0(e)\|^2\\ &+ \int_0^T \Big[\iint_\Omega f(t,x)^2\,\mathrm{d}x + \oint_{\Gamma_2} \big(e(s)g(t,s)\big)^2\,\mathrm{d}s\Big]\,\mathrm{d}t\\ &+ c_2\int_0^t \|w_\varepsilon'(\tau)\|_1^2\,\mathrm{d}\tau + c_3\int_0^t \|w_\varepsilon(\tau)\|^2\,\mathrm{d}\tau \quad \text{for all } t\in[0,T].\end{aligned}$$

Using (1.12), (2.6), (2.7), (2.21) we arrive at the inequality

$$\varphi_\varepsilon(t) \leqslant c_6 + c_7\int_0^t \varphi_\varepsilon(\tau)\,\mathrm{d}\tau \quad \text{for all } \varepsilon\in(0,\varepsilon_0),\ t\in[0,T].$$

The Gronwall lemma then implies

$$\varphi_\varepsilon(t) \leqslant c_6\exp(c_7 t), \quad \varepsilon\in(0,\varepsilon_0),\ t\in[0,T]$$

and by virtue of (1.12), (2.6), (2.7), (2.21) we obtain the estimate (2.16).

Further, we differentiate the equation (2.11) with respect to t and arrive at

$$\begin{aligned}(2.22)\quad &[B(e)+\varepsilon J]w_\varepsilon'''(t) + A_0(e,t)w_\varepsilon''(t)\\ &+ [A_0'(e,t) + A_1(e,t)]w_\varepsilon'(t) + A_1'(e,t)w_\varepsilon(t) + \frac{1}{\varepsilon}\beta\big(w_\varepsilon'(t)\big)'\\ &= F'(t) + G'(e,t) \quad \text{for a.e. } t\in[0,T].\end{aligned}$$

The function $\beta\big(w_\varepsilon'(.)\big)$ is almost everywhere differentiable in the space V^*, because V^* is reflexive and β is Lipschitz continuous (see [5], 143–145). The third derivative $w_\varepsilon''' \in V$ exits almost everywhere on $[0,T]$ due to the relation

$$\begin{aligned}w_\varepsilon''(t) = [B(e)+\varepsilon J]^{-1}\Big[&F(t) + G(e,t) - A_0(e,t)w_\varepsilon'(t)\\ &- \frac{1}{\varepsilon}\beta\big(w_\varepsilon'(t)\big) - A_1(e,t)w_\varepsilon(t)\Big], \quad t\in[0,T],\end{aligned}$$

and the differentiability of $F(t)$, $G(e,t)$.

After duality pairing of (2.22) with $w''_\varepsilon(t)$ we obtain in the same way as above the relation

$$(2.23)\quad \frac{\mathrm{d}}{\mathrm{d}t}\big[\big\langle (B(e)+\varepsilon J)w''_\varepsilon, w''_\varepsilon\big\rangle + \big\langle \big(A'_0(e,t)+A_1(e,t)\big)w'_\varepsilon(t), w'_\varepsilon(t)\big\rangle\big]$$
$$- \big\langle \big(A''_0(e,t)+A'_1(e,t)\big)w'_\varepsilon(t), w'_\varepsilon(t)\big\rangle + 2\langle A_0(e,t)w''_\varepsilon(t), w''_\varepsilon(t)\rangle$$
$$+ 2\langle A'_1(e,t)w_\varepsilon(t), w''_\varepsilon(t)\rangle + \frac{2}{\varepsilon}\langle \beta\big(w'_\varepsilon(t)\big)', w''_\varepsilon(t)\rangle$$
$$= 2\langle F'(t)+G'(e,t), w''_\varepsilon(t)\rangle \quad \text{for a.e. } t\in[0,T].$$

Monotonicity of β implies the inequality

$$(2.24)\qquad \big\langle \beta\big(w'_\varepsilon(t)\big)', w''_\varepsilon(t)\big\rangle \geqslant 0 \quad \text{for a.e. } t\in[0,T]$$

and integrating (2.23) we obtain, taking into account (1.11), (2.11), (2.16), (2.24), the inequality

$$(2.25)\quad \big\langle (B(e)+\varepsilon J)w''_\varepsilon, w''_\varepsilon\big\rangle + \big\langle \big(A'_0(e,t)+A_1(e,t)\big)w'_\varepsilon(t), w'_\varepsilon(t)\big\rangle$$
$$\leqslant \big\langle H(e), \big(B(e)+\varepsilon J\big)^{-1}H(e)\big\rangle_1 + \big\langle \big(A'_0(e,0)+A_1(e,0)\big)w_1, w_1\big\rangle$$
$$+ \int_0^t \big\langle \big(A''_0(e,\tau)+A'_1(e,\tau)\big)w'_\varepsilon(\tau), w'_\varepsilon(\tau)\big\rangle\,\mathrm{d}\tau$$
$$- 2\int_0^t \langle A'_1(e,\tau)w_\varepsilon(\tau), w''_\varepsilon(\tau)\rangle\,\mathrm{d}\tau + c_2\int_0^t \|w''_\varepsilon(\tau)\|_1^2\,\mathrm{d}\tau$$
$$+ \int_0^T \Big[\iint_\Omega f'(t,x)^2\,\mathrm{d}x + \oint_{\Gamma_2} \big(e(s)g'(t,s)\big)^2\,\mathrm{d}s\Big]\,\mathrm{d}t$$
$$\text{for a.e. } t\in[0,T],$$

where

$$(2.26)\qquad H(e) = F(0) + G(e,0) - A_0(e,0)w_1 - A_1(e,0)w_0(e).$$

Integration by parts yields the relation

$$(2.27)\quad -\int_0^t \langle A'_1(e,\tau)w_\varepsilon(\tau), w''_\varepsilon(\tau)\rangle\,\mathrm{d}\tau = \int_0^t \langle A''_1(e,\tau)w_\varepsilon(\tau), w'_\varepsilon(\tau)\rangle\,\mathrm{d}\tau$$
$$+ \int_0^t \langle A'_1(e,\tau)w'_\varepsilon(\tau), w'_\varepsilon(\tau)\rangle\,\mathrm{d}\tau - \langle A'_1(e,t)w_\varepsilon(t), w'_\varepsilon(t)\rangle$$
$$+ \langle A'_1(e,0)w_0(e), w_1\rangle.$$

Using (1.9), (1.19), (2.6), (2.7), (2.25), (2.27) we arrive at the inequality

$$c_8[\|w_\varepsilon''(t)\|_1^2 + \|w_\varepsilon'(t)\|^2] \leqslant c_9 + \int_0^T \Big[\iint_\Omega f'(t,x)^2\,\mathrm{d}x + \oint_{\Gamma_2} \big(e(s)g'(t,s)\big)^2\,\mathrm{d}s\Big]\,\mathrm{d}t$$

$$+ c_{10}\|w_\varepsilon(t)\|^2 + c_{11}\int_0^T \|w_\varepsilon(\tau)\|^2\,\mathrm{d}\tau$$

$$+ c_{12}\int_0^t [\|w_\varepsilon'(\tau)\|^2 + \|w_\varepsilon''(\tau)\|_1^2]\,\mathrm{d}\tau.$$

Considering (2.6) and the just verified estimate (2.16) we obtain the inequality

$$\|w_\varepsilon''(t)\|_1^2 + \|w_\varepsilon'(t)\|^2 \leqslant c_{12} + c_{13}\int_0^t [\|w_\varepsilon'(\tau)\|^2 + \|w_\varepsilon''(\tau)\|_1^2]\,\mathrm{d}\tau$$

for every $t \in [0,T]$.

The Gronwall inequality implies (2.17), which completes the proof. □

Using Theorem 2.1 we obtain existence, uniqueness and a priori estimates for a solution of the unilateral problem (2.1)–(2.4).

Theorem 2.2. *There exists a unique solution*

$$w(e) \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H^1(\Omega))$$

fulfilling the estimates

$$\|w'(e,t)\|_1^2 + \|w(e,t)\|^2 \leqslant M_1 \quad \textit{for all } t \in [0,T], \tag{2.28}$$

$$\|w''(e,t)\|_1^2 + \|w'(e,t)\|^2 \leqslant M_2 \quad \textit{for a.e. } t \in [0,T]. \tag{2.29}$$

Proof. We shall use a similar approach as in the proof of Th. 2.2 from [3].

a) *Existence.* The family of functions $\{w_\varepsilon\}$, $\varepsilon > 0$ from Theorem 2.1 is uniformly bounded with respect to ε in all spaces $W^{1,p}(0,T;V)$, $W^{2,p}(0,T;H^1(\Omega))$, $1 \leqslant p \leqslant \infty$. Further, the sets $\{w_\varepsilon(t)\}$, $\{w_\varepsilon'(t)\}$ are bounded in V for every $t \in [0,T]$. Moreover, we have $w_\varepsilon(0) = w_0(e)$, $w_\varepsilon'(0) = w_1$. Then there exist a sequence $\{\varepsilon_n\}$ ($\varepsilon_n > 0$) and a function $w \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H^1(\Omega))$ such that

$$\lim_{n\to\infty} \varepsilon_n = 0, \tag{2.30}$$

$$w_{\varepsilon_n}(t) \rightharpoonup w(t) \quad \text{(weakly) in } V \text{ for every } t \in [0,T], \tag{2.31}$$

$$w'_{\varepsilon_n}(t) \rightharpoonup w'(t) \quad \text{in } V \text{ for a.e. } t \in [0,T], \tag{2.32}$$

$$w'_{\varepsilon_n}(t) \rightharpoonup w'(t) \quad \text{in } H^1(\Omega) \text{ for every } t \in [0,T], \tag{2.33}$$

$$w_{\varepsilon_n} \overset{*}{\rightharpoonup} w \quad \text{(weakly star) in } L^\infty(0,T;V), \tag{2.34}$$

$$w'_{\varepsilon_n} \overset{*}{\rightharpoonup} w' \quad \text{in } L^\infty(0,T;V), \tag{2.35}$$

$$w''_{\varepsilon_n} \overset{*}{\rightharpoonup} w'' \quad \text{in } L^\infty(0,T;H^1(\Omega)). \tag{2.36}$$

The estimates (2.28), (2.29) then follow from the estimates (2.16), (2.17) and from the inequalities

$$\begin{aligned}
\|w''\|_{L^\infty(0,T;H^1(\Omega))} &\leqslant \liminf_{n\to\infty} \|w''_{\varepsilon_n}\|_{L^\infty(0,T;H^1(\Omega))},\\
\|w'\|_{L^\infty(0,T;V)} &\leqslant \liminf_{n\to\infty} \|w'_{\varepsilon_n}\|_{L^\infty(0,T;V)},\\
\|w'(t)\|_1 &\leqslant \liminf_{n\to\infty} \|w'_{\varepsilon_n}(t)\|_1 \quad \text{for every } t \in [0,T],\\
\|w(t)\| &\leqslant \liminf_{n\to\infty} \|w_{\varepsilon_n}(t)\|_1 \quad \text{for every } t \in [0,T].
\end{aligned}$$

It remains to verify that the function w is solution of the problem (2.1)–(2.4). We rewrite the penalized equation (2.11) in the form

$$\begin{aligned}
\beta\big(w'_{\varepsilon_n}(t)\big) = \varepsilon_n\big[F(t) + G(e,t) &- \big(B(e) + \varepsilon_n J\big)w''_{\varepsilon_n}(t)\\
&- A_0(e,t)w'_{\varepsilon_n}(t) - A_1(e,t)w_{\varepsilon_n}(t)\big] \quad \text{for all } t \in [0,T].
\end{aligned} \tag{2.37}$$

The sequences $\{w_{\varepsilon_n}(t)\}$, $\{w'_{\varepsilon_n}(t)\}$ and $\{w''_{\varepsilon_n}(t)\}$ are bounded in V and $H^1(\Omega)$, respectively. The sequence $\{B(e)w''_{\varepsilon_n}(t)\}$ is then bounded in V^* as the operator $B(e)$ belongs to $L(H^1(\Omega), V^*)$. Combining (2.25) and (2.27) we obtain boundedness of set $\{\sqrt{\varepsilon_n} w''_{\varepsilon_n}(t)\}$ in V and of $\{\sqrt{\varepsilon_n} J w''_{\varepsilon_n}(t)\}$ in V^*. The relations (2.30), (2.37) then imply

$$\lim_{n\to\infty} \beta\big(w'_{\varepsilon_n}(t)\big) = 0 \quad \text{in } V^* \text{ for all } t \in [0,T].$$

Monotonicity of β and the relation (2.32) then imply

$$\text{for a.e. } t \in [0,T] \quad \langle \beta(u), w'(t) - u\rangle \leqslant 0 \quad \text{for every } u \in V. \tag{2.38}$$

Inserting $u = w'(t) + sv$, $s > 0$, $v \in V$ into (2.38) we obtain

$$\big\langle \beta\big(w'(t) + sv\big), v\big\rangle \geqslant 0 \quad \text{for every } v \in V.$$

Due to the Lipschitz continuity of β the limiting process $s \to 0$ yields

$$\langle \beta(w'(t)), v \rangle \geqslant 0 \quad \text{for all } v \in V$$

and hence

$$\beta(w'(t)) = 0 \quad \text{for a.e. } t \in [0,T],$$

which due to (2.14 i) implies

$$w'(t) \in K \quad \text{for a.e. } t \in [0,T].$$

Further we verify the initial conditions (2.3), (2.4). After changing the function w on the set of zero measure in $[0,T]$ we obtain ([5])

$$w \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H^1(\Omega)) \cap C([0,T],V) \cap C^1([0,T],H^1(\Omega))$$

and

$$w(t) = w(0) + \int_0^t w'(\tau)\,\mathrm{d}\tau,$$

$$w'(t) = w'(0) + \int_0^t w''(\tau)\,\mathrm{d}\tau \quad \text{for every } t \in [0,T].$$

Simultaneously we have

$$w_{\varepsilon_n}(t) = w_0(e) + \int_0^t w'_{\varepsilon_n}(\tau)\,\mathrm{d}\tau,$$

$$w'_{\varepsilon_n}(t) = w_1 + \int_0^t w''_{\varepsilon_n}(\tau)\,\mathrm{d}\tau \quad \text{for every } t \in [0,T]$$

and comparison with (2.13)–(2.36) implies the fulfilling of the initial conditions

$$w(0) = w_0(e), \quad w'(0) = w_1. \tag{2.39}$$

We proceed with the proof of the variational inequality (2.1′) which is equivalent to (2.1). Let $v \in L^1(0,T;V)$ be an arbitrary function such that

$$v(t) \in K \quad \text{for a.e. } t \in [0,T].$$

The properties of the penalty operator β imply the inequalities

$$\text{for a.e. } t \in [0,T]: \quad \langle \beta(w'_{\varepsilon_n}(t)), v(t) - w'_{\varepsilon_n}(t) \rangle \leqslant 0 \quad n = 1,2,\ldots.$$

Then we obtain from (2.11) the inequalities

$$\begin{aligned}\text{for a.e. } t \in [0,T]: \quad & \langle [B(e) + \varepsilon_n J] w''_{\varepsilon_n}(t) + A_0(e,t) w'_{\varepsilon_n}(t) + A_1(e,t) e_{\varepsilon_n}(t) \\ & - F(t) - G(e,t), v(t) - w'_{\varepsilon_n}(t) \rangle \geqslant 0, \quad n = 1,2,\ldots.\end{aligned}$$

Integrating we arrive at the inequalities

$$\begin{aligned}(2.40) \quad & \langle B(e) w'_{\varepsilon_n}(t), w'_{\varepsilon_n}(t) \rangle + \langle A_1(e,t) w_{\varepsilon_n}(t), w_{\varepsilon_n}(t) \rangle \\ & + \int_0^t [2\langle A_0(e,\tau) w'_{\varepsilon_n}(\tau), w'_{\varepsilon_n}(\tau) \rangle - \langle A'_1(e,\tau) w_{\varepsilon_n}(\tau), w_{\varepsilon_n}(\tau) \rangle] \, \mathrm{d}\tau \\ & \leqslant \langle B(e) w_1, w_1 \rangle + \varepsilon_n \|w_1\|^2 + \langle A_1(e,0) w_0(e), w_0(e) \rangle \\ & + 2 \int_0^t \langle [B(e) + \varepsilon_n J] w''_{\varepsilon_n}(\tau) + A_0(e,\tau) e'_{\varepsilon_n}(\tau) + A_1(e,\tau) w_{\varepsilon_n}(\tau), v(\tau) \rangle \, \mathrm{d}\tau \\ & + 2 \int_0^t \langle F(\tau) + G(e,\tau), w'_{\varepsilon_n}(\tau) - v(\tau) \rangle \, \mathrm{d}\tau \quad \text{for every } n = 1,2,\ldots.\end{aligned}$$

The functionals on the left-hand side of (2.40) are weakly lower semicontinuous on the spaces $H^1(\Omega)$, V and $W^{1,2}(0,T;V)$ due to the assumptions (1.10), (1.11), (1.12), (1.14) and the form (2.7) of the operator $B(e)$. Further, we have the relation

$$(2.41) \quad \lim_{n\to\infty} \varepsilon_n \int_0^t \langle J w''_{\varepsilon_n}(\tau), v(\tau) \rangle \, \mathrm{d}\tau = 0 \quad \text{for every } v \in L^2(0,T;V)$$

which is a consequence of boundedness of the sequence $\{\sqrt{\varepsilon_n} J w''_{\varepsilon_n}(t)\}$ in V^* verified above. The relations (2.30)–(2.36), (2.39), (2.41) then imply the inequality

$$\begin{aligned}& \langle B(e) w'(t), w'(t) \rangle + \langle A_1(e,t) w(t), w(t) \rangle \\ & \quad + \int_0^t [2\langle A_0(e,\tau) w'(\tau), w'(\tau) \rangle - \langle A'_1(e,\tau) w(\tau), w(\tau) \rangle] \, \mathrm{d}\tau \\ & \leqslant \langle B(e) w'(0), w'(0) \rangle + \langle A_1(e,0) w(0), w(0) \rangle \\ & \quad + 2 \int_0^t [\langle B(e) w''(\tau) + A_0(e,\tau) w'(\tau) + A_1(e,\tau) w(\tau), v(\tau) \rangle \, \mathrm{d}\tau \\ & \quad + \langle F(\tau) + G(e,\tau), w'(\tau) - v(\tau) \rangle] \, \mathrm{d}\tau \quad \text{for a.e. } t \in [0,T],\end{aligned}$$

which can be rewritten in the form

$$(2.42)\qquad \int_0^t \langle B(e)w''(\tau) + A_0(e,\tau)w'(\tau) + A_1(e,\tau)w(\tau) - F(\tau) - G(e,\tau), v(\tau) - w'(\tau)\rangle \,\mathrm{d}\tau \geqslant 0$$

$$\text{for all } v \in L^1(0,T;V),\ v(\tau) \in K \text{ a.e. on } [0,t].$$

Using Proposition 3 from [4] (appendix I) we obtain that

for a.e. $t \in [0,T]$:
$\langle B(e)w''(t) + A_0(e,t)w'(t) + A_1(e,t)w(t) - F(t)G(e,t), v - w'(t)\rangle \geqslant 0$
for all $v \in K$

and the inequality (2.1′) as well as (2.1) is proved.

b) *Uniqueness.* Let w_1, w_2 be two solutions of the problem (2.1)–(2.4). Inserting successively $w = w_1$, $v = w_2'$ and $w = w_2$, $v = w_1'$ in (2.42) and adding we obtain the inequality

$$(2.43)\qquad \int_0^t \langle B(e)(w_1 - w_2)''(\tau) + A_0(e,\tau)(w_1 - w_2)'(\tau) + A_1(e,\tau)(w_1 - w_2)(\tau), (w_1 - w_2)'(\tau)\rangle \,\mathrm{d}\tau \leqslant 0.$$

Let us denote $u = w_1 - w_2$. We have $u(0) = u'(0) = 0$. The inequalities (1.14), (2.43) then imply

$$(2.44)\qquad \langle B(e)u'(t), u'(t)\rangle + 2\int_0^t \langle A_0(e,\tau)u'(\tau), u'(\tau)\rangle \,\mathrm{d}\tau + \langle A_1(e,t)u(t), u(t)\rangle \leqslant \int_0^t \langle A_1'(e,\tau)u(\tau), u(\tau)\rangle \,\mathrm{d}\tau \leqslant 0.$$

The operators $B(e)\colon H^1(\Omega) \to \left(H^1(\Omega)\right)^*$ and $A_1(e,t)\colon V \to V^*$ are positive definite, the operator $A_0(e,\tau)\colon V \to V^*$ is nonnegative. The relations (2.44) then imply $u = w_1 - w_2 = 0$ on $[0,T]$ and the proof of the theorem is complete. □

3. Solving an optimal control problem

We assume a variable thickness $e \in E$ of the plate in the role of a control parameter. Considering the operator form (2.1′) of the variational inequality (2.1) we shall deal with the initial value state problem

$$\text{for a.e. } t \in [0,T]:$$

$$\langle B(e)w''(e,t) + A_0(e,t)w'(e,t) + A_1(e,t)w(e,t) - F(t) - G(e,t), v - w'(t)\rangle \geqslant 0 \quad \text{for all } v \in K, \tag{3.1}$$

$$w'(e,t) \in K \quad \text{for a.e. } t \in [0,T], \tag{3.2}$$

$$w(e,0) = w_0(e) \in V, \tag{3.3}$$

$$w'(e,0) = w_1 \in K. \tag{3.4}$$

We associate with (3.1)–(3.4) the minimum problem

$$j\big(w(\bar{e}),\bar{e}\big) = \min_{e\in U_{\rm ad}} j\big(w(e),e\big), \tag{3.5}$$

where $U_{\rm ad}$ is defined by

$$U_{\rm ad} = \Big\{e \in H^2(\Omega)\colon 0 < e_1 \leqslant e(x) \leqslant e_2 \text{ for all } x \in \Omega, \|e\| \leqslant C_1, \iint_\Omega e(x)\,\mathrm{d}x = C_2\Big\} \tag{3.6}$$

and the functional

$$j\colon \big[W^{1,2}(0,T;V) \cap W^{2,2}\big(0,T;H^1(\Omega)\big)\big] \times H^2(\Omega) \to \mathbb{R}$$

is weakly lower semicontinuous, i.e.

$$w_n \rightharpoonup w \quad \text{in } W^{1,2}(0,T;V) \text{ and in } W^{2,2}\big(0,T;H^1(\Omega)\big), \quad e_n \rightharpoonup e \quad \text{in } H^2(\Omega) \Rightarrow j\big(w(e),e\big) \leqslant \liminf_{n\to\infty} j\big(w(e_n),e_n\big). \tag{3.7}$$

The data of (3.1)–(3.4) fulfill the assumptions

$$\langle B(e)u,v\rangle = \langle B(e)v,u\rangle, \tag{3.8}$$

$$\langle B(e)u,u\rangle \geqslant \beta_0\|u\|_1^2, \quad \beta_1 > 0 \tag{3.9}$$

$$\text{for all } u,v \in H^1(\Omega),\ t \in [0,T],\ e \in U_{\rm ad},$$

$$(3.10) \qquad A_0(e,0)w_1 + A_1(e,0)w_0(e) \in H^1(\Omega)^*,$$

$$(3.11) \qquad A_r(e,.) \in C^2\big([0,T], L(V,V^*)\big),$$

$$(3.12) \qquad \langle A_r(e,t)u, v\rangle = \langle A_r(e,t)v, u\rangle,$$

$$(3.13) \qquad \langle A_0(e,t)u, u\rangle \geqslant 0,$$

$$(3.14) \qquad \langle A_1(e,t)u, u\rangle \geqslant \alpha_1 \|u\|^2, \quad \alpha_1 > 0,$$

$$(3.15) \qquad \langle A_0'(e,t) + A_1(e,t)]u, u\rangle \geqslant \alpha_2 \|u\|^2, \quad \alpha_2 > 0,$$

$$(3.16) \qquad \langle A_1'(e,t)u, u\rangle \leqslant 0$$

for all $u, v \in V,\ t \in [0,T],\ e \in U_{\text{ad}};\ r = 0, 1,$

$$(3.17) \quad e_n \rightharpoonup e \quad \text{in } H^2(\Omega) \Rightarrow \begin{cases} \text{i)} & A_r(e_n,.) \to A_r(e,.),\ r = 0,1 \\ & \text{in } C^1\big([0,T], L(V,V^*)\big) \\ \text{ii)} & B(e_n) \to B(e) \text{ in } L\big(H^1(\Omega), (H^1(\Omega))^*\big) \\ \text{iii)} & G(e_n,.) \to G(e,.) \text{ in } \big(H^1(\Omega)\big)^* \\ \text{iv)} & w_0(e_n) \to w_0(e) \text{ in } V. \end{cases}$$

The property (3.17) is a consequence of the compact imbedding $H^2(\Omega) \subset\subset C(\overline{\Omega})$, of the theorem on traces in the space $H^1(\Omega)$ and of the relation (2.5) determining the initial function $w_0(e)$.

Now, we formulate and verify the existence theorem for the Optimal Control Problem (3.1)–(3.4).

Theorem 3.1. *There exists at least one solution of the Optimal Control Problem* (3.1)–(3.5).

P r o o f. Due to Theorem 2.2, for every $e \in U_{\text{ad}}$ there exists a unique solution $w(e) \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}\big(0,T;H^1(\Omega)\big)$ of the state initial value problem (3.1)–(3.4). We can define a functional

$$J\colon U_{\text{ad}} \to \mathbb{R}, \quad J(e) = j\big(e, u(e)\big).$$

Let $\{e_n\} \in U_{\text{ad}}$ be a minimizing sequence for J:

$$(3.18) \qquad \lim_{n\to\infty} J(e_n) = \inf_{e \in U_{\text{ad}}} J(e),$$

where we put $\inf\limits_{e\in U_{\text{ad}}} J(e) = -\infty$ if the set $\{J(e)\}$ is not lower bounded.

Since the set U_{ad} is bounded, convex and closed in the space $H^2(\Omega)$, there exist an element $\overline{e} \in U_{\text{ad}}$ and a subsequence of $\{e_n\}$ (denoted again by $\{e_n\}$) such that

$$(3.19) \qquad e_n \rightharpoonup e \quad \text{in } H^2(\Omega).$$

Denoting $w(e_n) = w_n$ we rewrite the state problem (3.1)–(3.4) for $e \equiv e_n$ in the following form:

$$\text{for a.e. } t \in [0,T]:$$

$$\langle B(e_n)w_n''(t) + A_0(e_n,t)w_n'(t) + A_1(e_n,t)w_n(t) - F(t) - G(e_n,t), v - w_n'(t)\rangle \geqslant 0 \quad \text{for all } v \in K, \tag{3.20}$$

$$w_n'(t) \in K \quad \text{for a.e. } t \in [0,T], \tag{3.21}$$

$$w_n(0) = w_0(e_n) \in V, \tag{3.22}$$

$$w_n'(0) = w_1 \in K. \tag{3.23}$$

Using the estimates (2.28), (2.29) we obtain an a priori estimates for w_n and w_n':

$$\|w_n\|_{W^{1,\infty}(0,T;V)} + \|w_n''\|_{L^\infty(0,T;H^1(\Omega))} \leqslant M_3, \quad n = 1,2,\ldots. \tag{3.24}$$

Then there exists a function $\overline{w} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H^1(\Omega))$ and a subsequence of $\{w_n\}$ (denoted again by $\{w_n\}$) such that

$$w_n \rightharpoonup \overline{w} \quad \text{in } W^{1,2}(0,T;V) \text{ and in } W^{2,2}(0,T;H^1(\Omega)), \tag{3.25}$$

$$w_n \overset{*}{\rightharpoonup} \overline{w}, w_n' \overset{*}{\rightharpoonup} \overline{w}' \quad \text{in } L^\infty(0,T;V), \tag{3.26}$$

$$w_n'' \overset{*}{\rightharpoonup} \overline{w}'' \quad \text{in } L^\infty(0,T;H^1(\Omega)), \tag{3.27}$$

$$w_n(t) \rightharpoonup \overline{w}(t) \quad \text{in } V \text{ for all } t \in [0,T], \tag{3.28}$$

$$w_n'(t) \rightharpoonup \overline{w}'(t) \quad \text{in } V \text{ for a.e. } t \in [0,T], \tag{3.29}$$

$$w_n'(t) \rightharpoonup \overline{w}'(t) \quad \text{in } H^1(\Omega) \text{ for all } t \in [0,T]. \tag{3.30}$$

The relations (3.21), (3.29) imply

$$\overline{w}'(t) \in K \quad \text{for a.e. } t \in [0,T]. \tag{3.31}$$

Further, from (3.17 iv), (3.20), (3.21), (3.25)–(3.30) we obtain the initial conditions

$$\overline{w}(0) = w_0(e), \quad \overline{w}'(0) = w_1, \tag{3.32}$$

Let $v \in L^1(0,T;V)$ be an arbitrary function such that

$$v(t) \in K \quad \text{for a.e. } t \in [0,T].$$

The inequality (3.20) implies

$$\int_0^t \langle B(e_n)w_n''(\tau) + A_0(e_n,\tau)w_n'(\tau) + A_1(e_n,\tau)w_n(\tau) - F(\tau) - G(e_n,\tau), V(\tau) - w_n'(\tau)\rangle \,\mathrm{d}\tau \geqslant 0, \quad n = 1,2,\ldots. \tag{3.33}$$

The inequality (3.33) can be expressed in the form

$$
\begin{aligned}
&\langle B(e_n)w_n'(t), w_n'(t)\rangle + \langle A_1(e_n,t)w_n(t), w_n(t)\rangle \\
&\qquad + \int_0^t [2\langle A_0(e_n,\tau)w_n'(\tau), w_n'(\tau)\rangle - \langle A_1'(e_n,\tau)w_n(\tau), w_n(\tau)\rangle]\,\mathrm{d}\tau \\
&\quad \leqslant \langle B(e_n)w_n'(0), w_n'(0)\rangle + \langle A_1(e_n,0)w_n(0), w_n(0)\rangle \\
&\qquad + 2\int_0^t \langle B(e_n)w_n''(\tau) + A_0(e_n,\tau)w_n'(\tau) + A_1(e_n,\tau)w_n(\tau), v(\tau)\rangle\,\mathrm{d}\tau \\
&\qquad + 2\int_0^t \langle F(\tau) + G(e_n,\tau), w_n'(\tau) - v(\tau)\rangle\,\mathrm{d}\tau
\end{aligned}
$$

and, further,

$$
\begin{aligned}
(3.34)\quad &\langle B(\bar{e})w_n'(t), w_n'(t)\rangle + \langle A_1(\bar{e},t)w_n(t), w_n(t)\rangle \\
&\qquad + \int_0^t \left[2\langle A_0(\bar{e},\tau)w_n'(\tau), w_n'(\tau)\rangle - \langle A_1'(\bar{e},\tau)w_n(\tau), w_n(\tau)\rangle\right]\mathrm{d}\tau \\
&\quad \leqslant \langle [B(\bar{e}) - B(e_n)]w_n'(t), w_n'(t)\rangle + \langle [A_1(\bar{e},t) - A_1(e_n,t)]w_n(t), w_n(t)\rangle \\
&\qquad + 2\int_0^t [\langle A_0(\bar{e},\tau) - A_0(e_n,\tau)]w_n'(\tau), w_n'(\tau)\rangle\,\mathrm{d}\tau \\
&\qquad - \int_0^t [\langle A_1'(\bar{e},\tau) - A_1'(e_n,\tau)]w_n(\tau)w_n(\tau)\rangle\,\mathrm{d}\tau \\
&\qquad + \langle B(e_n)w_1, w_1\rangle + \langle A_1(e_n,0)w_0(e_n), w_0(e_n)\rangle \\
&\qquad + 2\int_0^t \langle B(e_n)w_n''(\tau) + A_0(e_n,\tau)w_n'(\tau) + A_1(e_n,\tau)w_n(\tau), v(\tau)\rangle\,\mathrm{d}\tau \\
&\qquad + 2\int_0^t \langle F(\tau) + G(e_n,\tau), w_n'(\tau) - v(\tau)\rangle\,\mathrm{d}\tau, \quad n = 1, 2, \ldots.
\end{aligned}
$$

The functionals $v \to \langle B(\bar{e})v, v\rangle$, $v \to \langle A_1(\bar{e},t)v, v\rangle$

$$
w(.) \to \int_0^t [2\langle A_0(\bar{e},\tau)w'(\tau), w'(\tau)\rangle - \langle A_1'(\bar{e},\tau)w(\tau), w(\tau)\rangle]\,\mathrm{d}\tau
$$

on the left-hand side of the last inequality are weakly lower semicontinuous on the spaces $H^1(\Omega)$, V, $W^{1,2}(0,T;V)$ due to the assumptions (3.8), (3.9), (3.12), (3.14), (3.15), and we obtain applying (3.25), (3.28), (3.30) the inequality

$$\begin{aligned}
&\langle B(\overline{e})\overline{w}'(t), \overline{w}'(t)\rangle + \langle A_1(\overline{e},t)\overline{w}(t), \overline{w}(t)\rangle \\
&\qquad + \int_0^2 [2\langle A_0(\overline{e},\tau)\overline{w}'(\tau), \overline{w}'(\tau)\rangle - \langle A_1'(\overline{e},\tau)\overline{w}(\tau)\rangle]\,\mathrm{d}\tau \\
&\quad \leqslant \liminf_{n\to\infty} \Big\{\langle B(\overline{e})\overline{w}_n'(t), \overline{w}_n'(t)\rangle + \langle A_1(\overline{e},t)\overline{w}_n(t), \overline{w}_n(t)\rangle \\
&\qquad + \int_0^t [2\langle A_0(\overline{e},\tau)\overline{w}_n'(\tau), \overline{w}_n'(\tau)\rangle - \langle A_1'(\overline{e},\tau)\overline{w}_n(\tau), \overline{w}_n(\tau)\rangle]\,\mathrm{d}\tau\Big\}.
\end{aligned}$$

Applying the assumptions (3.17) and the relations (3.25)–(3.30), (3.32) to the right-hand side of (3.34) we arrive at the inequality

$$\begin{aligned}
&\langle B(\overline{e})\overline{w}'(t), \overline{w}'(t)\rangle + \langle A_1(\overline{e},t)\overline{w}(t), \overline{w}(t)\rangle \\
&\qquad + \int_0^t [2\langle A_0(\overline{e},\tau)\overline{w}'(\tau), \overline{w}'(\tau)\rangle - \langle A_1'(\overline{e},\tau)\overline{w}(\tau), \overline{w}(\tau)\rangle]\,\mathrm{d}\tau \\
&\quad \leqslant \langle B(\overline{e})\overline{w}'(0), \overline{w}'(0)\rangle + \langle A_1(\overline{e},0)\overline{w}(0), \overline{w}(0)\rangle \\
&\qquad + 2\int_0^t \langle B(\overline{e})\overline{w}''(\tau) + A_0(\overline{e},\tau)\overline{w}'(\tau) + A_1(\overline{e},\tau)\overline{w}(\tau), v(\tau)\rangle\,\mathrm{d}\tau \\
&\qquad + 2\int_0^t \langle F(\tau) + G(\overline{e},\tau), \overline{w}'(\tau) - v(\tau)\rangle\,\mathrm{d}\tau.
\end{aligned}$$

Using the initial conditions (3.32), the symmetry of operators $B(e)$, $A_0(\overline{e},t)$, $A_1(\overline{e},t)$ we arrive at the inequality

$$\begin{aligned}
(3.35)\qquad &\int_0^t \langle B(\overline{e})\overline{w}''(\tau) + A_0(\overline{e},\tau)\overline{w}'(\tau) + A_1(\overline{e},\tau)\overline{w}(\tau) \\
&- F(\tau) - G(\overline{e},\tau), v(\tau) - \overline{w}'(\tau)\rangle\,\mathrm{d}\tau \geqslant 0 \quad \text{for a.e. } t \in [0,T],
\end{aligned}$$

which implies due to [4] (Proposition 3, App. I) the inequality

$$\begin{aligned}
(3.36)\qquad &\langle B(\overline{e})\overline{w}''(t) + A_0(\overline{e},t)\overline{w}'(t) + A_1(\overline{e},t)\overline{w}(t) \\
&- F(t) - G(\overline{e},t), v(t) - \overline{w}'(t)\rangle \geqslant 0 \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

Hence the function $\overline{w}\colon [0,T] \to V$ solves the initial value problem (2.1)–(2.4) for $e \equiv \overline{e}$. The uniqueness of a solution of (2.1)–(2.4) then implies the relations

$$\overline{w}(t) = w(\overline{e}, t), \tag{3.37}$$

$$w(e_n) \rightharpoonup w(\overline{e}) \quad \text{in } W^{1,2}(0,T;V) \text{ and in } W^{2,2}\big(0,T;H^1(\Omega)\big). \tag{3.38}$$

The assumption (3.7) and the relation (3.18) imply

$$j\big(w(\overline{e}),\overline{e}\big) \leqslant \liminf_{n\to\infty} j\big(w(e_n),e_n\big) = \inf_{n\to\infty} j\big(w(e),e\big)$$

and the relation (3.5) follows, which completes the proof. □

Let X be any Hilbert space $z_d \in X$; let $\Phi\colon H^2(\Omega) \to \mathbb{R}$ be a weakly lower semi-continuous functional. Cost functionals can have the form

$$j_1\big(e,w(e)\big) = \|\mathcal{D}w(e) - z_d\|_X^2 + \Phi(e),$$

where $\mathcal{D}\colon W^{1,2}(0,T;V) \cap W^{2,2}\big(0,T;H^1(\Omega)\big) \to X$ is the linear bounded operator; or

$$j_2\big(e,w(e)\big) = \|Dw(e,T) - z_d\|_X^2 + \Phi(e), \quad D \in L(V,X),$$
$$j_3\big(e,w(e)\big) = \|D_1 w'(e,T)\|_X^2, \quad D_1 \in L(H,X).$$

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