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Pavel Stavinoha

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ON LIMITS OF L_p-NORMS OF AN INTEGRAL OPERATOR

PAVEL STAVINOHA, Praha

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Summary. A recurrence relation for the computation of the L_p-norms of an Hermitian Fredholm integral operator is derived and an expression giving approximately the number of eigenvalues which in absolute value are equal to the spectral radius is determined. Using the L_p-norms for the approximation of the spectral radius of this operator an a priori and an a posteriori bound for the error are obtained. Some properties of the a posteriori bound are discussed.

Keywords: L_p-norms of an integral operator, Hermitian Fredholm integral operator

AMS classification: 47A53, 47B15, 47A30, 47A10

1. INTRODUCTION

In the paper [9] we have proved that under certain conditions the L_p-norms of a linear operator converge to its spectral norm and the speed of the convergence was also estimated. The results were derived in a general form made possible by the theory of non-commutative integration (the fundaments of this theory are e.g. in [7], [3] and [6]). In [8] these general results were applied to a finite-dimensional and non-commutative case which represents the matrix algebra.

In this paper we will apply the results from [9] to a certain infinite dimensional non-commutative case. For the Fredholm integral operator and integral operators with weak singularities we thus get the well-known computational procedure for the determination of the spectral radius (see e.g. [5]), and the a priori error estimate for this method. Similarly as in [8] we obtain a recurrent algorithm for the calculation of the L_p-norm of an operator and an arithmetic expression converging to the number of eigenvalues equal to the absolute value of the spectral radius. We will then determine an a posteriori error estimate for the method mentioned and prove a number of its properties by which its quality is proved. Most of the results we obtain for the given Hermitian linear integral operators will be analogous to the results reached in [8] for matrices. This is quite natural, because from the point of view of the non-commutative integration theory, which was used for reaching general results in [9], matrices and the above mentioned integral operators differ very little.

The paper is organized as follows. In Part 2 some notions used in the sequel and the necessary notation will be explained. In Part 3 we will explain some concepts from the non-commutative integration theory on our particular example to which the general results from [9] can be applied and reformulated. In Part 4 the recurrent formulae for calculating the L_p-norms as well as an expression which converges to the number of eigenvalues which are equal to the absolute value of the spectral radius will be derived. In Part 5 we will investigate how quickly the L_p-norms of an integral operator converge, derive an a posteriori error estimate and show its properties. Part 6 contains a numerical illustration of the results just stated.

2. Terminology and notation

Let (a,b) be a finite or infinite interval of the real axis. By the symbol $L_2(a,b)$ or L_2, if there is no danger of misinterpretation, we will denote the complex Hilbert space of square integrable functions on (a,b). $B\big(L_2(a,b)\big)$ or briefly $B(L_2)$ will denote the space of all linear bounded operators defined on the whole $L_2(a,b)$. For $A \in B(L_2)$ the range of A will be denoted as $R(A)$, the point spectrum of A as $P_\sigma(A)$ and the spectral radius of the operator A as $r(A)$. As usual, A^* will denote the adjoint operator. If $A = A^*$ we will call A an Hermitian operator. The symbol $|A|$ will denote the operator $(A^*A)^{\frac{1}{2}}$. The concept of a projection will be used only for such an operator $P \in B(L_2)$ for which $P^2 = P$ and $P^* = P$ hold. If $A \in B(L_2)$ then $\|A\|_\infty$ will denote $\sup\limits_{\|x\|=1} \|Ax\|$. The symbol I will denote the identity operator.

The integral operator K on $L_2(a,b)$ which is defined by the rule $Kf = \int_a^b K(s,t)f(t)\,\mathrm{d}t$ shall be called the Fredholm integral operator if its kernel $K(s,t)$ fulfils the condition $K(s,t) \in L_2\big((a,b)\times(a,b)\big)$. If (a,b) is a finite interval and $K(s,t) = A(s,t)/|s-t|^\alpha$, where $A(s,t)$ is a bounded measurable function on $(a,b)\times(a,b)$, $\frac{1}{2} \leqslant \alpha < 1$, we will call K an integral operator with weak singularity. $K_n(s,t)$ will then denote the iterated kernels of the operator K.

3. The gage space $\Gamma = (L_2, B(L_2), \mathrm{tr})$

If $P \in B(L_2)$ is a projection, let us define $\mathrm{tr}(P) = \text{dimension}\,(R(P))$. It is well-known (see [6], Example 1.2) that $\Gamma = \left(L_2(a,b), B(L_2(a,b)), \mathrm{tr}\right)$ is a regular gage space. (The meaning of the terms from non-commutative integration theory used here can be found e.g. in [7], [3], [6], [9] or [1].) It can be proved easily that the system of measurable operators $\Lambda(\Gamma)$ in this case coincides with $B(L_2)$ and that the convergence almost everywhere is equivalent to the convergence in the norm $\| \; \|_\infty$. Then elementary operators are all operators from $B(L_2)$ the range of which is of finite dimension. It is easy to show that $T \in L_p(\Gamma)$ if and only if $T \in B(L_2)$, T is a compact operator and $\sum_{i=1}^{\infty} \lambda_i^p < \infty$, where $\{\lambda_i\}$ $(i = 1, 2, \ldots)$ is the sequence of eigenvalues of the operator $|T|$. (We always suppose that all eigenvalues are considered the number of times equal to their multiplicity.) If $A \in L_1(\Gamma)$ and $\{\lambda_i\}$ $(i = 1, 2, \ldots)$ is the sequence of eigenvalues of the operator A, then $\sum_{i=1}^{\infty} |\lambda_i| < \infty$ and $\mathrm{tr}(A) = \sum_{i=1}^{\infty} \lambda_i$. The spaces $L_p(\Gamma)$ therefore coincide with the well-known spaces c_p in [4]. If $1 \leqslant p \leqslant q \leqslant \infty$ and $T \in L_p(\Gamma)$ then $L_p(\Gamma) \subset L_q(\Gamma)$ and $\|T\|_q \leqslant \|T\|_p$ hold. It is easy to show that if K is an Hermitian Fredholm integral operator then $K \in L_2(\Gamma)$ and also for $n = 2, 3, \ldots$, $K^n \in L_1(\Gamma)$ and $\mathrm{tr}(K^n) = \int_a^b K_n(x,x)\,\mathrm{d}x$. Similarly it can be shown that if K is an Hermitian integral operator with weak singularity and $n > (3-\alpha)/(1-\alpha)$, n integer, then $K \in L_n(\Gamma)$, $K^n \in L_1(\Gamma)$ and $\mathrm{tr}(K^n) = \int_a^b K_n(x,x)\,\mathrm{d}x$. Now from [9, Corollary 3.2] the following theorem follows:

Theorem 3.1. *Let K be a Fredholm integral operator or an integral operator with weak singularity, $K = K^*$. Then*

$$\lim_{m\to\infty} \left(\int_a^b K_{2^m}(x,x)\,\mathrm{d}x \right)^{2^{-m}} = r(K).$$

The possibility of approximately calculating $r(K)$ of the Hermitian integral operator K in the way Theorem 3.1 indicates is well-known from [5, page 246] where a similar method of calculation is called the trace method by the authors. From the point of view of the calculation procedure using Theorem 3.1 it is evident that the difference between a Fredholm integral operator and an integral operator with weak singularity is inessential. We will therefore limit all further considerations to Fredholm integral operators.

Theorem 3.2. *Let K be a Fredholm integral operator, $K = K^*$ and $K \neq O$. Then for $m = 1, 2, \ldots$ we have*

$$\left| \left(\int_a^b K_{2^m}(x,x)\,\mathrm{d}x \right)^{2^{-m}} - r(K) \right| \leqslant 2^{-m} \Big\{ \|K_4\| \left(\|K\|_2^4 / \|K\|_4^4 \right)^{2^{-m}} \cdot \ln \left(\|K\|_2^2 / \|K\|_4^4 \right) \Big\},$$

where

$$\|K\|_2^2 = \int_a^b \int_a^b |K(s,t)|^2 \,\mathrm{d}s\,\mathrm{d}t$$

and

$$\|K\|_4^4 = \int_a^b \int_a^b |K_2(s,t)|^2 \,\mathrm{d}s\,\mathrm{d}t\,.$$

Proof. First let us verify the last equality:

$$\|K\|_2^2 = \mathrm{tr}(K^2) = \int_a^b \int_a^b K(s,t)K(t,s)\,\mathrm{d}t\,\mathrm{d}s = \int_a^b \int_a^b |K(s,t)|^2 \,\mathrm{d}s\,\mathrm{d}t\,.$$

Analogously the relation for $\|K\|_4^4$ can be verified. Then, according to [3, Corollary 1.1] $\|K\|_4^4 = \|K^2 \cdot K^2\|_1 \leqslant \|K\|_\infty^2 \cdot \|K^2\|_1 = \|K\|_\infty^2 \cdot \|K\|_2^2$ holds and so $1/\|K\|_\infty^2 \leqslant \|K\|_2^2/\|K\|_4^4$. Let $\{\lambda_i\}$ $(i = 1, 2, \ldots)$ be the sequence of eigenvalues of the operator K and let it be ordered so that $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_t| > |\lambda_{t+1}| \geqslant \ldots \geqslant 0$. Let us denote $\alpha_i = \left|\frac{\lambda_i}{\lambda_1}\right|$ $(i = 1, 2, \ldots)$, then

$$\begin{aligned}
\|K\|_2^4/\|K\|_4^4 &= \left(\sum_{i=1}^{\infty} \lambda_i^2 \right)^2 \Big/ \sum_{i=1}^{\infty} \lambda_i^4 \\
&= \left\{ t^2 + 2t \sum_{i=t+1}^{\infty} \alpha_i^2 + \left(\sum_{i=t+1}^{\infty} \alpha_i^2 \right)^2 \right\} \Big/ \left(t + \sum_{i=t+1}^{\infty} \alpha_i^4 \right) \geqslant t.
\end{aligned}$$

If we denote by S the projection onto the eigenspace corresponding to the eigenvalue $|\lambda_1| = \|K\|_\infty$ of the operator $|K|$, then $\mathrm{tr}(S) = t$. From this result we obtain $\|K\|_2^4/\|K\|_4^4 \geqslant \mathrm{tr}(S) \geqslant 1$. Let $R = \|K\|_2^2/\|K\|_\infty^2$, then from the relations proved earlier we obtain $R \leqslant \|K\|_2^4/\|K\|_4^4$. Now [9, Corollary 3.5], where we substitute $q = 2$, implies the desired inequality. □

4. Computation of the L_p-norm of an integral operator

Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. To calculate $\mathrm{tr}(A^{2^k})$ $(k = 1, 2, \ldots)$ it is necessary to determine a sequence of operators A^{2^k} $(k = 1, 2, \ldots)$. In the case that $\|A^{2^k}\|_\infty \to \infty$ for $k \to \infty$, calculating the kernel $A_{2^k}(s,t)$ of the operators A^{2^k} $(k = 1, 2, \ldots)$ could become impracticable for quickly increasing coefficients. In order to avoid this phenomen, instead of the sequence of operators A^{2^k} $(k = 1, 2, \ldots)$ we shall determine a sequence of operators B_m $(m = 1, 2, \ldots)$, every element of which is some multiple of a certain element in the sequence A^{2^k} $(k = 1, 2, \ldots)$, and we will show that $\|B_m\|_\infty$ $(m = 1, 2, \ldots)$ is then a bounded sequence. Let us define the sequence of operators B_m $(m = 1, 2, \ldots)$ by the relations

$$B_1 = A, \tag{1}$$

$$B_{2k} = B_{2k-1}^2, \tag{2}$$

$$B_{2k+1} = B_{2k}/c_k, \tag{3}$$

where $k = 1, 2, \ldots$ and $c_k = \mathrm{tr}(B_{2k})$.

By similar considerations as in [8] we could show that this choice of c_k is useful because the relations between $\mathrm{tr}(A^{2^k})$ and $\mathrm{tr}(B_{2k})$ become simpler. From the relation (1), (2) and (3) further relations follow:

$$B_2 = A^2, \tag{4}$$

$$B_{2k} = A^{2^k}/c_1^{2^{k-1}} \cdot c_2^{2^{k-2}} \ldots c_{k-1}^2, \quad k = 2, 3, \ldots, \text{ and} \tag{5}$$

$$B_{2k+1} = A^{2^k}/c_1^{2^{k-1}} \cdot c_2^{2^{k-2}} \ldots c_k^2, \quad k = 1, 2, \ldots. \tag{6}$$

Theorem 4.1. *Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. Let B_m $(m = 1, 2, \ldots)$ be the sequence of operators defined by the relations* (1), (2) *and* (3), *where*

$$c_k = \mathrm{tr}(B_{2k}). \tag{9}$$

Then

(1) $c_k \neq 0$ *for* $k = 1, 2, \ldots$.

(2) $\|A\|_{2^k}$ *for* $k = 1, 2, \ldots$ *can be calculated according to the recurrent relation*

$$d_{k+1} = \big(\mathrm{tr}(B_{2k})\big)^{2^{-k}} \cdot d_k, \text{ where} \tag{10}$$
$$d_1 = 1,\ d_{k+1} = \|A\|_{2^k} \quad (k = 1, 2, \ldots).$$

(3) *There exists a constant M for which*

$$\|B_m\|_\infty \leqslant M \quad (m = 1, 2, \ldots).$$

Proof. The fact that c_k $(k = 1, 2, \ldots)$ are non-zero and the sequence of the norms $\|B_m\|_\infty$ is bounded can be proved completely analogously to the proof of Theorem 4.1 from [8]. The relations (10) and (11) can be easily derived from the relations (5) and (9). □

Theorem 4.2. *Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. Let the number of eigenvalues of the operator A, the absolute value of which is equal to $\mathrm{r}(A)$, be t. Then*

(1) $t \leqslant 1/\operatorname{tr}(B_{2k})$ for $k = 2, 3, \ldots$.

(2) $\lim 1/\operatorname{tr}(B_{2k}) = t$.

(3) The sequence $1/\operatorname{tr}(B_{2k})$, $k = 2, 3, \ldots$ is non-increasing.

Proof. Can be done similarly to the proof of Theorem 4.2 from [8].

Let the assumptions of Theorem 4.1 hold and let $B_m(s,t)$ denote the kernels of a Hermitian Fredholm operator B_m. For the actual calculation of $\operatorname{tr}(B_{2k})$ $(k = 1, 2, \ldots)$ it is suitable to use the following expression:

$$\begin{aligned}
\operatorname{tr}(B_{2k}) &= \int_a^b B_{2k}(s,s)\,\mathrm{d}s = \int_a^b\int_a^b \left|B_{2k-1}(s,t)\right|^2 \mathrm{d}t\,\mathrm{d}s = \\
&= \int_a^b \Big(\int_a^t \left|B_{2k-1}(s,t)\right|^2 \mathrm{d}s \Big)\,\mathrm{d}t + \int_a^b \Big(\int_t^b \left|B_{2k-1}(t,s)\right|^2 \mathrm{d}s \Big)\,\mathrm{d}t = \\
&= 2\int_a^b \Big(\int_a^t \left|B_{2k-1}(s,t)\right|^2 \mathrm{d}s \Big)\,\mathrm{d}t .
\end{aligned}$$

□

5. Estimation of the rate of convergence

Theorem 5.1. *Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. Then*

$$\Big| \|A\|_{2^k} - \mathrm{r}(A) \Big| \leqslant 2^{-k} \Big\{ \|A\|_{2^k} \cdot \ln\big(1/\operatorname{tr}(B_{2k})\big) \Big\}$$

holds for $k = 2, 3, \ldots$.

Proof. Can be done by a method similar to that used for Theorem 5.1 of [8].

We will also omit the proofs of the following two theorems because they are completely analogous to the proofs of Theorems 5.2 and 5.3 from [8]. □

Theorem 5.2. *Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. Let $\{\lambda_i\}$ $(i = 1, 2, \ldots)$ be the eigenvalues of the operator A and let $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \ldots \geqslant 0$. Then*

(1) $\left| \|A\|_{2^k} - r(A) \right| = O\left(\frac{1}{2^k} \left|\frac{\lambda_2}{\lambda_1}\right|^{2^k}\right)$,

(2) $2^{-k}\left\{ \|A\|_{2^k} \cdot \ln\left(1/\operatorname{tr}(B_{2k})\right) \right\} = O\left(\frac{1}{2^{k-1}} \left|\frac{\lambda_2}{\lambda_1}\right|^{2^{k-1}}\right)$.

Theorem 5.3. *Let A denote a Fredholm integral operator, $A = A^*$ and $A \neq O$. Let $\{\lambda_i\}$ $(i = 1, 2, \dots)$ be the eigenvalues of the operator A and let $|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant \dots \geqslant 0$. Let us denote*

$$E(A, k) = \frac{1}{2^k}\left\{ \|A\|_{2^k} \cdot \ln\left(1/\operatorname{tr}(B_{2k})\right) \right\}.$$

Then

(1) $\|A\|_{2^k} - r(A) \leqslant E(A, k) \leqslant \|A\|_{2^{k-1}} - r(A)$.

(2) *If* $|\lambda_1| > |\lambda_2| > 0$ *then*

$$\lim_{k \to \infty} \frac{E(A, k)}{\|A\|_{2^{k-1}} - r(A)} = 1.$$

(3) *If* $|\lambda_1| = |\lambda_2|$ *then*

$$\lim_{k \to \infty} \frac{E(A, k)}{\|A\|_{2^k} - r(A)} = 1.$$

6. Numerical illustration

In the following examples we will always first present the kernel $A(s, t)$ inducing an Hermitian Fredholm integral operator, an interval (a, b) which determines the range of the operator A and the two eigenvalues λ_1, λ_2 of the operator A which have the largest absolute value. The calculations were done using a program written in REDUCE 2 programming language on an EC 1040 computer. In Example 6.1 a well-known elementary formula was used to find the primitive function to the power function. In Example 6.2 the Cambridge university analytic integration program (see [2]) was used to determine a primitive function.

Example 6.1. Let A be an Hermitian Fredholm integral operator induced by the kernel $A(s, t) = \min(s, t)$ and let $(a, b) = (0, 1)$. Then according to [5], $\lambda_1 \doteq$

0.40528 47346 and $\lambda_2 \doteq 0.04503\,16372$.

Table 1

k	$\|A\|_{2^k}$	$1/\operatorname{tr}(B_{2k})$	computing time in s
1	0.40824 82905		8.56
2	0.40530 04566	1.02941	10.90
3	0.40528 47357	1.00031	24.08

Table 2

k	approximation error r(A) using $\|A\|_{2^k}$	a posteriori estimate of the error
1	0.30×10^{-2}	
2	0.16×10^{-4}	0.29×10^{-2}
3	0.12×10^{-8}	0.16×10^{-4}

Example 6.2. Let A be an Hermitian Fredholm integral operator induced by the kernel $A(s,t)$ where $A(s,t) = -\sqrt{st} \cdot \ln t$ for $s \leqslant t$ and $A(s,t) = -\sqrt{st} \cdot \ln s$ for $s \geqslant t$. Let $(a,b) = (0,1)$. According to [5], $\lambda_1 \doteq 0.1729\,1507$ and $\lambda_2 \doteq 0.0328\,1781$.

Table 3

k	$\|A\|_{2^k}$	$1/\operatorname{tr}(B_{2k})$	computing time in s
1	0.1767 7670		10.76
2	0.1729 7282	1.0909	21.52
3	0.1729 1511	1.0027	70.98

Table 4

k	approximation error r(A) using $\|A\|_{2^k}$	a posteriori estimate of the error
1	0.39×10^{-2}	
2	0.58×10^{-4}	0.38×10^{-2}
3	0.36×10^{-7}	0.58×10^{-4}

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Author's address: *Pavel Stavinoha*, Stavební fakulta ČVUT, katedra matematiky a deskriptivní geometrie, Thákurova 7, 166 29 Praha.