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SPECTRAL METHODS FOR SINGULAR PERTURBATION PROBLEMS

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Summary. We study spectral discretizations for singular perturbation problems. A special technique of stabilization for the spectral method is proposed. Boundary layer problems are accurately solved by a domain decomposition method. An effective iterative method for the solution of spectral systems is proposed. Suitable components for a multigrid method are presented.

Keywords: spectral methods, singular perturbation, stabilization, domain decomposition, iterative solver, multigrid method.

AMS classification: 65N35

1. INTRODUCTION

We consider singular perturbation problems (or advection-diffusion equations) which can in the one-dimensional case be written as

$$\begin{aligned} -\varepsilon u'' + u' &= f \quad \text{in } (-a, b), \\ u(-a) = u(b) &= 0, \end{aligned} \tag{1.1}$$

where $\varepsilon > 0$, $a > 0$, $b > 0$ and f is a given function defined in $(-a, b)$. In the two-dimensional case we consider problems given by

$$\begin{aligned} -\varepsilon \Delta u + u_x &= g \quad \text{in } \Omega = (-a, b) \times (-1, 1), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where g is a given function defined on Ω. In particular, one is interested in problems with small $\varepsilon > 0$. The problem (1.1) is discretized by a spectral collocation

method (see [1, 2]). One of the main features of such problems is the occurence of thin boundary layers which have a width of order $O(\varepsilon)$ for $\varepsilon \to 0$. The "maximum principle" in the physical space does not hold for spectral approximations. Nevertheless, Canuto [1] has proved a certain sort off "maximum principle" in the frequency space, in the sense that all Chebyshev coefficients of the spectral solution are strictly positive for all ε and N (N: maximal degree of polynomials). By using this property it is possible to derive convergence estimates. One of the main results is the following: if $\varepsilon N^2 \to \infty$ as $N \to \infty$ and $\varepsilon \to 0$ in such a way that $\varepsilon N^2 = N^\alpha$ for a suitable $\alpha > 0$, then the spectral discretization error tends to 0 faster than an power of N^{-1} (see [1, 3.2A]). In particular, it comes out that the spectral scheme is stable for $\varepsilon = N^{-1}$. This is a property which is also known from central difference discretizations of (1.1) or (1.2). In the case of very small $\varepsilon \ll N^{-1}$, artificial viscosity with $\tilde{\varepsilon} = N^{-1}$ is introduced in order to stabilize the scheme (see [3, 4, 7]). An obvious disadvantage of this approach lies in the fact that the method now becomes only first order accurate. For finite differences the second order accuracy can be obtained by applying a "Mehrstellen"-operator to the right hand side (see [4, 10, 12]). By similar technique we stabilize our spectral scheme.

First we introduce artifical viscosity and then we perturb the right hand side f by adding the first and second derivatives. Here it is assumed that f is sufficiently smooth. This assumption is reasonable since in connection with spectral methods we are mainly interested in the case of smooth data. This method of stabilization leads to an approximation error which is perturbed by quantities of order $O(\varepsilon\tilde{\varepsilon})$ or $O(\varepsilon^2\tilde{\varepsilon})$ for small ε. A similar treatment is performed in the two-dimensional case. We present numerical results which substantiate this improvement.

In Section 3 and 4 we consider a domain decomposition method for singular perturbation problems in one and two dimensions. We present an iterative procedure with interface relaxation. Here we mainly follow the treatment in [6]. During each iteration, at every interface between two subdomains, first Dirichlet boundary conditions are imposed on the on side, and the Neumann boundary conditions are imposed on the other. In order to ensure convergence, a relaxation of the Dirichlet data at the interface is used. We analyze this algorithm and prove convergence for relaxation parameters which lie in a certain interval. In the one-dimensional case the optimal relaxation parameter can be explicitly calculated and this choice leads to convergence in two steps. In the case of two dimensions the number of iterations depends on the location of the interface. If the interface lies near the boundary layer, convergence in a few steps (less then 10) can be achieved. If the interface is located far away from the boundary layer, the number of iterations becomes dependent on ε and asymptotically behaves like $O(\varepsilon^{-2})$ for $\varepsilon \to 0$. Furthermore, we investigate a strategy for an automatic selection of relaxation parameters (see [6]). In the one-dimensional

case it will converge after three steps (independent by of the starting relaxation parameter). In the two-dimensional case it also yields very satisfying results and is in general faster than the method with a fixed relaxation parameter. Numerical analysis is performed in both the continuous and the discrete case. Numerical results are presented which confirm the expected behaviour from the theoretical analysis.

In Section 5 we introduce an effective iterative method for the solution of singular perturbed spectral systems. Special techniques of preconditioning are proposed. Numerical results are presented. It is shown how the relaxation method can be imbedded in a multigrid V-cycle.

2. Stabilization of the spectral method

We consider the one-dimensional advection-diffusion equation (1.1). Let L_ε denote the singular perturbed operator, i.e. $L_\varepsilon u = -\varepsilon u'' + u'$. By L_ε^N we denote the corresponding collocation operator with nodes $x_j = \cos \frac{j\pi}{N}$ $(j = 0, \ldots, N)$. After an elimination of the boundary conditions the collocation problem now reads as follows:

$$L_\varepsilon^N u_N = f_N,$$

where $f_N = \big(f(x_j)\big)_{j=1,\ldots,N-1}$. For small ε the operator L_ε becomes instable and hence we introduce a stabilized operator $L_{\tilde{\varepsilon}}^N$ which corresponds to L_ε^N with ε replaced by $\tilde{\varepsilon} \gg \varepsilon$. From the results in Canuto [1] and from numerical experiments it becomes obvious that $L_{\tilde{\varepsilon}}^N$ with $\tilde{\varepsilon} = 1/N$ is stable for all N. Unfortunately, by a simple replacement of $L_{\tilde{\varepsilon}}^N$ by L_ε^N we get a first order method and the high accuracy of the spectral method is lost. Hence we propose a simultaneous change of the right hand side. This is an idea which is closely related to the "Mehrstellen"-method (see [4, 7, 10, 12]). Doerfer et al. [4] introduced this method for central finite difference discretizations. In general the introduction of artifical viscosity makes the method only first order accurate. By means of the "Mehrstellen"-method it becomes second order accurate. Now we introduce two perturbations of the right hand side f:

$$f_\varepsilon^1 = f + (\varepsilon - \tilde{\varepsilon}) f',$$
$$f_\varepsilon^2 = f + (\varepsilon - \tilde{\varepsilon}) f' + (\varepsilon - \tilde{\varepsilon}) \varepsilon f''.$$

Here we assume that the second derivative of f exists. The corresponding collocation problems read

$$L_{\tilde{\varepsilon}}^N v_N = f_{\varepsilon,N}^1$$

and

$$L_{\tilde{\varepsilon}}^N w_N = f_{\varepsilon,N}^2,$$

where

$$f^1_{\varepsilon,N} = f_N + (\varepsilon - \tilde{\varepsilon}) f'_N$$

and

$$f^2_{\varepsilon,N} = f_N + (\varepsilon - \tilde{\varepsilon}) f'_N + (\varepsilon - \tilde{\varepsilon}) \varepsilon f''_N.$$

Here we write $f'_N = \left(f'(x_j)\right)_{j=1,\ldots,N-1}$, $f''_N = \left(f''(x_j)\right)_{j=1,\ldots,N-1}$. Now we are interested in the error functions $u_N - v_N$ and $u_N - w_N$. An easy calculation yields

$$\begin{aligned} -\tilde{\varepsilon}(u_N - v_N)'' + (u_N - v_N)' &= -\varepsilon u''_N + u'_N + \tilde{\varepsilon} v''_N - v'_N + (\varepsilon - \tilde{\varepsilon}) u''_N \\ &= f_N - f^1_{\varepsilon,N} + (\varepsilon - \tilde{\varepsilon}) u''_N \\ &= -(\varepsilon - \tilde{\varepsilon}) f'_N + (\varepsilon - \tilde{\varepsilon}) u''_N \\ &= -(\varepsilon - \tilde{\varepsilon})(-\varepsilon u'''_N + u''_N) + (\varepsilon - \tilde{\varepsilon}) u''_N \\ &= (\varepsilon - \tilde{\varepsilon}) \varepsilon u'''_N, \end{aligned}$$

which behaves like $O(\tilde{\varepsilon}\varepsilon)$ for small ε. Since $L^N_{\tilde{\varepsilon}}$ is a stable operator we conclude that

$$\|u_N - v_N\|_2 = O(\tilde{\varepsilon}\varepsilon),$$

where $\| \ \|_2$ denotes the discrete L^2-norm, i.e.

$$\|w\|_2 = \frac{1}{\sqrt{N}} \sqrt{\sum_{i=0}^{N} w(x_i)^2}.$$

By similar calculation one obtains

$$\begin{aligned} -\tilde{\varepsilon}(u_N - w_N)'' + (u_N - w_N)' &= (\varepsilon - \tilde{\varepsilon}) \varepsilon u'''_N - (\varepsilon - \tilde{\varepsilon}) \varepsilon (-\varepsilon u^{IV}_N + u'''_N) \\ &= (\varepsilon - \tilde{\varepsilon}) \varepsilon^2 u^{IV}_N, \end{aligned}$$

which behaves like $O(\varepsilon^2 \tilde{\varepsilon})$ for small ε. Hence we obtain

$$\|u_N - w_N\|_2 = O(\varepsilon^2 \tilde{\varepsilon}).$$

This means that the spectral accuracy is perturbed by $O(\tilde{\varepsilon}\varepsilon)$ or $O(\varepsilon^2 \tilde{\varepsilon})$, respectively, which is quite small for small ε.

A similar treatment can also be transferred to the two-dimensional singular perturbation problem (1.2). Let u_N denote the solution of the corresponding collocation problem with respect to the nodes $(x_i, y_j) = (\cos \frac{i\pi}{N}, \cos \frac{j\pi}{N})$ $(i, j = 0, \ldots, N)$. This means that

$$u_N \in \mathbb{P}^0_N = \{p_N : p_N \text{ polynomial of degree } \leqslant N \text{ in } x, y \text{ and } p_N = 0 \text{ on } \partial\Omega\}$$

and

$$(-\varepsilon\Delta u_N + u_{N,x})(x_i, y_j) = f(x_i, y_j) \quad (i, j = 1, \ldots, N-1).$$

The corresponding stabilized problems define functions $v_N, w_N \in \mathbb{P}_N^0$ as follows:

$$(-\tilde{\varepsilon}v_{N,xx} - \varepsilon v_{N,yy} + v_{N,x})(x_i, y_j) = (f + (\varepsilon - \tilde{\varepsilon})f_x)(x_i, y_j)$$

and

$$(2.1) \quad (-\tilde{\varepsilon}w_{N,xx} - \varepsilon w_{N,yy} + w_{N,x})(x_i, y_i) = (f + (\varepsilon - \tilde{\varepsilon})f_x + (\varepsilon - \tilde{\varepsilon})\varepsilon\Delta f)(x_i, y_j)$$

for $i, j = 1, \ldots, N-1$. Since only the first x-derivative is involved we only have to introduce artificial viscosity in the x-direction. Once more we are able to estimate $v_N - u_N$ and $w_N - u_N$. We easily obtain

$$-\tilde{\varepsilon}(v_N - u_N)_{xx} - \varepsilon(v_N - u_N)_{yy} + (v_N - u_N)_x = (\tilde{\varepsilon} - \varepsilon)\varepsilon(u_{N,xxx} + u_{N,yyx}),$$

which behaves like $O(\varepsilon^2\tilde{\varepsilon})$ for small ε.

By similar calculation one obtains

$$-\tilde{\varepsilon}(w_N - u_N)_{xx} - \varepsilon(w_N - u_N)_{yy} + (w_N - u_N)_x = (\tilde{\varepsilon} - \varepsilon)\varepsilon^2(u_{N,xxxx} + 2u_{N,xxyy} + u_{N,yyyy}),$$

which behaves like $O(\varepsilon^2\tilde{\varepsilon})$ for small ε.

Since the modified operator is stable, it follows that

$$\|v_N - u_N\|_2 = O(\tilde{\varepsilon}\varepsilon)$$

and

$$\|w_N - u_N\|_2 = O(\varepsilon^2\tilde{\varepsilon}),$$

where $\| \ \|_2$ denotes the discrete L^2-norm in two dimensions. In practice, the derivatives of f are also calculated by applying spectral collocation derivative operators. The previous analysis can be adopted in a straightforward manner. We apply these methods to some numerical examples.

In Tables 2.1 and 2.2 we present numerical results which show the stabilization of our treatment. For the numerical tests we consider singular perturbed problems with the exact solution

$$u(x) = \sin(\pi x)$$

in the one-dimensional case and

$$u(x, y) = \sin(\pi x)\sin(\pi y)$$

in the two-dimensional case. We fix $\varepsilon = 10^{-6}$.

N	$\|u_N - u\|_2$	$\|v_N - u\|_2$	$\|w_N - u\|_2$
4	$.196 \cdot 10^{5}$	$.194 \cdot 10^{0}$	$.194 \cdot 10^{0}$
8	$.122 \cdot 10^{2}$	$.401 \cdot 10^{-3}$	$.401 \cdot 10^{-3}$
12	$.201 \cdot 10^{-2}$	$.522 \cdot 10^{-6}$	$.127 \cdot 10^{-6}$
14	$.176 \cdot 10^{-4}$	$.445 \cdot 10^{-6}$	$.145 \cdot 10^{-8}$

Table 2.1. Numerical results in the one-dimensional case.

N	$\|u_N - u\|_2$	$\|v_N - u\|_2$	$\|w_N - u\|_2$
4	$.550 \cdot 10^{4}$	$.109 \cdot 10^{0}$	$.109 \cdot 10^{0}$
8	$.649 \cdot 10^{1}$	$.754 \cdot 10^{-2}$	$.250 \cdot 10^{-3}$
12	$.117 \cdot 10^{-2}$	$.646 \cdot 10^{-6}$	$.795 \cdot 10^{-7}$
14	$.104 \cdot 10^{-4}$	$.556 \cdot 10^{-6}$	$.904 \cdot 10^{-9}$

Table 2.2. Numerical results in the two-dimensional case.

It becomes obvious that the singular perturbed problem is unstable for small N. For increasing N this effect of instability is somewhat disturbed by the high consistency of the spectral method. Our methods of stabilization work very well. The improvement of the approximation w_N over v_N becomes visible for $N = 14$ where the spectral accuracy is sufficiently high.

We further remark that a similar treatment is not possible if the first order terms in both directions, i.e.

$$\alpha u_x + \beta u_y$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, $\beta \neq 0$ are added to the second order operator. In this case the above treatment always introduces the mixed derivative u_{xy} which cannot be eliminated. More general problems result from a rotation of the rectangle $R = (-1,1)^2$. The rotation by an angle φ is given by

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = T_\varphi \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$T_\varphi = \begin{pmatrix} \cos\varphi, & \sin\varphi \\ -\sin\varphi, & \cos\varphi \end{pmatrix}.$$

We denote the rotated rectangle by R_φ. The transformed problem now reads as follows:

$$(2.2) \qquad -\varepsilon\tilde{\Delta}z + \cos\varphi z_{\tilde{x}} - \sin\varphi z_{\tilde{y}} = f \quad \text{in } R_\varphi,$$
$$z = 0 \quad \text{on } \partial R_\varphi, \text{ where } \tilde{\Delta}z = z_{\tilde{x}\tilde{x}} + z_{\tilde{y}\tilde{y}}.$$

For such an equation the corresponding stabilization can be determined by a rotation of the corresponding stabilized problems in R. It is explicitly given by

$$\begin{aligned} -\tilde{\varepsilon}(\cos^2\varphi\tilde{w}_{N,\tilde{x}\tilde{x}} - 2\cos\varphi\sin\varphi\tilde{w}_{N,\tilde{x}\tilde{y}} + \sin^2\varphi\tilde{w}_{N,\tilde{y}\tilde{y}}) \\ - \varepsilon(\sin^2\varphi\tilde{w}_{N,\tilde{x}\tilde{x}} + 2\cos\varphi\sin\varphi\tilde{w}_{N,\tilde{x}\tilde{y}} + \cos^2\varphi\tilde{w}_{N,\tilde{y}\tilde{y}}) \\ + \cos\varphi\tilde{w}_{N,\tilde{x}} - \sin\varphi\tilde{w}_{N,\tilde{y}} \\ = f + (\varepsilon - \tilde{\varepsilon})(\cos\varphi f_{\tilde{x}} - \sin\varphi f_{\tilde{y}}) + (\varepsilon + \tilde{\varepsilon})\varepsilon\tilde{\Delta}f \quad \text{in } R_\varphi \\ \tilde{w}_N = 0 \quad \text{on } \partial R_\varphi. \end{aligned}$$

The corresponding error $\|z - \tilde{w}_N\|_2$ behaves like $O(\varepsilon^2\tilde{\varepsilon})$ $(\varepsilon \to 0)$.

Finally, we numerical observed that problems like (2.2) defined on R instead of R_φ instead of R_φ are stable for all $\varphi \in [0, 2\pi)$, $\varphi \neq k\pi/2$, $k \in \{0, 1, 2, 3\}$. The solution of the continuous problem is denoted by $\tilde{z}$ and the solution of the corresponding spectral problem is denoted by z_N. By $\tilde{z}_N$ we further denote the spectral approximation of the stabilized version.

N	4	8	12	14
$\|\tilde{z} - z_N\|_2$	$0,209 \cdot 10^0$	$0.529 \cdot 10^{-3}$	$0,195 \cdot 10^{-6}$	$0.231 \cdot 10^{-8}$
$\|\tilde{z} - \tilde{z}_N\|_2$	$0.153 \cdot 10^0$	$0,443 \cdot 10^{-3}$	$0.141 \cdot 10^{-6}$	$0.160 \cdot 10^{-8}$

Table 2.3. Numerical results for $\varphi = \pi/4$.

From the numerical results in Table 2.3 it becomes obvious that the spectral problem is already stable. The spectral approximation from the stabilized problem only yields a slight improvement in accuracy. Hence for $\varphi \neq k\pi/2$, $k \in \{0, 1, 2, 3\}$ it is not necessary to resort to the stabilized version. At the moment it is not clear why this phenomen of instability occurs only if the direction of flow is parallel to the coordinate axes.

3. Domain decomposition in one dimension

3.1. The continuous case. We once more consider the one-dimensional advection-diffusion equation (1.1). In particular, we are interested in the special equation with

$$f = -\frac{1}{a+b}.$$

By the transform

$$\tilde{u} = u + \frac{x+a}{b+a} \quad \text{in } (-a, b),$$

problem (1.1) is now equivalent to the boundary value problem

$$
\begin{aligned}
-\varepsilon \tilde{u}_{xx} + \tilde{u}_x &= 0 \quad \text{in } (-a, b), \\
\tilde{u}(-a) = 0, &\quad \tilde{u}(b) = 1.
\end{aligned} \tag{3.1}
$$

The exact solution of (3.1) is given by

$$
\tilde{u}(x) = \frac{\mathrm{e}^{\frac{x+a}{\varepsilon}} - 1}{\mathrm{e}^{\frac{b+a}{\varepsilon}} - 1}.
$$

The boundary layer exhibited near $x = 0$ when $\varepsilon \to 0$ has now a width of order $O(\varepsilon)$.

We are interested in a domain decomposition approach to this class of problems. If we denote by v the restriction of u to the interval $(-a, 0)$, and by w its restriction to $(0, b)$, it follows that (1.1) is equivalent to the split problem

$$
\begin{aligned}
-\varepsilon v_{xx} + v_x &= f \quad \text{in } (-a, 0), \\
v(-a) &= 0, \\
v(0) &= w(0), \\
v_x(0) &= w_x(0), \\
w(b) &= 0, \\
-\varepsilon w_{xx} + w_x &= f \quad \text{in } (0, b).
\end{aligned}
$$

A similar statement holds for decompositions of $(-a, b)$ into a finite number of subintervals.

In analogy to the treatment in [6] we define the following two sequences $v^{(n)}$, $n \geqslant 1$ in $(-a, 0)$ and $w^{(n)}$, $n \geqslant 1$ in $(0, b)$ which solve the problems

$$
\begin{aligned}
-\varepsilon v_{xx}^{(n)} + v_x^{(n)} &= f \quad \text{in } (-a, 0), \\
v^{(n)}(-a) &= 0, \\
v^{(n)}(0) &= \lambda_n
\end{aligned} \tag{3.2}
$$

and

$$
\begin{aligned}
-\varepsilon w_{xx}^{(n)} + w_x^{(n)} &= f \quad \text{in } (0, b), \\
w^{(n)}(b) &= 0, \\
w_x^{(n)}(0) &= v_x^{(n)}(0).
\end{aligned} \tag{3.3}
$$

Here λ_1 is given a number, and the λ_n $(n \geqslant 2)$ are defined by the formula

$$
\lambda_{n+1} = \theta w^{(n)}(0) + (1 - \theta)\lambda_n, \quad n \geqslant 1.
$$

$\theta \in [0,1]$ denotes the relaxation parameter. For a further study of the interface relaxation let us introduce the error functions

$$e_v^{(n)}(x) = v^{(n)}(x) - v^{(n-1)}(x) \quad \text{in } (-a,0)$$

and

$$e_w^{(n)}(x) = w^{(n)}(x) - w^{(n-1)}(x) \quad \text{in } (0,b)$$

for $n \geqslant 2$. In terms of the error functions the interface relaxation procedure now reads as follows:

$$\begin{aligned}
-\varepsilon e_{v,xx}^{(n)} + e_{v,x}^{(n)} &= 0 \quad \text{in } (-a,0),\\
e_v^{(n)}(-a) &= 0,\\
e_v^{(n)}(0) &= \theta e_w^{(n-1)}(0) + (1-\theta) e_v^{(n-1)}(0),\\
-\varepsilon e_{w,xx}^{(n)} + e_{w,x}^{(n)} &= 0 \quad \text{in } (0,b),\\
e_w^{(n)}(b) &= 0,\\
e_{w,x}^{(n)}(0) &= e_{v,x}^{(n)}(0).
\end{aligned}$$

Convergence analysis of this relaxation procedure yields the following result:

Proposition 3.1. *The interface relaxation* (3.2), (3.3) *converges provided* θ *satisfies*

$$0 < \theta < \theta^*(\varepsilon),$$

where

$$\theta^*(\varepsilon) = 2\big(1 + \varrho(\varepsilon)\big)^{-1}$$

with

$$\varrho(\varepsilon) = \mathrm{e}^{\frac{a}{\varepsilon}} \frac{\mathrm{e}^{\frac{b}{\varepsilon}} - 1}{\mathrm{e}^{\frac{a}{\varepsilon}} - 1}.$$

The optimal choice for θ *is given by* $\theta_{\mathrm{opt}} = \frac{1}{2}\theta^*(\varepsilon)$. *With this choice the procedure* (3.2), (3.3) *converges after two steps.*

Proof. The error functions $e_v^{(n)}$, $e_w^{(n)}$ can explicitly be written as

$$\begin{aligned}
e_v^{(n)}(x) &= e_v^{(n)}(0) \frac{\mathrm{e}^{\frac{x+a}{\varepsilon}} - 1}{\mathrm{e}^{\frac{a}{\varepsilon}} - 1},\\
e_w^{(n)}(x) &= e_{w,x}^{(n)}(0) \varepsilon \mathrm{e}^{\frac{b}{\varepsilon}} (\mathrm{e}^{\frac{x-b}{\varepsilon}} - 1).
\end{aligned}$$

Using

$$\begin{aligned}
e_w^{(n)}(0) &= e_{v,x}^{(n)}(0) \varepsilon \mathrm{e}^{\frac{b}{\varepsilon}} (\mathrm{e}^{-\frac{b}{\varepsilon}} - 1),\\
e_{v,x}^{(n)}(0) &= e_v^{(n)}(0) \frac{1}{\varepsilon} \mathrm{e}^{\frac{a}{\varepsilon}} \frac{1}{\mathrm{e}^{\frac{a}{\varepsilon}} - 1}
\end{aligned}$$

we derive

$$e_w^{(n)}(0) = \mathrm{e}^{\frac{a+b}{\varepsilon}} \frac{\mathrm{e}^{-\frac{b}{\varepsilon}} - 1}{\mathrm{e}^{\frac{a}{\varepsilon}} - 1} e_v^{(n)}(0).$$

Hence

$$\begin{aligned} e_v^{(n+1)}(0) &= \theta e_w^{(n)}(0) + (1-\theta) e_v^{(n)}(0) \\ &= \big[1 - \theta\big(1 + \varrho(\varepsilon)\big)\big] e_v^{(n)}(0) \\ &= [1 - 2\theta^*(\varepsilon)^{-1}\theta] e_v^{(n)}(0). \end{aligned}$$

Therefore the necessary and sufficient condition for convergence is

$$|1 - 2\theta^*(\varepsilon)^{-1}\theta| < 1,$$

which yields $0 < \theta < \theta^*(\varepsilon)$. Furthermore, it can be seen that $v^{(3)} = v^{(2)} = u$ in $(-a, 0)$ and $w^{(3)} = w^{(2)} = u$ in $(0, b)$ if $\theta = \theta_{\mathrm{opt}}$. □

This means that with the choice $\theta = \theta_{\mathrm{opt}}$ we can always guarantee convergence in 2 steps. Nevertheless it is interesting to see what happens to the interval of convergence (in θ) when ε tends to zero. For $\varepsilon \to 0$ we obtain

$$\theta^*(\varepsilon) \sim 2\mathrm{e}^{-b/\varepsilon} \to 0.$$

This means that the interval length $\theta^*(\varepsilon)$ tends exponencially fast to zero.

Now we fix a independent of ε and take $b = O(\varepsilon)$, say $b = \sigma\varepsilon$ ($\sigma > 0$, constant). For $\varepsilon \to 0$ we then obtain

$$\theta^*(\varepsilon) \sim 2\mathrm{e}^{-\sigma}.$$

This means that now the interval length is fixed independently of ε. Hence we observe that a large interval of convergence is obtained if the interface lies near the boundary layer. This results is very satisfactory since this is just the case one is mainly interested in.

We are also interested in an automatic generation of the relaxation parameter. Now we allow θ to change as n changes. As introduced in [6] an effective way to update θ is to take $\theta = \theta_n$ with

$$\theta_n = \frac{\lambda_n - \lambda_{n-1}}{(\lambda_n - \lambda_{n-1}) - (w^{(n)}(0) - w^{(n-1)}(0))}, \quad n \geqslant 2.$$

The value θ_1 is a given parameter. By this automatic choice one gets $\theta_2 = \theta_{\mathrm{opt}}$ and therefore convergence after 3 steps, no matter how θ_1 is chosen.

3.2. Spectral collocation method. A spectral collocation approximation to the split problem is defined as follows: Let N, M denote positive integers and let denote by $\mathbb{P}_N$ the space of polynomials of degree $\leqslant N$. We look for two polynomials $v_N \in \mathbb{P}_N$ and $w_M \in \mathbb{P}_M$ such that

$$
\begin{aligned}
(3.4) \qquad (-\varepsilon v_{N,xx} + v_{N,x})(x_j^{(1)}) &= f(x_j^{(1)}) \quad \text{for } 1 \leqslant j \leqslant N-1, \\
v_N(-a) &= 0, \\
v_N(0) &= w_M(0), \\
v_{N,x}(0) &= w_{M,x}(0), \\
w_M(b) &= 0, \\
(-\varepsilon w_{M,xx} + w_{M,x})(x_j^{(2)}) &= f(x_j^{(2)}) \quad \text{for } 1 \leqslant j \leqslant M-1.
\end{aligned}
$$

Here we use the nodes

$$
\begin{aligned}
x_j^{(1)} &= \frac{a}{2}(\xi_j^N - 1), \\
x_j^{(2)} &= \frac{b}{2}(\xi_j^M + 1), \quad \text{where } \xi_j^N = \cos\frac{j\pi}{N}.
\end{aligned}
$$

This type of approximation was first introduced by Orszag [11]. There it was called the patching-collocation method. The interface relaxation procedure for (3.4) is now defined as follows: Let λ_1 be a given number. We define two sequences $v_N^{(n)} \in \mathbb{P}_N$ and $w_M^{(n)} \in \mathbb{P}_M$, $n \geqslant 1$, such that

$$
\begin{aligned}
(3.5) \qquad (-\varepsilon v_{N,xx}^{(n)} + v_{N,x}^{(n)})(x_j^{(1)}) &= f(x_j^{(1)}) \quad \text{for } 1 \leqslant j \leqslant N-1, \\
v_N^{(n)}(-a) &= 0, \\
v_N^{(n)}(0) &= \lambda_n
\end{aligned}
$$

and

$$
\begin{aligned}
(3.6) \qquad (-\varepsilon w_{M,xx}^{(n)} + w_{M,x}^{(n)})(x_j^{(2)}) &= f(x_j^{(2)}) \quad \text{for } 1 \leqslant j \leqslant M-1, \\
w_M^{(n)}(b) &= 0, \\
w_{M,x}^{(n)}(0) &= v_{N,x}^{(n)}(0).
\end{aligned}
$$

The λ_n ($n \geqslant 2$) are defined by the following formula:

$$
\lambda_{n+1} = \theta w^{(n)}(0) + (1-\theta)\lambda_n, \quad n \geqslant 1.
$$

Here $\theta \in [0,1]$ denotes the relaxation parameter. If $\theta = 1$ there is no relaxation. The choice of the dynamic parameter is given by

$$
\theta_n = \frac{-\lambda_n - \lambda_{n-1}}{(\lambda_n - \lambda_{n-1}) - (w_M^{(n)}(0) - w_M^{(n-1)}(0))}, \quad n \geqslant 2.
$$

Now we are interested in an analysis of convergence of the interface relaxation procedure (3.5), (3.6). For this purpose we introduce error functions

$$e_v^{(n)} = v_N^{(n)} - v_N^{(n-1)}$$

and

$$e_w^{(n)} = w_M^{(n)} - w_M^{(n-1)} \quad \text{for } n \geqslant 2.$$

We deduce that $e_v^{(n)} \in \mathbb{P}_N$, $e_w^{(n)} \in \mathbb{P}_M$, and consequently

$$\begin{aligned}(-\varepsilon e_{v,xx}^{(n)} + e_{v,x}^{(n)})(x_j^{(1)}) &= 0 \quad \text{for } 1 \leqslant j \leqslant N-1,\\ e_v^{(n)}(-a) &= 0,\\ e_v^{(n)}(0) &= \delta_n\end{aligned}$$

and

$$\begin{aligned}(-\varepsilon e_{w,xx}^{(n)} + e_{w,x}^{(n)})(x_j^{(2)}) &= 0 \quad \text{for } 1 \leqslant j \leqslant M-1,\\ e_w^{(n)}(b) &= 0,\\ e_{w,x}^{(n)}(0) &= e_{v,x}^{(n)}(0),\end{aligned}$$

where we have set $\delta_n = \lambda_n - \lambda_{n-1}$ for $n \geqslant 2$. For fixed θ we have the relation

$$\delta_{n+1} = \theta e_w^{(n)}(0) + (1-\theta) e_v^{(n)}(0) \quad \text{for } n \geqslant 2.$$

Let us further define error functions e_v, e_w defined on $(-a, 0)$, $(0, b)$ and $\hat{e}_v$, $\hat{e}_w$ defined on $(-1, 1)$. e_v, e_w result from $\hat{e}_v$, $\hat{e}_w$ by the coordinate transform

$$e_v(x) = \hat{e}_v\Big(\frac{2}{a}x + 1\Big), \quad e_w(x) = \hat{e}_w\Big(-\frac{2}{b}x + 1\Big).$$

$\hat{e}_v$ is defined by

$$\begin{aligned}\Big(-\varepsilon\frac{2}{a}\hat{e}_{v,\xi\xi} + \hat{e}_{v,\xi}\Big)(\xi_j^N) &= 0, \quad 1 \leqslant j \leqslant N-1,\\ \hat{e}_v(-1) &= 0,\\ \hat{e}_v(1) &= 1;\end{aligned}$$

$\hat{e}_w$ is defined by

$$\begin{aligned}\Big(-\varepsilon\frac{2}{b}\hat{e}_{w,\xi\xi} + \hat{e}_{w,\xi}\Big)(\xi_j^M) &= 0, \quad 1 \leqslant j \leqslant M-1,\\ \hat{e}_w(-1) &= 0,\\ \hat{e}_{w,\xi}(1) &= -\frac{b}{a}\hat{e}_{v,\xi}(1).\end{aligned}$$

Furthermore, for δ_n we have the recurrence relation

$$\begin{aligned}\delta_{n+1} &= (\theta e_w(0) + 1 - \theta)\delta_n \\ &= (\theta \hat{e}_w(1) + 1 - \theta)\delta_n.\end{aligned}$$

This means that the iterative procedure converges iff

$$|\theta \hat{e}_w(1) + 1 - \theta| < 1.$$

In order to give a more precise criterion for convergence we have to determine the functions $\hat{e}_v$, $\hat{e}_w$. This can be accomplished by a technique introduced by Canuto [1]. We set $\varepsilon_a = \frac{2}{a}\varepsilon$ and obtain from [1, (3.11)]

$$\hat{e}_v(\xi) = \lambda + \mu \hat{w}(\xi), \tag{3.7}$$

where

$$\hat{w}(\xi) = \sum_{m=1}^{N} w_m T_m(\xi), \quad T_m(\xi) = \cos(m \arccos \xi)$$

with

$$\begin{cases} \hat{w}_N = 1, \\ \hat{w}_m = \dfrac{2\varepsilon_a}{c_m} \displaystyle\sum_{\substack{k \geqslant m+1 \\ |k-m| \text{ odd}}} k\hat{w}_k + \frac{2N\varepsilon_a}{\sigma_{m,N}}, \quad 1 \leqslant m \leqslant N-1 \end{cases}$$

and

$$\sigma_{m,N} = \begin{cases} 1, & |m-N| \text{ odd}, \\ 0, & \text{else}, \end{cases}$$

$$c_m = \begin{cases} 2, & m = 0, \\ 1, & m > 0. \end{cases}$$

λ, μ are determined by the boundary conditions. One obtains

$$\lambda = \frac{1}{2}\left[1 - \frac{k \sum_{\text{even } k \neq 0}^{N} \hat{w}_k}{k \sum_{k \text{ odd}} \hat{w}_k}\right],$$

$$\mu = \frac{1}{2 \sum_{k \text{ odd}}^{N} \hat{w}_k}.$$

From (3.7) and by using again the equality $T_m'(1) = m^2$ we derive

$$\begin{aligned}\hat{e}_{v,\xi}(1) = \mu \hat{w}_\xi(1) &= \mu \sum_{m=1}^{N} \hat{w}_m T_m'(1) = \mu \sum_{m=1}^{N} \hat{w}_m m^2 \\ &= \frac{1}{2} \frac{\sum_{m=1}^{N} \hat{w}_m m^2}{\sum_{k \text{ odd}}^{N} \hat{w}_k} =: G_N(\varepsilon_a).\end{aligned}$$

By a similar treatment we can calculate $\hat{e}_w$. Setting $\varepsilon_b = \frac{2}{b}\varepsilon$ we obtain

$$\hat{e}_w(\xi) = \varrho + \sigma\hat{z}(\xi),$$

where

$$\hat{z}(\xi) = \sum_{m=1}^{M} \hat{z}_m T_m(\xi)$$

with

$$\hat{z}_M = 1,$$

$$\hat{z}_m = -\frac{2\varepsilon_b}{x_m} \sum_{\substack{k \geqslant m+1 \\ |k-m| \text{ odd}}}^{M} k\hat{z}_k - \frac{2M\varepsilon_b}{c_m}\sigma_{m,N}, \quad 1 \leqslant m \leqslant M-1.$$

The parameters ϱ and σ are determined so that the boundary conditions are fulfilled. This means:

$$\hat{e}_w(-1) = \varrho + \sigma \sum_{m=1}^{M} \hat{z}_m(-1)^m = 0,$$

$$\hat{e}_{w,\xi}(1) = \varrho \sum_{m=1}^{M} \hat{z}_m m^2 = -\frac{b}{a} G_N(\varepsilon_a).$$

This implies

$$\sigma = -\frac{b}{a}\frac{G_N(\varepsilon_a)}{\sum_{m=1}^{M} \hat{z}_m m^2},$$

$$\varrho = \frac{b}{a}\frac{\sum_{m=1}^{M} \hat{z}_m(-1)^m}{\sum_{m=1}^{M} \hat{z}_m m^2} G_N(\varepsilon_a).$$

By these formulas we obtain

$$\begin{aligned}
\hat{e}_w(1) &= \varrho + \sigma \sum_{m=1}^{M} \hat{z}_m \\
&= \frac{b}{a}\left(\frac{\sum_{m=1}^{M} \hat{z}_m(-1)^m}{\sum_{m=1}^{M} \hat{z}_m m^2} - \frac{\sum_{m=1}^{M} \hat{z}_m}{\sum_{m=1}^{M} \hat{z}_m m^2}\right) G_N(\varepsilon_a) \\
&= \frac{b}{a}\frac{2\sum_{m \text{ odd}}^{M} \hat{z}_m}{\sum_{m=1}^{M} \hat{z}_m m^2} G_N(\varepsilon_a) \\
&= \frac{b}{a}\frac{G_N(\varepsilon_a)}{G_M(\varepsilon_b)},
\end{aligned}$$

where we have set $G_M(\varepsilon_b) = \frac{\sum_{m=1}^{M} \hat{z}_m m^2}{2\sum_{m \text{ odd}}^{M} \hat{z}_m}$. Let us further define

$$\varrho_{N,M}(\varepsilon) = \frac{b}{a}\frac{G_N(\varepsilon_a)}{G_M(\varepsilon_b)}.$$

From the above consideration we now derive the following proposition:

Proposition 3.2. *The interface relaxation procedure* (3.5), (3.6) *converges provided* θ *satisfies*

$$0 < \theta < \theta^*_{N,M}(\varepsilon),$$

where

$$\theta_{N,M}(\varepsilon) = 2\big(1 + \varrho_{N,M}(\varepsilon)\big)^{-1}.$$

The optimal relaxation parameter is given by

$$\theta_{\text{opt}} = \tfrac{1}{2}\theta^*_{N,M}(\varepsilon).$$

With this choice the procedure (3.5), (3.6) *converges after two steps.*

Remark 3.1. If there is no relaxation ($\theta = 1$) the interface relaxation procedure converges iff

$$\frac{b}{G_M(\varepsilon_b)} < \frac{a}{G_N(\varepsilon_a)}.$$

For the automatic parameter choice we obtain $\theta_2 = \theta_{\text{opt}}$ and hence convergence after 3 steps, no matter how λ_1 is chosen.

Remark 3.2. Obviously, for increasing N, M the function $\varrho_{N,M}(\varepsilon)$ approximates $\varrho(\varepsilon)$. Hence we also obtain

$$\theta^*(\varepsilon) = \lim_{N,M\to\infty} \theta_{N,M}(\varepsilon) \quad \text{for all } \varepsilon > 0.$$

Therefore, in general (if N and M are not too small), the—easily available—relaxation parameter from the continuous case can be adopted.

4.1. The continuous case. We now consider the two-dimensional singular perturbed problem (1.2). We are interested in a decomposition of problem (1.2) into two subdomains, say $\Omega_a = (-a, 0) \times (-1, 1)$ and $\Omega_b = (0, b) \times (-1, 1)$. Let Γ denote the interface between Ω_a and Ω_b, i.e. $\Gamma = \{0\} \times (-1, 1)$. If we further denote by v the restriction of u to Ω_a and by w the restriction of u to Ω_b, it follows that (1.2) is equivalent to the split problem

$$\begin{aligned} -\varepsilon \Delta v + v_x &= g \quad \text{in } \Omega_a, \\ v &= 0 \quad \text{on } \partial\Omega_a \setminus \Gamma, \\ v &= w \quad \text{on } \Gamma, \\ v_x &= w_x \quad \text{on } \Gamma, \\ w &= 0 \quad \text{on } \partial\Omega_b \setminus \Gamma, \\ -\varepsilon \Delta w + w_x &= g \quad \text{in } \Omega_b. \end{aligned}$$

The interface relaxation procedure now reads as follows: λ_1 is a given function defined on Γ. We define sequences of functions $v^{(n)}$ on Ω_a and $w^{(n)}$ on Ω_b, $n \geqslant 1$, by the iteration

$$\begin{aligned} -\varepsilon \Delta v^{(n)} + v_x^{(n)} &= g \quad \text{in } \Omega_a, \\ v^{(n)} &= 0 \quad \text{on } \partial\Omega_a \setminus \Gamma, \\ v^{(n)} &= \lambda_n \quad \text{on } \Gamma \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} -\varepsilon \Delta w^{(n)} + w_x^{(n)} &= g \quad \text{in } \Omega_b, \\ w^{(n)} &= 0 \quad \text{on } \partial\Omega_b \setminus \Gamma, \\ w_x^{(n)} &= v_x^{(n)} \quad \text{on } \Gamma. \end{aligned} \tag{4.2}$$

λ_n ($n \geqslant 2$) are defined by the recursive formula

$$\lambda_{n+1} = \theta w^{(n)} + (1 - \theta)\lambda_n \quad \text{on } \Gamma, \ n \geqslant 1.$$

Here $\theta \in (0, 1]$ denotes the relaxation parameter. Following the analysis in one dimension we once more introduce error functions

$$e_v^{(n)} = v^{(n)} - v^{(n-1)}, \quad e_w^{(n)} = w^{(n)} - w^{(n-1)}, \quad n \geqslant 2.$$

By using the error functions the interface relaxation procedure can now equivalently be written as

$$
\begin{aligned}
-\varepsilon \Delta e_v^{(n)} + e_{v,x}^{(n)} &= 0 \quad \text{in } \Omega_a, \\
e_v^{(n)} &= 0 \quad \text{on } \partial\Omega_a \setminus \Gamma, \\
e_v^{(n)} &= \delta_n \quad \text{on } \Gamma
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
-\varepsilon \Delta e_w^{(n)} + e_{w,x}^{(n)} &= 0 \quad \text{in } \Omega_b, \\
e_w^{(n)} &= 0 \quad \text{on } \partial\Omega_b \setminus \Gamma, \\
e_{w,x}^{(n)} &= e_{v,x}^{(n)} \quad \text{on } \Gamma.
\end{aligned} \tag{4.4}
$$

δ_n are defined by the recursive formula

$$
\delta_{n+1} = \theta e_w^{(n)} + (1-\theta) e_v^{(n)} \quad \text{on } \Gamma, \ n \geqslant 2.
$$

By means of the representation with the error functions it is possible to derive a convergence result.

Proposition 4.1. *The interface relaxation procedure* (4.1), (4.2) *converges provided* θ *satisfies*

$$
0 < \theta < \theta^*(\varepsilon),
$$

where

$$
\theta^*(\varepsilon) = 2 \inf_{k=1,2,\ldots} \{\varrho_k(\varepsilon)\}.
$$

and

$$
\varrho_k(\varepsilon) = 1 - \frac{\alpha_k^+ \mathrm{e}^{\beta_k a} - \alpha_k^-}{\mathrm{e}^{\beta_k a} - 1} \cdot \frac{\mathrm{e}^{\beta_k b} - 1}{\alpha_k^- \mathrm{e}^{\beta_k b} - \alpha_k^+}
$$

with

$$
\begin{aligned}
\alpha_k^{\pm} &= \tfrac{1}{2}\varepsilon^{-1} \pm \sqrt{\tfrac{1}{4}\varepsilon^{-2} + \tfrac{1}{4}k^2\pi^2}, \\
\beta_k &= \alpha_k^+ - \alpha_k^-.
\end{aligned}
$$

Proof. We solve the problem (4.3), (4.4) by separation of variables. Let $\{\psi_k, \ k \geqslant 1\}$ be the eigensystem of the eigenvalue problem

$$
\begin{aligned}
-\psi_k''(y) &= \left(\frac{k\pi}{2}\right)^2 \psi_k(y), \quad -1 \leqslant y \leqslant 1, \\
\psi_k(-1) &= \psi_k(1) = 0.
\end{aligned}
$$

Let us further denote by $\{\varphi_k, k \geqslant 1\}$ and $\{\chi_k, k \geqslant 1\}$ the eigensystems to the problems

$$
\begin{aligned}
&-\varepsilon\varphi_k''(x) + \varphi_k(x) = -\varepsilon\Big(\frac{k\pi}{2}\Big)^2 \varphi_k(x), \\
&\varphi_k(-a) = 0, \quad \varphi_k(0) = 1
\end{aligned}
$$

and

$$
\begin{aligned}
&-e\chi_k''(x) + \chi_k'(x) = -\varepsilon\Big(\frac{k\pi}{2}\Big)^2 \chi_k(x), \\
&\chi_k(b) = 0, \quad \chi_k'(0) = 1.
\end{aligned}
$$

The eigenfunctions are explicitly given by

$$
\psi_k(y) = \sin\frac{k\pi}{2}(y+1), \quad k \geqslant 1
$$

and

$$
\begin{aligned}
\varphi_k(x) &= \frac{1}{\mathrm{e}^{-\alpha_k^- a} - \mathrm{e}^{-\alpha_k^+ a}}(\mathrm{e}^{\alpha_k^+ x - \alpha_k^- a} - \mathrm{e}^{\alpha_k^- x - \alpha_k^+ a}), \\
\chi_k(x) &= \frac{1}{\alpha_k^+ \mathrm{e}^{\alpha_k^- b} - \alpha_k^- \mathrm{e}^{\alpha_k^+ b}}(\mathrm{e}^{\alpha_k^+ x + \alpha_k^- b} - \mathrm{e}^{\alpha_k^- x + \alpha_k^+ b}).
\end{aligned}
$$

Now $e_v^{(n)}$, $e_w^{(n)}$ can be expanded as follows:

$$
e_v^{(n)} = \sum_{k=1}^{\infty} \beta_k^{(n)} \varphi_k(x)\psi_k(y),
$$

where

$$
\beta_k^{(n)} = \int_{-1}^{1} \delta_n(y)\psi_k(y)\,\mathrm{d}y, \quad k \geqslant 1
$$

and

$$
e_w^{(n)} = \sum_{k=1}^{\infty} \gamma_k^{(n)} \chi_k(x)\psi_k(y),
$$

where

$$
\gamma_k^{(n)} = \int_{-1}^{1} e_{v,x}^{(n)}(0,y)\psi_k(y)\,\mathrm{d}y\,.
$$

By the interface condition on the x—derivative we immediately obtain

$$
\gamma_k^{(n)} = \varphi_k'(0)\beta_k^{(n)}.
$$

This leads to the recurence relation

$$\beta_k^{n+1} = \big(\theta\varphi_k'(0)\chi_k(0) + (1-\theta)\varphi_k(0)\big)\beta_k^n \quad \text{for } k \geqslant 1,\ n \geqslant 2.$$

We have $\varphi_k(0) = 1$ and an explicit calculation yields

$$\varphi_k'(0)\chi_k(0) = \frac{\mathrm{e}^{\alpha_k^- b} - \mathrm{e}^{\alpha_k^+ b}}{\mathrm{e}^{-\alpha_k^- a} - \mathrm{e}^{-\alpha_k^+ a}} \frac{\alpha_k^+ \mathrm{e}^{-\alpha_k^- a} - \alpha_k^- \mathrm{e}^{-\alpha_k^+ a}}{\alpha_k^+ \mathrm{e}^{\alpha_k^- b} - \alpha_k^- \mathrm{e}^{\alpha_k^+ b}}$$

By simple transforms one finally obtains

$$\beta_k^{n+1} = \big(1 - \theta\varrho_k(\varepsilon)\big)\beta_k^n.$$

Hence the interface relaxation procedure converges iff

$$|1 - \theta\varrho_k(\varepsilon)| < 1 \quad \text{for all } k \geqslant 1.$$

This leads to the convergence criterion stated in the proposition. □

Now let us discuss the behaviour of $\varrho_k(\varepsilon)$ and the resulting restriction on the relaxation parameter θ. In particular, we are interested in situations with small ε. Because of the exponencial terms we approximately obtain

$$\varrho_k(\varepsilon) \sim 1 - \frac{\alpha_k^+}{\alpha_k^-}.$$

Since $\alpha_k^+ \alpha_k^- = -\frac{k^2\pi^2}{4}$ we further derive

$$\varrho_k(\varepsilon) \sim 1 + \frac{4}{k^2\pi^2}(\alpha_k^+)^2.$$

This implies

$$\varrho_k(\varepsilon) \sim 2 + \frac{2\varepsilon^{-2}}{k^2\pi^2} + \frac{4\varepsilon^{-1}}{k^2\pi^2}\sqrt{\tfrac{1}{4}\varepsilon^{-2} + \frac{k^2\pi^2}{4}}.$$

For fixed k and $\varepsilon \to 0$ we obtain

$$\varrho_k(\varepsilon) \sim O(\varepsilon^{-2}) \quad (\varepsilon \to 0).$$

On the other hand, for fixed ε and $k \to \infty$ we derive

$$\varrho_k(\varepsilon) \sim 2.$$

Altogether, we obtain

$$2 < \varrho_k(\varepsilon) < O(\varepsilon^{-2}).$$

This bound on $\varrho_k(\varepsilon)$ leads to a qualitative bound in θ of the form

$$0 < \theta < O(\varepsilon^2).$$

Once more we are especially interested in the case when the interface lies near the boundary layer. We consider the case when $b = O(\varepsilon)$, say $b = \sigma\varepsilon$, $\sigma > 0$. For $\varepsilon \to 0$ we obtain

$$\varrho_k(\varepsilon) \sim 1 - \alpha_k^+ \frac{\mathrm{e}^\sigma - 1}{\alpha_k^- \cdot \mathrm{e}^\sigma - \alpha_k^+} \sim \mathrm{e}^\sigma.$$

This means that now $\varrho_k(\varepsilon)$ becomes independent of ε. The corresponding optimal relaxation parameter is $\theta_{\mathrm{opt}} = \mathrm{e}^{-\sigma}$. With this choice one should get convergence of the interface relaxation procedure in a few steps (see also Table 4.1).

For a definition of the dynamical parameter choice we introduce the inner product of the space $L^2(\Gamma)$

$$(u, v) = \int_{-1}^{1} u(0, y)v(0, y)\,\mathrm{d}y$$

and its associated norm $\|u\| = (u, u)^{1/2}$. As was already mentioned in [6] the automatic parameters are given by

$$\theta_n = \frac{(e_v^{(n)}, e_v^{(n)} - e_w^{(n)})}{\|e_v^{(n)} - e_w^{(n)}\|^2}, \quad n \geqslant 2.$$

The first relaxation parameter θ_1 should be prescribed. A geometric interpretation of this choice is given in [6].

A similar treatment can be applied if the singular perturbed operator is of the "stabilized" form (see (2.1))

$$-\tilde{\varepsilon}u_{xx} - \varepsilon u_{yy} + u_x,$$

where $\tilde{\varepsilon} \gg \varepsilon > 0$. The statement of Proposition 4.1 still holds with $\alpha_k^\pm$ replaced by $\tilde{\alpha}_k^\pm$, where

$$\tilde{\alpha}_k^\pm = \tfrac{1}{2}\tilde{\varepsilon}^{-1} \pm \sqrt{\tfrac{1}{4}\tilde{\varepsilon}^{-2} + \tfrac{1}{4}k^2\pi^2\varepsilon\tilde{\varepsilon}^{-1}}.$$

Hence we obtain convergence iff

$$0 < \theta < \tilde{\theta}^*(\varepsilon),$$

where

$$\tilde{\theta}^*(\varepsilon) = 2 \cdot \inf_{k=1,2,\ldots} \{\tilde{\varrho}_k(\varepsilon)\}.$$

Here $\tilde{\varrho}_k(\varepsilon)$ is constructed in the same way as $\varrho_k(\varepsilon)$ with $\alpha_k^{\pm}$ replaced by $\tilde{\alpha}_k^{\pm}$. A study of the qualitative behaviour of $\tilde{\varrho}_k(\varepsilon)$ for $\varepsilon \to 0$ yields

$$\tilde{\varrho}_k(\varepsilon) \sim 1 + \frac{4}{k^2\pi^2}(\varepsilon\tilde{\varepsilon})^{-1},$$

which implies

$$0 < \theta < O(\varepsilon\tilde{\varepsilon})^{-1} \quad \text{for } \varepsilon \to 0.$$

If we allow b to depend on ε, say $b = \sigma\varepsilon$ $(\sigma > 0)$, then we obtain

$$\tilde{\varrho}_k(\varepsilon) \sim 1$$

and therefore the optimal relaxation parameter becomes $\theta_{\text{opt}} = 1$ (no relaxation). The numerical results in Section 4.3 confirm our theoretical considerations.

4.2. Spectral collocation method. For an introduction of the spectral collocation method we define a subspace of polynomials

$$\mathbf{P}_{N,L} = \{p_k(x)p_\ell(y)\colon p_k, p_\ell \text{ are polynomials of degree } \leqslant k \text{ and } \leqslant \ell, \text{ respectively, for } 0 \leqslant k \leqslant N,\ 0 \leqslant \ell \leqslant L\}.$$

Let the nodes $x_j^{(1)}$ $(j = 0, \ldots, N)$, $x_j^{(2)}$ $(j = 0, \ldots, M)$ be defined as in Section 3.2. In $(-1, 1)$ we further define nodes $y_\ell = \cos\frac{\ell\pi}{L}$, $\ell = 0, \ldots, L$. The spectral collocation method corresponding to our split problem now reads as follows: Find $v_N \in \mathbf{P}_{N,L}$, $w_M \in \mathbf{P}_{M,L}$ such that

$$\begin{aligned}
(-\varepsilon\Delta v_N + v_{N,x})(x_j^{(1)}, y_\ell) &= g(x_j^{(1)}, y_\ell) \quad \text{for } 1 \leqslant j \leqslant N-1,\ 1 \leqslant \ell \leqslant L-1,\\
v_N(x_j^{(1)}, y_\ell) &= 0 \quad \text{for } j = N,\ 0 \leqslant \ell \leqslant L,\ \ell = 0, L,\ 0 \leqslant j \leqslant N,\\
v_N(0, y_\ell) &= w_M(0, y_\ell) \quad \text{for } 1 \leqslant \ell \leqslant L-1,\\
v_{N,x}(0, y_\ell) &= w_{M,x}(0, y_\ell) \quad \text{for } 1 \leqslant \ell \leqslant L-1,\\
w_M(x_j^{(2)}, y_\ell) &= 0 \quad \text{for } j = 0,\ 0 \leqslant \ell \leqslant L,\ \ell = 0, L,\ 0 \leqslant j \leqslant M,\\
(-\varepsilon\Delta w_M + w_{M,x})(x_j^{(2)}, y_\ell) &= g(x_j^{(2)}, y_\ell) \quad \text{for } 1 \leqslant j \leqslant M-1,\ 1 \leqslant \ell \leqslant L-1.
\end{aligned}$$

The corresponding interface relaxation procedure now reads as follows: Find two sequences, $v_N^{(n)} \in \mathbf{P}_{N,L}$ and $w_M^{(n)} \in \mathbf{P}_{M,L}$, for $n \geqslant 1$, by solving the collocation

problems

$$
\begin{aligned}
(-\varepsilon\Delta v_{N,x}^{(n)} + v_{N,x}^{(n)})(x_j^{(1)}, y_\ell) &= g(x_j^{(1)}, y_\ell) \quad \text{for } 1 \leqslant j \leqslant N-1,\ 1 \leqslant \ell \leqslant L-1, \\
v_n^{(n)}(x_j^{(1)}, y_\ell) &= 0 \quad \text{for } j = N,\ 0 \leqslant \ell \leqslant L,\ \ell = 0, L,\ 0 \leqslant j \leqslant N, \\
v_n^{(n)}(0, y_\ell) &= \lambda_n(y_\ell) \quad \text{for } 1 \leqslant \ell \leqslant L-1,
\end{aligned}
$$

and

$$
\begin{aligned}
(-\varepsilon\Delta w_{M,x}^{(n)} + w_{M,x}^{(n)})(x_j^{(2)}, y_\ell) &= g(x_j^{(2)}, y_\ell) \quad \text{for } 1 \leqslant j \leqslant M-1,\ 1 \leqslant \ell \leqslant L-1, \\
w_M^{(n)}(x_j^{(2)}, y_\ell) &= 0 \quad \text{for } j = 0,\ 0 \leqslant \ell \leqslant L,\ \ell = 0, L,\ 0 \leqslant j \leqslant M, \\
w_{M,x}^{(n)}(0, y_\ell) &= v_{N,x}^{(n)}(0, y_\ell) \quad \text{for } 1 \leqslant \ell \leqslant L-1.
\end{aligned}
$$

Here λ_1 is a polynomial of degree $\leqslant L$ which is defined on the interface Γ. The λ_n ($n \geqslant 2$) are recursively defined by the formula

$$
\lambda_{n+1} = \theta w_M^{(n)} + (1-\theta)\lambda_n \quad \text{on } \Gamma,\ n \geqslant 1.
$$

Here $\theta \in (0, 1]$ is a fixed relaxation parameter. In order to give a definition of the dynamical parameter choice we once more introduce the error functions

$$
e_{v,N}^{(n)} = v_N^{(n)} - v_N^{(n-1)} \in \mathbb{P}_{N,L}, \quad e_{w,M}^{(n)} = w_M^{(n)} - w_M^{(n-1)} \in \mathbb{P}_{M,L}.
$$

The discrete inner product in $L^2(\Gamma)$ is given by

$$
(u, v)_2 = \sum_{\ell=0}^{L} u(0, y_\ell) v(0, y_\ell) \omega_\ell,
$$

where ω_ℓ ($\ell = 0, \ldots, L$) denote the Chebyshev-Gauss-Lobatto weights. Now, let λ_1 be a given parameter. Then θ_n ($n \geqslant 2$) are automatically given by

$$
\theta_n = \frac{(e_{v,N}^{(n)}, e_{v,N}^{(n)} - e_{w,M}^{(n)})_2}{\|e_{v,N}^{(n)} - e_{w,M}^{(n)}\|_2^2}, \tag{4.5}
$$

where $\|u\|_2^2 = (u, u)_2$. The error function $e_{v,N}^{(n)}$ can be expanded as follows:

$$
e_{v,N}^{(n)}(x, y) = \sum_{k=1}^{L-1} \beta_k^{(n)} \varphi_k(x) \psi_k(y).
$$

Here φ_k $(k = 1, \ldots, L-1)$ is a polynomial of degree N and ψ_k $(k = 1, \ldots, L-1)$ is a polynomial of degree L. The polynomials ψ_k are solutions of the discrete eigenvalue problem

$$\psi_k''(y_l) = \tilde{\mu}_k \psi_k(y_l), \quad l = 1, \ldots, L-1,$$
$$\psi_k(-1) = \psi_k(1) = 0.$$

The eigenvalues $\tilde{\mu}_k$ $(k = 1, \ldots, L-1)$ are real and positive. The polynomials φ_k are the solutions to the collocation problem

$$-\varepsilon\varphi_k''(x_j^{(1)}) + \varphi_k'(x_j^{(1)}) = -\varepsilon\tilde{\mu}_k\varphi_k(x_j^{(1)}), \quad j = 1, \ldots, N-1,$$
$$\varphi_k(-a) = 0, \quad \varphi_k(0) = 1.$$

The coefficients $\beta_k^{(n)}$ are determined by the relation

$$\sum_{k=1}^{L-1} \beta_k^{(n)} \psi_k(y) = e_{v,N}^{(n)}(0, y), \quad n \geqslant 2.$$

In the same manner $e_{w,M}^{(n)}$ can be expanded as follows:

$$e_{w,M}^{(n)}(x, y) = \sum_{k=1}^{L-1} \gamma_k^{(n)} \chi_k(x) \psi_k(y).$$

The functions χ_k are polynomials of degree $\leqslant M$ and eigenfunctions of the discrete eigenvalue problem

$$-\varepsilon\chi_k''(x_j^{(2)}) + \chi_k'(x_j^{(2)}) = -\varepsilon\tilde{\mu}_k\chi_k(x_j^{(2)}), \quad j = 1, \ldots, M-1,$$
$$\chi_k(b) = 0, \quad \chi_k'(0) = 1.$$

The coefficients $\gamma_k^{(n)}$ are determined by the relation

$$\sum_{k=1}^{L-1} \gamma_k^{(n)} \psi_k(y) = \frac{\partial e_{w,M}^{(n)}}{\partial x}(0, y).$$

Hence one obtains the recurrence relation

$$\beta_k^{(n+1)} = \left[1 - \theta\left(1 - \varphi_k'(0)\chi_k(0)\right)\right]\beta_k^{(n)} \quad \text{for } 1 \leqslant k \leqslant L-1,\ n \geqslant 2$$

and convergence is achieved iff

$$\left|1 - \theta\left(1 - \varphi_k'(0)\chi_k(0)\right)\right| < 1 \quad \text{for all } k = 1, 2, \ldots.$$

By means of the polynomials φ_k and χ_k it is now possible to derive explicit conditions on convergence (compare the one-dimensional case). For the sake of brevity we leave this out. For increasing N, M the polynomials φ_k, χ_k approximate the corresponding continuous eigenfunctions. Then the more practical criterion from the continuous case can be adopted.

4.3. Numerical results. We consider problem (1.2) with

$$g(x,y) = \left[\varepsilon\pi^2 \frac{e^{\frac{x-b}{\varepsilon}} - e^{-\frac{a+b}{\varepsilon}}}{1 - e^{-\frac{a+b}{\varepsilon}}} - \frac{1}{a+b}\right] \sin(\pi y).$$

Hence the exact solution is given by

$$u(x,y) = \left[\frac{e^{\frac{x-b}{\varepsilon}} - e^{\frac{a+b}{\varepsilon}}}{1 - e^{\frac{a+b}{\varepsilon}}} - \frac{x+a}{b+a}\right] \sin(\pi y).$$

u has a boundary layer of width ε near $x = b$. We examine the interface relaxation procedure for $a = 1$ and $b = 1$ or $b = \varepsilon$. We fix $N = M = L = 12$ and study the stable method with $\varepsilon = 1/N$. Different kinds of relaxation parameters are examined:

- fixed relaxation parameter with

$\theta = \varepsilon^2$ for $b = 1$,

$\theta = e^{-1}$ for $b = \varepsilon$;

- dynamical parameter choice (4.5).

We further measure the discretization errors ERV and ERW on Ω_a and Ω_b, respectively:

$$\text{ERV} = \|v_N - v\|_{2,a}, \quad \text{ERW} = \|w_M - w\|_{2,b}.$$

Here $\| \ \|_{2,a}$ and $\| \ \|_{2,b}$ denote respectively the discrete L^2 norms on Ω_a and Ω_b, i.e., in particular,

$$\|v\|_{2,a} = \left[\sum_{j,\ell=0}^{N} u(x_j^{(1)}, y_\ell)^2\right]^{1/2} / N.$$

In Table 4.1 we present numerical results for different values of b and different relaxation parameters. We give the number NIT of interface relaxations which is necessary to reach an accuracy of at least 10 digits. As already results from the numerical analysis, the choice $b = 1$ requires many more relaxations than the choice $b = \varepsilon$. For $b = \varepsilon$ we always obtain convergence after 9 steps.

a	b	θ	NIT	ERV	ERW
1	1	ε^2	557	$.28 \cdot 10^{-5}$	$.50 \cdot 10^{-5}$
1	1	(4.5)	44	$.28 \cdot 10^{-5}$	$.50 \cdot 10^{-5}$
1	ε	e^{-1}	9	$.75 \cdot 10^{-6}$	$.12 \cdot 10^{-5}$
1	ε	(4.5)	9	$.75 \cdot 10^{-6}$	$.12 \cdot 10^{-5}$

Table 4.1. Interface relaxation for problem (1.2).

Now, we consider the two-dimensional example introduced in Section 2. We apply the domain decomposition technique to the stabilized problem with the right hand side $f^2_{\varepsilon,N}$. We once more compare the dynamical parameter choice with the fixed relaxation parameters from the theoretical analysis. We fix $a = 1$ and choose $b = 1$ or $b = \varepsilon$. In the case $b = 1$ the fixed relaxation parameter is given by $\theta = \varepsilon\tilde{\varepsilon}$ and in the case $b = \varepsilon$ the fixed relaxation parameter is $\theta = 1$. In order to reduce the computing time we fix $N = 8$, $\varepsilon = 10^{-2}$, $\tilde{\varepsilon} = 1/N$. In Table 4.2 we collect the numerical results for this example. If $a = b = 1$ the number of iterations is dramatically reduced by the parameter choice (4.5). Furthermore, for $b = \varepsilon$ the number of iterations 3 or 4, respectively. The relaxation parameter generated by (4.5) is approximately equal to that which analytically turned out to be the optimal choice.

a	b	θ	NIT
1	1	$\varepsilon\tilde{\varepsilon}$	1830
1	1	(4.5)	133
1	ε	1	3
1	ε	(4.5)	4

Table 4.2. Interface relaxation for the example in Section 2.

5. Iterative solver and multigrid method

We consider the two-dimensional singular perturbation problem (1.2) and the stabilized version (2.1). We fix $\tilde{\varepsilon} = 1/N$ and $\varepsilon = 10^{-6}$. We intend to solve the spectral problems by an iterative method. In this context the Richardson relaxation (see [8, 9, 13, 14] is an appropriate scheme. Iterative methods for spectral problems are only efficient if fast (Fourier) transform are available. For this purpose the Richardson method is very well suited in contrast to other methods used for finite difference of finite element methods. Because of the high condition number of the spectral operators some kind of preconditioning is necessary. In order to make the preconditioning not too expensive one prefers finite difference of finite element discretizations.

These low order problems themselves are not solved exactly but by a few steps of an appropriate iterative solver.

Here we consider finite difference preconditioning. For the second order derivatives we employ the standard central differences (see [8]). For the first order derivative we also use a central finite difference formula, i.e.

$$u'(x_i) \simeq \frac{1}{2(x_{i-1}-x_i)}u(x_{i-1}) - \frac{1}{2}\Big(\frac{1}{x_{i-1}-x_i} - \frac{1}{x_i - x_{i+1}}\Big)u(x_i)$$
$$- \frac{1}{2(x_i - x_{i+1})}u(x_{i+1}) \quad \text{in } x_i = \cos\frac{i\pi}{N},\ i = 1,\ldots,N-1.$$

Unfortunately, upwind finite differences have very poor preconditioning properties (see [5]).

For preconditioning of problem (1.2) we employ one step of the alternating zebra line relaxation (ZLR) method (see [8, 9]). It consists of relaxing along lines of constant x and y in an alternating manner. Vectorization is achieved by solving first for the odd and then for the even lines. The resulting systems are tridiagonal and symmetric. Hence they can efficiently be solved by means of a Cholesky decomposition. An alternative to this method results from an incomplete LU (ILU)-decomposition. This approach yields similar preconditioning properties as the ZLR. Since the alternating ILU needs twice as much work for the decomposition (see [8]) we prefer the alternating ZLR.

For preconditioning of problem (2.1) we employ one step of ZLR only in the x-direction. Another step in the y-direction is not needed since the coefficient ε is much smaller than $\tilde{\varepsilon}$. In Tables 5.1 and 5.2 we present the minimal and maximal eigenvalues of the precondictioned spectral operators. It becomes obvious that our technique of preconditioning works very well also for increasing N. For the stationary Richardson relaxation with optimal relaxation parameter $\omega_{\text{opt}} = \frac{2}{\lambda_{\max}+\lambda_{\min}}$ we obtain a convergence factor $\varrho_{\text{opt}} = \frac{\lambda_{max}-\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$. We also present ω_{opt}, ϱ_{opt} for different N. It becomes obvious that preconditioning is very effective for problem (2.1). For problem (1.2) the convergence factors are not satisfactory.

N	$\lambda_{\min}$	$\lambda_{\max}$	ω_{opt}	ϱ_{opt}
4	1.18	7.31	0.2356	0.7220
8	0.65	17.1	0.1127	0.9268
16	0.58	65.8	0.0301	0.9825

Table 5.1. $\lambda_{\min}$, $\lambda_{\max}$, ω_{opt}, ϱ_{opt} for the problem (1.2).

N	$\lambda_{\min}$	$\lambda_{\max}$	ω_{opt}	ϱ_{opt}
4	1.10	1.87	0.6734	0.2593
8	1.03	2.20	0.6192	0.3622
16	0.90	2.47	0.5935	0.4659
32	0.42	2.85	0.6116	0.7431

Table 5.2. $\lambda_{\min}$, $\lambda_{\max}$, ω_{opt}, ϱ_{opt} for the problem (2.1).

Finally, we have found suitable components for a *multigrid* method. For the construction of an effective multigrid method we have to consider the stabilized problem (2.1). For relaxation we chose the iterative method which has already been described. The transfer operators for restriction and interpolation are chosen in a standard way (see [8, 13]). The most important aspect for the multigrid method is the construction of the coarse grid problems. Here we tried a lot of different variants but only one gave satisfying results. Here we choose $\tilde{\varepsilon} = 1$ *on all levels.* On the coarse grids the right hand side is *not* perturbed by adding derivatives. It is exactly the restricted residual. Hence on the fine grid we obtain on $O(\varepsilon^2)$ perturbation of the exact solution whereas on the coarse grids we only have an $O(1)$ approximation of the solution. Also different $\tilde{\varepsilon}$, for instance $\tilde{\varepsilon} = 1/N$, result in bad convergence results (with factors of about $0.9 - 1.0$).

In order to estimate the convergence properties of the spectral multigrid method we compute the spectral radius ϱ of the multigrid operator by means of the power method. By $\varrho_W = \varrho^{1/W}$ we define the convergence factor per work unit (see [8]). The standard work unit is the amount of work involved in one relaxation sweep on the finest grid. We compare stationary and nonstationary Richardson (SR and NSR) relaxation. In NSR we allow the parameter to change in each step. For SR relaxation we employ one relaxation before coarse grid correction. For NSR relaxation we employ three relaxation steps before coarse grid correction. The optimal parameters are given in [8].

In the numerical computations we have use a V-cycle with four grids, i.e. $N = 4$, 8, 16 and 32. The convergence factors ϱ_W (see Table 5.3) are acceptable and quite close to the values we obtained for the Laplace operator. We finally remark that a similar treatment is possible for the more general problem (2.2). Here one also has to resort to the corresponding stabilized version.

Relaxation	ϱ	ϱ_W
SR (1 Relax.)	0.4230	0.5232
NSR (3 Relax.)	0.0217	0.3823

Table 5.3. ϱ, ϱ_W for the V-cycle.

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