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ON NUMERICAL SOLUTION TO THE PROBLEM OF REACTOR KINETICS WITH DELAYED NEUTRONS BY MONTE CARLO METHOD

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Summary. In this paper, the linear problem of reactor kinetics with delayed neutrons is studied whose formulation is based on the integral transport equation. Besides the proof of existence and uniqueness of the solution, a special random process and random variables for numerical elaboration of the problem by Monte Carlo method are presented. It is proved that these variables give an unbiased estimate of the solution and that their expectations and variances are finite.

Keywords: Reactor kinetics, integral transport equation

INTRODUCTION

The problem can be formulated as follows: Find functions $\varphi(x, E, \omega, t)\colon E_3 \times (0,\infty) \times \Omega \times [0,\tau) \to [0,\infty)$ and $N_i(x,t)\colon E_3 \times [0,\tau) \to [0,\infty)$, $i = 1, 2, \ldots, n$ so that the equations

$$
\begin{aligned}
\text{(1a)}\quad \varphi(x, E, \omega, t) = {} & \int_0^t \mathrm{d}r \exp\Big(- \int_0^r \mathrm{d}s\, \sqrt{2E}\Sigma\big(x(u), E, \omega, u\big)\Big) \\
& \Big\{ F\big(x(r), E, \omega, r\big) + \frac{\sqrt{2E}}{4\pi} \sum_{i=1}^{n} \lambda_i \chi_i(E) N_i\big(x(r), r\big) \\
& + \sqrt{2E} \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega'\, \varphi\big(x(r), E', \omega', r\big) \Big[\Sigma_s\big(x(r), E' \to E, \omega' \to \omega, r\big) \\
& + \frac{1-\beta}{4\pi} \chi(E) \nu(E') \Sigma_f\big(x(r), E', r\big)\Big]\Big\} \\
& + \varphi_0\big(x(0), E, \omega\big) \exp\Big(- \int_0^t \mathrm{d}s\, \sqrt{2E}\Sigma\big(x(s), E, \omega, s\big)\Big)
\end{aligned}
$$

and

$$\text{(1b)}\ N_i(x,t) = \int_0^t \mathrm{d}s \exp\left(\lambda_i(s-t)\right)\beta_i \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\omega'\, \nu(E')\varphi(x,E',\omega',s)\Sigma_f(x,E',s) + N_{0i}(x)\exp(-\lambda_i t), \qquad i = 1,2,\ldots,n$$

are satisfied.

Here Ω denotes the surface of the unit sphere, τ is a positive number and $x(s) \equiv x - \sqrt{2E}\omega(t-s)$, $s \in [0,t]$. The other symbols have the following interpretation:

$x \in E_3$	...	vector of location,
$E \in (0,\infty)$	...	kinetic energy,
$\omega \in \Omega$	...	direction of velocity,
$t \in [0,\tau)$	...	time,
φ	...	differential neutron flux,
φ_0	...	initial value of the flux,
F	...	external source of neutrons,
Σ	...	total macroscopic effective cross-section,
Σ_s	...	differential macroscopic scattering effective cross-section,
Σ_f	...	fission macroscopic effective cross-section,
$\nu(E)$	...	the number of neutrons produced as a result of a fission by a neutron of the energy E (prompt and delayed),
$\chi(E)$	...	energy spectrum of prompt fission neutrons,
n	...	the number of different emitters,
N_i	...	concentration of the emitter i of delayed neutrons,
N_{0i}	...	initial value of concentration of the emitter,
λ_i	...	decay constant,
β_i	...	fraction of delayed neutrons,
$\chi_i(E)$	...	energy spectrum of delayed neutrons,
β	...	total fraction of delayed neutrons, $\beta = \sum_{i=1}^{n} \beta_i < 1$.

Usually, the problem of reactor kinetics is described by a set of integro-differential equations, but, for many reasons, integral form (1) is advantageous. For instance, it admits more general behavior of independent physical quantities mentioned above.

Existence and uniqueness of the solution to problem (1) have been shown for less or more general cases [1–3]. As concerns numerical elaboration of equations (1), some difficulties occur. Let us recall the stiffness or the difficulties connected with multidimensionality of the problem. In this situation, the use of Monte Carlo method seems to be more convenient: There exist computer Monte Carlo codes by which in principle Eq. (1a) can be solved with respect to φ for N_i known (see [4]). Using these codes we can try to solve Eqs. (1) by iterations in the following way: In the first step, set $N_i(\mathbf{x},t) = N_{0i}(\mathbf{x})$, $i = 1, 2, \ldots, n$ and compute the function φ from Eq. (1a). Then express functions N_i from Eqs. (1b). In the second step, substitute the new functions N_i into Eq. (1a) and compute the new φ. Express $N_i(\mathbf{x},t)$ by Eqs. (1b) using φ just computed. Go to the next step ..., etc.

Assume that the process is convergent. In any step of computation, the functions φ and N_i are estimated by arithmetic mean values of a finite number of results obtained in mutually independent trials. In general, therefore, the method gives a biased estimation of the solution to problem (1).

In this paper we will study problem (1) with emphasis to its numerical solution by Monte Carlo method. The main goal is to show how to avoid the bias problem just mentioned. The plan is a follows: First of all, basic physical properties of the medium will be stated in the form of a generalizing assumption and a class of real functions will be chosen with respect to these properties. Next, a solution to problem (1) will be found in the form of a convergent series and its uniqueness will be proved (Theorem 1). Then a special Monte Carlo game will be constructed by means of which numerical solution to the problem can be estimated (Theorems 2 and 3).

BASIC RELATIONS

Assumption. a) *The differential effective cross-section Σ_s is either a real function,*

$$\Sigma_s\colon E_3 \times (0,\infty) \times (0,\infty) \times \Omega \times \Omega \times [0,\infty) \to [0,\infty)$$

or a combination of such a function with the Dirac δ-function of energy and angular variables.

b) *The effective cross-sections Σ and Σ_f are real functions, $\Sigma\colon E_3 \times (0,\infty) \times \Omega \times [0,\infty) \to [0,\infty)$, $\Sigma_f\colon E_3 \times (0,\infty) \times [0,\infty) \to [0,\infty)$, which satisfy the inequalities*

$$\Sigma_f(\mathbf{x},E,t) \leqslant \Sigma(\mathbf{x},E,\boldsymbol{\omega},t) \leqslant b + a/\sqrt{2E}$$

where a and b are finite constants. The integral scattering effective cross-section

$$\tilde{\Sigma}_s(\mathbf{x},E,\boldsymbol{\omega},t) \equiv \int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\boldsymbol{\omega}\, \Sigma_s(\mathbf{x},E \to E',\boldsymbol{\omega} \to \boldsymbol{\omega}',t)$$

is a real function, $\tilde{\Sigma}_s\colon E_3 \times (0,\infty) \times \Omega \times [0,\infty) \to [0,\infty)$. It satisfies the inequality

$$\tilde{\Sigma}_s(\mathbf{x},E,\boldsymbol{\omega},t) \leqslant \Sigma(\mathbf{x},E,\boldsymbol{\omega},t) - \Sigma_f(\mathbf{x},E,t).$$

c) *The quantity* $\nu(E)$ *is a real bounded function,* $\nu\colon (0,\infty) \to [0,\infty)$.

d) *There exist a function* $f(E)\colon (0,\infty) \to (0,\infty)$ *and a constant* $C \in (0,\infty)$ *such that the inequalities*

$$\sqrt{2E}\chi_i(E)/f(E) \leqslant C, \quad i = 1,2,\dots,n,$$
$$\sqrt{2E}\chi(E)/f(E) \leqslant C,$$
$$\int_0^\infty \mathrm{d}E\, f(E)\Sigma(\boldsymbol{x},E,\boldsymbol{\omega},t) \leqslant C$$

and

$$\sqrt{2E}\int_0^\infty \mathrm{d}E' \int_\Omega \mathrm{d}\boldsymbol{\omega}'\, \Sigma_s(\boldsymbol{x},E' \to E,\boldsymbol{\omega}' \to \boldsymbol{\omega},t)f(E')/f(E) \leqslant C$$

hold for any $\boldsymbol{x} \in E_3$, $\boldsymbol{\omega} \in \Omega$ *and* $t \in [0,\infty)$.

We assume in general that the macroscopic cross-sections are time dependent (this makes it possible, for example, to consider the movement of the reactor control rods). For any fixed values of the variables $\boldsymbol{x}$ and t assumptions a), b) and c) are in good agreement with the properties of the cross-section models known from literature (see [5], [6], II, §5 and IX, §7 or [7], IV, §1). For both these models and the fission spectra models ([7], V, §2), also validity of supposition d) can be verified (it is sufficient to set

$$f(E) = \sqrt{E}\exp\left(-\alpha\sqrt{E}\right)$$

where $\alpha > 0$ is a constant).

Definition. Let τ be a positive number and $f(E)$ the function from Assumption d). Denote $M_\tau \equiv \{1,2\} \times E_3 \times (0,\infty) \times \Omega \times [0,\tau)$. We say that a function $\Phi(i,\boldsymbol{x},E,\boldsymbol{\omega},t)\colon M_\tau \to E_1$ belongs to the linear space $m\{f,\tau\}$ if its norm $\|\Phi\|_\tau$,

$$\|\Phi\|_\tau \equiv \sup_{M_\tau}|\Phi/f|$$

is finite.

Theorem 1. *Let* $\varphi_0/f\colon E_3 \times (0,\infty) \times \Omega \to E_1$, $F/\big(f\Sigma\sqrt{2E}\big)\colon E_3 \times (0,\infty) \times \Omega \times [0,\infty) \to E_1$ *and* $N_{0i}\colon E_3 \to E_1$, $i = 1, 2, \dots, n$ *be bounded functions. Then problem* (1) *has a solution* $\varphi\colon E_3 \times (0,\infty) \times \Omega \times [0,\infty) \to E_1$ *and* $N_i\colon E_3 \times [0,\infty) \to E_1$ *such that the inequality*

$$\max_{i\leqslant n}\Big(\sup_{E_3\times(0,\infty)\times\Omega\times[0,\tau)}|\varphi/f|,\ \sup_{E_3\times[0,\tau)}|N_i|\Big) < \infty. \tag{2}$$

holds for any $\tau \in (0,\infty)$. There is only one solution to the problem which has, at the same time, property (2).

Proof. Consider $\tau \in (0,\infty)$ arbitrary and put

$$
\begin{aligned}
(3)\quad &\Phi(1,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \varphi(\boldsymbol{x},E,\boldsymbol{\omega},t),\\
&\Phi(2,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \frac{\sqrt{2E}}{4\pi}\sum_{i=1}^{n}\lambda_i\chi_i(E)N_i(\boldsymbol{x},t),\\
&S(1,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \int_0^t \mathrm{d}r\exp\Big(-\int_r^t \mathrm{d}s\,\sqrt{2E}\Sigma\big(x(s),E,\boldsymbol{\omega},s\big)\Big)F\big(\boldsymbol{x}(r),E,\boldsymbol{\omega},r\big)\\
&\qquad\qquad + \varphi_0\big(\boldsymbol{x}(0),E,\boldsymbol{\omega}\big)\exp\Big(-\int_0^t \mathrm{d}s\,\sqrt{2E}\Sigma\big(\boldsymbol{x}(s),E,\boldsymbol{\omega},s\big)\Big),\\
&S(2,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \frac{\sqrt{2E}}{4\pi}\sum_{i=1}^{n}\lambda_i\chi_i(E)N_{0i}(\boldsymbol{x})\exp(-\lambda_i t)
\end{aligned}
$$

on the set M_τ, and for any $\Psi \in m\{f,\tau\}$, put

$$
\begin{aligned}
(4)\quad &\mathbb{A}_s\Psi(1,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \int_0^t \mathrm{d}r\exp\Big(-\int_r^t \mathrm{d}u\,\sqrt{2E}\Sigma\big(\boldsymbol{x}(u),E,\boldsymbol{\omega},u\big)\Big)\\
&\times\Big[\Psi\big(2,\boldsymbol{x}(r),E,\boldsymbol{\omega},r\big) + \sqrt{2E}\int_0^\infty \mathrm{d}E'\int_\Omega \mathrm{d}\boldsymbol{\omega}'\,\Sigma_s\big(\boldsymbol{x}(r),E'\to E,\boldsymbol{\omega}'\to\boldsymbol{\omega},r\big)\\
&\qquad\times\Psi\big(1,\boldsymbol{x}(r),E',\boldsymbol{\omega}',r\big)\Big],\\
&\mathbb{A}_s\Psi(2,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv 0,\\
&\mathbb{A}_f\Psi(1,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \int_0^t \mathrm{d}r\exp\Big(-\int_r^t \mathrm{d}u\,\sqrt{2E}\Sigma\big(\boldsymbol{x}(u),E,\boldsymbol{\omega},u\big)\Big)\\
&\times\frac{1-\beta}{4\pi}\sqrt{2E}\chi(E)\int_0^\infty \mathrm{d}E'\int_\Omega \mathrm{d}\boldsymbol{\omega}'\,\nu(E')\Sigma_f\big(\boldsymbol{x}(r),E',r\big)\Psi\big(1,\boldsymbol{x}(r),E',\boldsymbol{\omega}',r\big),\\
&\mathbb{A}_f\Psi(2,\boldsymbol{x},E,\boldsymbol{\omega},t) \equiv \frac{\sqrt{2E}}{4\pi}\sum_{i=0}^{n}\beta_i\lambda_i\chi_i(E)\int_0^t \mathrm{d}u\exp\big(\lambda_i(u-t)\big)\\
&\times\int_0^\infty \mathrm{d}E'\int_\Omega \mathrm{d}\boldsymbol{\omega}'\,\nu(E')\Sigma_f(\boldsymbol{x},E',u)\Psi(1,\boldsymbol{x},E',\boldsymbol{\omega}',u).
\end{aligned}
$$

First, by the rule (4), the linear bounded operators

$$\mathbb{A}_r\colon m\{f,\tau\}\to m\{f,\tau\},\quad r=s,f$$

are defined. Indeed, let Ψ be an element belonging to the space $m\{f,\tau\}$. Then, by Assumption and the rule (4), the inequalities

$$|\mathbb{A}_s\Psi(1,x,E,\omega,t)| \leqslant tf(E)\|\Psi\|_\tau(C+1),$$
$$|\mathbb{A}_f\Psi(1,x,E,\omega,t)| \leqslant tf(E)\|\Psi\|_\tau C^2C_1$$

and

$$|\mathbb{A}_f\Psi(2,x,E,\omega,t)| \leqslant tf(E)\|\Psi\|_\tau C^2C_1\sum_{i=1}^{n}\lambda_i\beta_i$$

hold for any $t \in [0,\tau)$. Here C_1 is a finite constant such that

$$\sup_E \nu(E) \leqslant C_1$$

(see Assumption c)). So a constant $C_2 < \infty$ can be found such that the inequality

(5a) $$|\mathbb{A}_r\Psi(\ldots,t)| \leqslant tC_2f(E)\|\Psi\|_\tau$$

is satisfied for all $t \in [0,\tau)$, $r = s,\ f$. Similarly, on the basics of definition (4) and of guess (5a), we get

$$|\mathbb{A}_r^2\Psi(\ldots,t)| \leqslant (tC_2)^2/2f(E)\|\Psi\|_\tau$$

and, in general,

(5b) $$|\mathbb{A}_r^m\Psi(\ldots,t)| \leqslant \frac{(tC_2)^m}{m!}f(E)\|\Psi\|_\tau$$

for all $t \in [0,\tau)$, $r = s,\ f$ and for any nonnegative integer m. Therefore

(5c) $$\left\|\sum_{m=0}^{\infty}\mathbb{A}_r^m\right\|_\tau \leqslant \sum_{m=0}^{\infty}\|\mathbb{A}_r^m\|_\tau \leqslant \exp(C_2\tau) < \infty$$

and we have come the following conclusion: The operators

$$(\mathbb{I}-\mathbb{A}_s)^{-1} \quad \text{and} \quad \left(\mathbb{I}-(\mathbb{I}-\mathbb{A}_s)^{-1}\mathbb{A}_f\right)^{-1}$$

are bounded ($\mathbb{I}$ is the unit operator) and

(6) $$(\mathbb{I}-\mathbb{A}_s)^{-1} = \sum_{k=0}^{\infty}\mathbb{A}_s^k, \quad \left(\mathbb{I}-(\mathbb{I}-\mathbb{A}_s)^{-1}\mathbb{A}_f\right)^{-1} = \sum_{l=0}^{\infty}\left(\sum_{m=0}^{\infty}\mathbb{A}_s^m\mathbb{A}_f\right)^l.$$

Second, according to Assumption and the assumption of the theorem, $S \in m\{f, \tau\}$. Considering (2), problem (1) can be rewritten into the form

$$\Phi = (\mathbb{A}_s + \mathbb{A}_f)\Phi + S, \quad \Phi \in m\{f, \tau\} \tag{7}$$

This implies

$$\Phi = \left(\mathbb{I} - (\mathbb{I} - \mathbb{A}_s)^{-1}\mathbb{A}_f\right)^{-1}(\mathbb{I} - \mathbb{A}_s)^{-1}S$$

and, using (6), we get a solution of problem (7) in the form

$$\Phi = \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty}\mathbb{A}_s^l\mathbb{A}_f\right)^k\sum_{m=0}^{\infty}\mathbb{A}_s^m S. \tag{8}$$

As is seen from inequalities (5c), the right hand side of Eg. (8) is defined for any $\tau < \infty$.

Uniqueness of the solution is a consequence of linearity of the operators $\mathbb{A}_s$ and $\mathbb{A}_f$ and of inequalities (5): Let φ' and N_i', $i = 1, 2, \ldots, n$ be another solution to problem (1) having property (2). Consider a time interval $[0, \tau]$, $\tau < \infty$, and put

$$\Phi'(1, x, E, \omega, t) = \varphi(x, E, \omega, t) - \varphi'(x, E, \omega, t),$$
$$\Phi'(2, x, E, \omega, t) = \frac{\sqrt{2E}}{4\pi}\sum_{i=0}^{n}\lambda_i\chi_i(E)\big(N_i(x, t) - N_i'(x, t)\big).$$

Clearly, $\Phi' \in m\{f, \tau\}$ and

$$\Phi' = (\mathbb{A}_s + \mathbb{A}_f)^m\Phi', \quad m = 1, 2, 3, \ldots.$$

Therefore, by (5b)

$$\Phi' \equiv 0$$

for any positive τ. The theorem is proved. □

For any $\tau \in (0, \infty)$, problem (7) can be written in the form

$$\Phi(x) = \int_{M_\tau} \mathrm{d}x'[K_s(x, x') + K_f(x, x')]\Phi(x') + S(x).$$

Here $x \equiv (i, \mathbf{x}, E, \boldsymbol{\omega}, t)$ and $x' \equiv (i', \mathbf{x}', E', \boldsymbol{\omega}', t')$ are points of the set M_τ and integration over the set M_τ means both the summation over the set $\{1, 2\}$ in the first component of x' and the integration over the set $E_3 \times (0, \infty) \times \Omega \times [0, \tau)$ in the other components. The integral kernels K_s and K_f correspond to the operators $\mathbb{A}_s$ and $\mathbb{A}_f$, respectively.

Consider a particle in the following process of random collisions in the set M_τ. The particle starts its history by the first collision event. It means that an integer $i \in \{1, 2\}$, the location $\mathbf{x} \in E_3$, the velocity $(E, \boldsymbol{\omega}) \in E_3$ and the time parameter $t \in [0, \tau)$ are assigned to the particle in a random way (in what follows the integer i assigned to the particle will be called the state i of the particle).

The particle collides with elements of a medium and is either scattered (i.e. its state, position and velocity together with time are changed) or absorbed and then possibly reproduced. Velocity of the particle between any two consecutive collisions is constant.

History of the particle is determined by its track α_m (the sequence of collision points in the set M_τ),

$$\alpha_m \equiv (\beta_1, \beta_2, \ldots, \beta_m), \quad m = 1, 2, \ldots, \tag{9a}$$

where $\beta_i \equiv (x_1^i, \ldots, x_{n_i}^i)$ denotes a stage of the track and $x_j^i \in M_\tau$, $j = 1, 2, \ldots, n_i$ are the points of collisions. Specifically, $x_{n_i}^i$ is the point of absorption and x_j^i, $j \neq n_i$ are the places of scattering collisions in the stage β_i. Any stage is terminated by absorption. The particle may pass into the next stage only by means of reproduction. The point x_1^{i+1} corresponds to the place of the first collision of the particle reproduced.

The random walk is described by the following quantities:

(9b) $p_1(x)$ ... probability density of the first collision,

$p(x,y)$ ... probability density of the scattering (i.e., $p(x,y)\,dy$ is the probability that after the collision at the point x, the particle will be scattered and its new collision will occur in the neighbourhood dy of the point y),

$p_a(x)$ ... absorption probability,

$q_r(x)$ ... probability of reproduction of the particle after its absorption,

$q(x,y)$ ... probability density of the transition (i.e., $q(x,y)\,dy$ is the probability that ofter the reproduction at the point x, the first collision of the particle will occur in the neighbourhood dy of the point y).

We aim at using random process (9) for numerical solution of problem (7) and, for this purpose, we put several restrictions on the behavior of the quantities p_1, p_a, p, q_r and q.

First, we assume that the relations

$$\int_{M_\tau} dy\, p(x,y) \equiv 1 - p_a(x), \quad \int_{M_\tau} dy\, q(x,y) \equiv 1 \quad \text{and} \quad \sup_{M_\tau} q_r(x) \equiv Q < 1 \tag{10}$$

are fulfilled. Next, we demand any stage β of the history to be terminated after a finite number of collisions with probability 1, i.e.

$$\lim_{n\to\infty} \int_{M_\tau} dx_1 \ldots \int_{M_\tau} dx_n\, p(x,x_1) \prod_{j=1}^{n-1} p(x_j,x_{j+1}) = 0 \tag{11}$$

for any $x \in M_\tau$. Finally, we suppose that the implications

$$S(x) \neq 0 \Rightarrow p_1(x) \neq 0, \quad K_s(y,x) \neq 0 \Rightarrow p(x,y) \neq 0 \quad \text{and} \quad K_f(y,x) \neq 0 \Rightarrow p_a(x)q_r(x)q(x,y) \neq 0 \tag{12}$$

hold for any x, $y \in M_\tau$.

Let $g\colon M_\tau \to E_1$ be a function for which

$$\int_{M_\tau} \mathrm{d}x\, |g(x)| f(E) < \infty. \tag{13}$$

Simultaneously with process (9), we will consider a random variable $\eta(\alpha)$,

$$\eta(\alpha_m) = \tilde{S}(x_1^1) \prod_{i=1}^{m-1} \prod_{j=1}^{n_i-1} u(x_j^i, x_{j+1}^i) v(x_{n_i}^i) w(x_{n_i}^i, x_1^{i+1}) \tag{14}$$

$$\sum_{j=1}^{n_m} \prod_{k=1}^{j-1} u(x_k^m, x_{k+1}^m) \frac{g(x_j^m)}{1 - q_r(x_{n_m}^m)},$$

where

$$\tilde{S}(x) = \begin{cases} S(x)/p_1(x) & \\ 0 & \text{if } p_1(x) = 0, \end{cases}$$

$$u(x,y) = \begin{cases} K_s(y,x)/p(x,y) & \\ 0 & \text{if } p(x,y) = 0, \end{cases}$$

$$v(x) = \begin{cases} \dfrac{1}{p_a(x) q_r(x)} & \\ 0 & \text{if } p_a(x) q_r(x) = 0 \end{cases}$$

and

$$w(x,y) = \begin{cases} K_f(y,x)/q(x,y) & \\ 0 & \text{if } q(x,y) = 0. \end{cases}$$

In expression (14) (and, similarly, in the following text), it is understood that

$$\prod_{i=j}^{k} h(x_i) \equiv 1 \quad \text{for } k < j.$$

Theorem 2. *Let implications* (12) *hold on the set* M_τ. *Then*

A) *Expectation* $M\eta$ *of the variable* η *in the random process* (9) *is finite and*

$$M\eta = \int_{M_\tau} \mathrm{d}x\, g(x) \Phi(x)$$

where Φ *is the solution to problem* (7).

B) *Let, moreover, the inequalities*

$$\begin{aligned} C_1 &\geqslant |g(x)|/\Sigma(x), \\ C_2 p_1(x) &\geqslant \Sigma(x)|S(x)|, \\ C_3 p(x,y) &\geqslant K_s(y,x)\Sigma(y)/\Sigma(x) \end{aligned} \tag{15}$$

and

$$C_4 p_a(x) q_r(x) q(x,y) \geqslant K_f(y,x) \frac{\Sigma(y)}{\Sigma(x)}$$

be satisfied where C_i, $i = 1, 2, 3, 4$ are finite positive constants and

$$\Sigma(x) \equiv \Sigma(j, \boldsymbol{x}, E, \boldsymbol{\omega}, t) = \begin{cases} \Sigma(\boldsymbol{x}, E, \boldsymbol{\omega}, t) \\ 1 \qquad \text{if } j = 2. \end{cases}$$

Then the dispersion $D\eta$ of the random variable η is finite.

Proof. The general definition of expectation yields

$$M\eta = \sum_{\alpha} P(\alpha)\eta(\alpha) \tag{16}$$

where summation (integration) is taken over all histories corresponding to a given process of random collisions and $P(\alpha)$ is the probability of occurrence of the track α.

Next, using relations (10) and (11), we get the identity

$$\begin{aligned} p_a(x) &+ \int_{M_T} dy_1\, p(x, y_1) p_a(y_1) \\ &+ \int_{M_\tau} dy_1\, p(x, y_1) \int_{M_\tau} dy_2\, p(y_1, y_2) p_a(y_2) + \dots \\ &= 1 - \int_{M_\tau} dy_1\, p(x, y_1) + \int_{M_\tau} dy_1\, p(x, y_1) \\ &- \int_{M_\tau} dy_1\, p(x, y_1) \int_{M_\tau} dy_2\, p(y_1, y_2) + \dots = 1. \end{aligned} \tag{17}$$

In definition (16), express $P(\alpha)$ and $\eta(\alpha)$ in terms of (9b) and (14). Using (17) we can write

(18)

$$
\begin{aligned}
M\eta = & \sum_{m=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty}\int_{M_r} \mathrm{d}x_1^1 \ldots \int_{M_r} \mathrm{d}x_{n_1}^1 \ldots \int_{M_r} \mathrm{d}x_1^m \ldots \int_{M_r} \mathrm{d}x_{n_m}^m \, p_1(x_1^1) \\
& \times \prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} p(x_j^i, x_{j+1}^i) p_a(x_{n_i}^i) q_r(x_{n_i}^i) q(x_{n_i}^i, x_1^{i+1}) \prod_{k=1}^{n_m-1} p(x_k^m, x_{k+1}^m) \\
& \times p_a(x_{n_m}^m)\big(1 - q_r(x_{n_m}^m)\big)\tilde{S}(x_1^1) \prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} u(x_j^i, x_{j+1}^i) v(x_{n_i}^i) w(x_{n_i}^i, x_1^{i+1}) \\
& \times \sum_{k=1}^{n_m}\prod_{l=1}^{k-1} u(x_l^m, x_{l+1}^m)\frac{g(x_k^m)}{1 - q_r(x_{n_m}^m)} \\
= & \sum_{m=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{k=1}^{\infty}\cdots\sum_{n_m=k}^{\infty}\int_{M_r} \mathrm{d}x_1^1 \ldots \int_{M_r} \mathrm{d}x_{n_1}^1 \ldots \int_{M_r} \mathrm{d}x_1^m \ldots \int_{M_r} \mathrm{d}x_{n_m}^m \, S(x_1^1) \\
& \times \prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} K_s(x_{j+1}^i, x_j^i) K_f(x_1^{i+1}, x_{n_i}^i) \prod_{l=1}^{k-1} u(x_l^m, x_{l+1}^m) g(x_k^m) \\
& \times \prod_{l=1}^{n_m-1} p(x_l^m, x_{l+1}^m) p_a(x_{n_m}^m) \\
= & \sum_{m=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_{m-1}=1}^{\infty}\sum_{k=1}^{\infty}\int_{M_r} \mathrm{d}x_1^1 \ldots \int_{M_r} \mathrm{d}x_{n_1}^1 \ldots \int_{M_r} \mathrm{d}x_k^m \, S(x_1^1) \\
& \times \prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} K_s(x_{j+1}^i, x_j^i) K_f(x_1^{i+1}, x_{n_i}^i) \prod_{l=1}^{k-1} K_s(x_{l+1}^m, x_l^m) g(x_k^m) \\
& \times \left\{ p_a(x_k^m) + \int_{M_r} \mathrm{d}x_{k+1}^m \, p(x_k^m, x_{k+1}^m) p_a(x_{k+1}^m) + \ldots \right\} \\
= & \int_{M_r} \mathrm{d}x \, g(x) \sum_{k=0}^{\infty}\left(\sum_{l=0}^{\infty} \mathbb{A}_s^l \mathbb{A}_f\right)^k \sum_{m=0}^{\infty} \mathbb{A}_s^m S(x) \\
= & \int_{M_r} \mathrm{d}x \, g(x)\Phi(x)
\end{aligned}
$$

where relation (8) is used. Now, it is sufficient to take into account inequality (13) and part A) of the theorem is proved.

To prove the second part set

$$u'(x,y) = \Sigma(y)/\Sigma(x) u(x,y) \quad \text{and} \quad w'(x,y) = \Sigma(y)/\Sigma(x) w(x,y).$$

Using the inequalities (15) we get

$$|\eta^2(\alpha_m)| \leqslant C_2 \frac{\Sigma(x_1^1)}{p_1(x_1^1)} |S(x_1^1)| \prod_{i=1}^{m-1} C_4 v(x_{n_i}^i) w'(x_{n_i}^i, x_1^{i+1}) \prod_{j=1}^{n_i-1} C_3 u'(x_j^i, x_{j+1}^i)$$
$$\times \left(\sum_{k=1}^{n_m} \prod_{j=1}^{k-1} u'(x_j^m, x_{j+1}^m) \frac{|g(x_k^m)|}{\Sigma(x_k^m)\big(1 - q_r(x_{n_m}^m)\big)} \right)^2.$$

Next, set

$$S(m) = \sum_{k=1}^{m} \prod_{i=1}^{k-1} u'(x_i, x_{i+1}), \quad m = 1, 2, \ldots, \quad S(0) = 0.$$

Using (15) we obtain

$$S(m) \leqslant \sum_{i=1}^{m} C_3^{i-1} \leqslant (C_3 + 1)^m$$

and

$$\big(S(m+1)\big)^2 - \big(S(m)\big)^2 = \prod_{i=1}^{m} u'(x_i, x_{i+1}) [2S(m) + \prod_{i=1}^{m} u'(x_i, x_{i+1})]$$
$$\leqslant \prod_{i=1}^{m} u'(x_i, x_{i+1}) [2(C_3 + 1)^m + C_3^m].$$

Therefore the inequality

$$\big(S(m)\big)^2 = \left(\sum_{k=1}^{m} \prod_{i=1}^{k-1} u'(x_i, x_{i+1}) \right)^2 \leqslant \sum_{k=1}^{m} \prod_{i=1}^{k-1} \big(C_6 u'(x_i, x_{i+1})\big)$$

is satisfied for all $m = 1, 2, \ldots$ where $C_6 = 3C_3 + 3$ and, therefore,

$$|\eta^2(\alpha_m)| \leqslant C_1 C_2 \frac{\Sigma(x_1^1)}{p_1(x_1^1)} |S(x_1^1)| \prod_{i=1}^{m-1} C_4 v(x_{n_i}^i) w'(x_{n_i}^i, x_1^{i+1})$$
$$\prod_{j=1}^{n_i-1} C_3 u'(x_j^i, x_{j+1}^i) \sum_{k=1}^{m} \prod_{j=1}^{k-1} C_6 u'(x_j^m, x_{j+1}^m) \frac{|g(x_k^m)|}{\Sigma(x_k^m)} \frac{1}{(1 - Q)\big(1 - q_r(x_{n_m}^m)\big)}.$$

Substituting this estimate into the general formula

$$M\eta^2 = \sum_{\alpha} P(\alpha) \eta^2(\alpha)$$

and proceeding in the same way as in the case of the estimate $M\eta$ (see relation (18)) we get

$$M\eta^2 \leqslant \int_{M_\tau} \mathrm{d}x\, |g(x)| \sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} (C_6 \mathbb{A}_s)^l C_4 \mathbb{A}_f \right)^k \sum_{m=0}^{\infty} (C_6 \mathbb{A}_s)^m |S|(x) \frac{C_1 C_2}{1-Q}.$$

Relations (5) and (6) imply that the function

$$\sum_{k=0}^{\infty} \left(\sum_{l=0}^{\infty} (C_6 \mathbb{A}_s)^l C_4 \mathbb{A}_f \right)^k \sum_{m=0}^{\infty} (C_6 \mathbb{A}_s)^m |S|(x)$$

belongs to the space $m\{f, \tau\}$. Therefore, by (13),

$$M\eta^2 < \infty$$

and

$$D\eta = M\eta^2 - (M\eta)^2 < \infty.$$

□

Next, considering random process (9), let us define a random variable ζ by

$$\zeta(\alpha_m) = \tilde{S}(x_1^1) \prod_{i=1}^{m-1} \prod_{j=1}^{n_i - 1} u(x_j^i, x_{j+1}^i) v(x_{n_i}^i) w(x_{n_i}^i, x_1^{i+1}) \prod_{k=1}^{n_m - 1} u(x_k^m, x_{k+1}^m) \tilde{g}(x_{n_m}^m) \tag{19}$$

where

$$\tilde{g}(x) = \begin{cases} \dfrac{g(x)}{p_a(x)\big(1 - q_r(x)\big)} & \\ 0 & \text{if } p_a(x)\big(1 - q_r(x)\big) = 0. \end{cases}$$

Theorem 3. *Let relations* (12) *and the implication*

$$g(x) \neq 0 \Rightarrow p_a(x) \neq 0$$

hold on the set M_τ. *Then*

A) *Expectation* $M\zeta$ *of the random variable* ζ *is finite and*

$$M\zeta = \int_{M_\tau} \mathrm{d}x\, g(x) \Phi(x)$$

where Φ is the solution to problem (7).

B) *If, moreover, relations* (15) *and the inequality*

$$C_5 p_a(x) \geqslant |g(x)|/\Sigma(x) \tag{20}$$

are satisfied on M_τ ($C_5 < \infty$ is a constant) then the dispersion $D\zeta$ of the random variable ζ is finite.

Proof. Similarly as in the previous theorem we have

$$\begin{aligned}
M\zeta = &\sum_{m=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty}\int_{M_\tau}\mathrm{d}x_1^1\cdots\int_{M_\tau}\mathrm{d}x_{n_1}^1\cdots\int_{M_\tau}\mathrm{d}x_1^m\cdots\int_{M_\tau}\mathrm{d}x_{n_m}^m\, p_1(x_1^1)\\
&\times\prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} p(x_j^i, x_{j+1}^i)p_a(x_{n_i}^i)q_r(x_{n_i}^i)q(x_{n_i}^i, x_1^{i+1})\\
&\times\prod_{k=1}^{n_m-1} p(x_k^m, x_{k+1}^m)p_a(x_{n_m}^m)\big(1-q_r(x_{n_m}^m)\big)\tilde{S}(x_1^1)\\
&\times\prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} u(x_j^i, x_{j+1}^i)v(x_{n_i}^i)w(x_{n_i}^i, x_1^{i+1})\prod_{k=1}^{n_m-1} u(x_k^m, x_{k+1}^m)\tilde{g}(x_{n_m}^m)\\
= &\sum_{m=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{n_m=1}^{\infty}\int_{M_\tau}\mathrm{d}x_1^1\cdots\int_{M_\tau}\mathrm{d}x_{n_1}^1\cdots\int_{M_\tau}\mathrm{d}x_k^m\, S(x_1^1)\\
&\times\prod_{i=1}^{m-1}\prod_{j=1}^{n_i-1} K_s(x_{j+1}^i, x_j^i)K_f(x_1^{i+1}, x_{n_i}^i)\prod_{k=1}^{n_m-1} K_s(x_{k+1}^m, x_k^m)g(x_{n_m}^m)\\
= &\int_{M_\tau}\mathrm{d}x\, g(x)\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty}\mathbb{A}_s^l\mathbb{A}_f\Big)^k\sum_{m=0}^{\infty}\mathbb{A}_s^m S(x) = \int_{M_\tau}\mathrm{d}x\, g(x)\Phi(x) < \infty.
\end{aligned}$$

Next, according to inequalities (15) and (20), we have

$$\zeta^2(\alpha_m) \leqslant |\zeta(\alpha_m)|\frac{C_5C_2}{1-Q}\prod_{i=1}^{m-1}C_4\prod_{j=1}^{n_i-1}C_3\prod_{k=1}^{n_m-1}C_3.$$

Expressing the expectation $M\zeta^2$ in terms of (9b) and (19) and using the estimate of ζ^2 just obtained, we get

$$\begin{aligned}
M\zeta^2 \leqslant &\int_{M_\tau}\mathrm{d}x\, |g(x)|\sum_{k=0}^{\infty}\Big(\sum_{l=0}^{\infty}(C_3\mathbb{A}_s)^l C_4\mathbb{A}_f\Big)^k\\
&\times\sum_{m=0}^{\infty}(C_3\mathbb{A}_s)^m|S|(x)\frac{C_5C_2}{1-Q} < \infty.
\end{aligned}$$

Then

$$D\zeta = M\zeta^2 - (M\zeta)^2 < \infty$$

and the theorem is proved. □

Let us simulate the behavior of the random variables η or ζ in process (9) by N mutually independent trials. As the result of any of the trials, we will record the values η_i or ζ_i, $i = 1, 2, \ldots, N$ according to (14) or (19), respectively. By part A) of Theorems 2 and 3, we have

$$|M\eta_i| = |M\zeta_i| = \left| \int_{M_\tau} \mathrm{d}x \, g(x)\Phi(x) \right| < \infty, \quad i = 1, 2, \ldots, N.$$

Therefore, according to Kchinchin's theorem ([8], §32), the relations

$$\lim_{N\to\infty} P\left(\left| \sum_{i=1}^{N} \eta_i / N - \int_{M_\tau} \mathrm{d}x \, g(x)\Phi(x) \right| > \varepsilon \right) = 0 \tag{21}$$

and

$$\lim_{N\to\infty} P\left(\left| \sum_{i=1}^{N} \zeta_i / N - \int_{M_\tau} \mathrm{d}x \, g(x)\Phi(x) \right| > \varepsilon \right) = 0$$

hold for any $\varepsilon > 0$ (P is the probability of the corresponding event). Consequently, Theorem 1 together with assertion A) of Theorem 2 or Theorem 3 can serve as a basis for numerical solution of problem (1) by Monte Carlo method.

In accordance with (21), the integral $\int_{M_\tau} \mathrm{d}x \, g(x)\Phi(x)$ is approximated by the mean arithmetic value of the results obtained in mutually independent trials (corresponding to random variables (14) or (19) in the random process (9)–(11)) with an accuracy which grows with the number of the trials. The detailed behavior of the solution Φ to problem (7) can be found out by a suitable choice of the function g. Knowing Φ we express the solution to problem (1) in terms of (3).

As for the speed of convergence of the method (i.e. dependence of the approximation on the number of trials considered), it can be estimated using the following obvious consequence of Theorems 2B) and 3B) and of Lyapunov's or Chebyshev's theorem ([8], §32 and §42):

If the dispersion $D\eta$ is nonzero then, for any $x \geqslant 0$,

$$\lim_{N\to\infty} P\left(\left| \sum_{i=1}^{N} \eta_i / N - \int_{M_\tau} \mathrm{d}y \, g(y)\Phi(y) \right| < x\sqrt{(D\eta/N)} \right) = \sqrt{\frac{2}{\pi}} \int_0^x \mathrm{d}t \exp(-t^2/2). \tag{22}$$

In the case $D\eta = 0$, the equation

$$P\left(\left|\eta_i - \int_{M_\tau} \mathrm{d}x\, g(x)\Phi(x)\right| > 0\right) = 0 \tag{23}$$

holds for any $i = 1, 2, \ldots, N$.

Relations (22) and (23) remain true if the quantities η and η_i are replaced by the quantities ζ and ζ_i, $1 = 1, 2, \ldots, N$.

Conclusions

We have seen that the random variables η and ζ in the process (9) of random collisions give unbiased estimates to the solution of the problem (7). So Theorems 2 and 3 together with formula (22) form a basis and instructions to numerical solution of problem (1) by Monte Carlo method.

As for computational applications of the method suggested, recall the following fact: In current cases, the time interval between two consecutive collisions of the neutron with the medium is very small (it is of order 10^{-5}s and smaller if the kinetic energy of the neutron is greater than 0.0255 eV). Then the use of the analog process of random collisions would probably lead to time consuming computations. Therefore we expect that it is the nonanalog processes (i.e. those in which the probabilistic functions $p_1(x)$, $p_a(x)$, $q_r(x)$, $p(x, y)$ and $q(x, y)$ are appropriately chosen) that would be able to play a substantial role in practical calculations.

Formulating the problem of reactor kinetics we have assumed that the spatial region considered is the whole space E_3. However, in practice the following situation frequently occurs: the material medium is contained in a bounded convex spatial region D surrounded by vacuum. External sources of neutrons are placed in D and no neutrons enter the medium from outside. Find the solution φ and N_i, $i = 1, 2, \ldots, n$ of the problem (1) restricted to the domain $D \times (0, \infty) \times \Omega \times [0, \tau)$ and $D \times [0, \tau)$, respectively.

As the macroscopic cross-sections and the densities $N_i(x, t)$, $i = 1, 2, \ldots, n$ identically vanish in vacuum, Eqs. (1) imply that the simplified problem just formulated is equivalent to the original one. Consequently, Theorems 1, 2 and 3 remain true if the space E_3 is replaced by the region D.

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