Adolf Karger

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SPECIAL MOTIONS OF ROBOT-MANIPULATORS

ADOLF KARGER, Praha

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Summary. There exist many examples of closed kinematical chains which have a freedom of motion, but there are very few systematical results in this direction. This paper is devoted to the systematical treatment of 4-parametric closed kinematical chains and we show that the so called Bennet's mechanism is essentially the only 4-parametric closed kinematical chain which has the freedom of motion. According to [3] this question is connected with the problem of existence of asymptotic geodesic lines on robot-manipulators considered as submanifolds of a pseudo-Riemannian space. All computations were performed by the help of a formal manipulation system on a PC-computer.

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THE THEORETICAL PART

The motion of a p-parametric robot-manipulator determined by axes $X_1, \ldots, X_p$ is described by the matrix

$$g(u_1, \ldots, u_p) = g_1(u_1) \cdot \ldots \cdot g_p(u_p), \; g_i(u_i) = \exp(u_i X_i),$$

where $g_i(u_i)$ denotes either the revolution around the straight line determined by its Plücker coordinates X_i or the translation in the direction determined by X_i. In what follows se shall for simplicity consider robot-manipulators with rotational links only and so $\exp(u_i X_i)$ means always the revolution around the straight line X_i with the angle of revolution u_i.

The motion $\hat{g}(u_1, \ldots, u_p)$ of the effector space with respect to the base space is expressed by the formula

$$\hat{g}(u_1, \ldots, u_p)(\overline{R}) = R \cdot g(u_1, \ldots, u_p),$$

where $\bar{R}$ is an orthonormal frame in the effector space, R is an orthonormal frame in the base space and we may suppose that $R = \bar{R}$ at the point $[0, \ldots, 0]$ of the parameter space (in the starting position of the robot-manipulator). This condition yields $g(0, \ldots, 0) = E$.

The instantaneous position $Y_1, \ldots, Y_p$ of axes $X_1, \ldots, X_p$ of a robot-manipulator is given by the formula

$$(1) \qquad Y_1 = X_1,\ Y_2 = \mathrm{Ad}(g_1)X_2,\ \ldots,\ Y_p = \mathrm{Ad}(g_1 \ldots g_{p-1})X_p,$$

where $\mathrm{Ad}(g)X$ denotes the induced (adjoint) action of the matrix g of a space congruence on a straight line determined by its Plücker coordinates X.We can also say that the induced action is the action of the group of space congruences in the space of screws.

Details concerning the above introduced formalism can be found for instance in [1] or [2]. The proof of the formula (1) is easy: Let us consider the motion $h(t) = g(u_1^0, \ldots, u_{i-1}^0, t, u_{i+1}^0, \ldots, u_p^0)$, which is the revolution around Y_i. Then $h'(t)h^{-1}(t) = \mathrm{Ad}(g_1 \ldots g_{i-1})X_i$.

A motion $g(t) = g_1(u_1(t)) \ldots g_p(u_p(t))$ of the p-parametric robot-manipulator, determined by functions $u_1(t), \ldots, u_p(t)$ of one variable t determines a one-parametric motion of the end-effector. Such a motion $g(t)$ of the robot-manipulator will be called a motion of a closed kinematical chain iff $g_1(u_1(t)) \ldots g_p(u_p(t)) = E$. The basic properties of closed kinematical chains are described in [2], the definition complies with the intuitive meaning that during the motion of a closed kinematical chain the distance, angle and offset between the last and first axes of the robot-manipulator remain fixed.

Let a robot-manipulator be given by axes $X_1, \ldots, X_p$. A position of this robot-manipulator determined by the position $Y_1, \ldots, Y_p$ of its axes is called singular iff $\mathrm{rank}(Y_1, \ldots, Y_p) < p$.

Theorem 1. *Motions of closed kinematical chains are possible only in singular positions.*

P r o o f. We have $g_1(t) \ldots g_p(t) = E$. The derivative of this equation with respect to t yields $g_1' \cdot g_2 \ldots g_p + \ldots + g_1 \ldots g_{p-1} \cdot g_p' = 0$. We obtain $Y_1 v_1 + \ldots + Y_p v_p = 0$, where $v_i = u_i'(t)$ and we see that vectors $Y_1, \ldots, Y_p$ are linerly dependent. □

Theorem 2. *Let $g_1(t) \ldots g_p(t) = E$ be a motion of a closed kinematical chain. Then the $p+1$-parametric robot-manipulator $g_1(u_1) \ldots g_p(u_p) \cdot g_{p+1}(u_{p+1})$, where $g_{p+1}(u_{p+1}) = g_1(u_{p+1})$, satisfies the equation $Y_1(t) = Y_{p+1}(t)$ during this motion, $u_{p+1}(t)$ can be arbitrary.*

Proof. We have $Y_{p+1}(t) = \mathrm{Ad}(g_1(t)\dots g_p(t))X_{p+1} = X_{p+1} = X_1 = Y_1$, because $g_1(t)\dots g_p(t) = E$. □

Theorem 3. *Let a $(p+1)$-parametric robot-manipulator satisfy the equation $Y_1 = Y_{p+1}$ during some motion $g(t)$ given by $u_i = u_i(t)$, $i = 1,\dots,p+1$. Then this motion is a motion of a closed kinematical chain with the last link an arbitrary screw motion around the last axis.*

Proof. $Y_1 = Y_{p+1}$ implies $Y_{p+1} = \mathrm{Ad}(g_1(t)\dots g_p(t))X_{p+1} = X_1$. Let us write $X_1 = \mathrm{Ad}\,\gamma X_{p+1}$ for some fixed congruence γ. Then we have $\mathrm{Ad}(g_1(t)\dots g_p(t))X_{p+1} = \mathrm{Ad}\,\gamma X_{p+1}$. This yields $g_1(t)\dots g_p(t)\cdot h(t) = \gamma$, where $h(t)$ is a screw motion around the axis X_{p+1}. Let us choose one position of the robot-manipulator determined by $t = t_0$. Then we have $\gamma = g_1(t_0)\dots g_p(t_0)\cdot h(t_0)$ and we obtain the following equation:

$$g_1(t)\dots g_p(t)\cdot h(t) = g_1(t_0)\dots g_p(t_0)\cdot h(t_0).$$

This equation can be written in the form

$$g_1(t)\dots g_p(t)h(t)h^{-1}(t_0)g_p^{-1}(t_0)\dots g_1^{-1}(t_0) = E.$$

Let us denote

$$k_i(t) = g_1(t_0)\dots g_{i-1}(t_0)\cdot g_i(t)\cdot g_i^{-1}(t_0)\cdot g_{i-1}^{-1}(t_0)\dots g_1^{-1}(t_0),$$
$$m(t) = g_1(t_0)\dots g_p(t_0)\cdot h(t)h(t_0)^{-1}\cdot g_p^{-1}(t_0)\dots g_1^{-1}(t_0).$$

Then we have

$$k_1(t)\dots k_p(t).m(t) = E$$

and $k_i(t)$ is a revolution around the axis $Z_i = \mathrm{Ad}(g_1(t_0)\dots g_{i-1}(t_0))X_i$, $m(t)$ is a screw motion around the axis $Z_{p+1} = \mathrm{Ad}(g_1(t_0)\dots g_p(t_0))X_{p+1}$. □

Theorem 4. *The motion $u_i = u_i(t)$, $i = 1,\dots,p$ of a p-parametric robot-manipulator is a screw motion around some axis X_{p+1} iff the robot manipulator determined by $X_1,\dots,X_{p+1}$ has a motion of a closed kinematical chain with the last link a screw motion.*

Proof. Let $g_1(t)\dots g_p(t) = h(t)$, where $h(t)$ is a screw motion around X_{p+1}. Then we have

$$g_1(t)\dots g_p(t)\cdot h^{-1}(t) = E$$

and the statement follows from the previous considerations.

From Theorems 1, 2, 3, 4 we can deduce the following facts: If we find all solutions of the equation $Y_1 = Y_{p+1}$, we can compute the motion $h(t)$ for each such solution and we have solved the following problems:

a) For $h(t)$ a general screw motion we have find all motions of $p-1$-parametric robot-manipulators, which yield a screw motion of the end-effector (with rotation and translation as special cases). We have also obtained examples of $p+1$-parametric closed kinematical chains with the freedom of motion such that they have a translation and rotation with the same axis.

b) If $h(t)$ is a rotation, we can suppose $h(t) = E$ by changing the representation of the motion as rotations around the same axis commute. We have found all closed kinematical chains with p links, which have a freedom of motion. □

Theorem 5. *The only solution of the equation $Y_1 = Y_5$ is the Bennet's mechanism.*

Remark. The Bennet's mechanism is the closed kinematical chain with four rotational links oriented in such a way that the following relations for Denavit-Hartenberg parameters (see below) are satisfied:

$$d_i = 0,\ i = 1, \ldots 4,\ \alpha_1 = \alpha_3,\ \alpha_2 = \alpha_4,\ m_1 = m_3,\ m_2 = m_4,\ a_1^2 S_2^2 = a_2^2 S_1^2.$$

The trivial cases of all axes parallel and of all axes passing through one point are considered as special cases of the Bennet's mechanism.

Corollaries of Theorem 5.

a) *The Bennet's mechanism is the only 4-parametrical closed kinematical chain with the freedom of motion.*

b) *If one of the links of the Bennet's mechanism is allowed to slide (to perform an arbitrary screw motion), nothing will change and the concerned link will remain rotational.*

c) *There exists not a 3-parametrical robot-manipulator with rotational links, which can perform a translation or a screw motion of the end-effector appart from the revolutions around of its axes.*

Remark. Corollaries follow from Theorems 1 to 4. The Bennet's mechanism is known for a long time already, but its uniquennes was not shown before. The question of the classification of all closed kinematical chains with freedom of movement for 5 links remains open. The computations below show that the solution of such a problem will be extremely complicated even on the assumption that the formal manipulation with equations will be done on a computer as was the case also in the presented paper.

We shall solve the equation $Y_1 = Y_{p+1}$ for $p = 4$. For this purpose one has to compute the instantaneous position of axes of a robot-manipulator. This has been done in [2] for a 6-parametric robot manipulator and we shall use the result of this computation.

For symmetry and simplicity reasons it is convenient to choose as the reference frame the orthonormal frame located between axes X_3 and X_4 in symmetrical position (the origin is at the middle distance between X_3 and X_4, the z axis is perpendicular to both X_3 and X_4 and the direction of x and y axes is in the middle between X_3 and X_4).

The computation yields for Plücker coordinates of axes $Y_1, \ldots, Y_6$, where $Y_i = (y_i; z_i)$:

$$(2) \qquad y_4 = \begin{pmatrix} \kappa \\ \sigma \\ 0 \end{pmatrix}, \quad z_4 = \frac{1}{2}a_3 \begin{pmatrix} -\sigma \\ \kappa \\ 0 \end{pmatrix}, \quad y_5 = \begin{pmatrix} \kappa C_4 - \sigma c_4 S_4 \\ \sigma C_4 + \kappa c_4 S_4 \\ s_4 S_4 \end{pmatrix},$$

$$z_5 = \begin{pmatrix} -\kappa G_4 + \sigma H_4 \\ -\sigma G_4 + \kappa H_4 \\ R_4 \end{pmatrix},$$

where

$$(3) \quad G_4 = S_4\left(a_4 + \frac{1}{2}a_3 c_4\right), \; H_4 = s_4 S_4 d_4 - C_4\left(\frac{1}{2}a_3 + a_4 c_4\right), \; R_4 = d_4 c_4 S_4 + a_4 s_4 C_4,$$

$\kappa = \cos(\frac{1}{2}\alpha_3)$, $\sigma = \sin(\frac{1}{2}\alpha_3)$, and as usually $C_i = \cos\alpha_i$, $S_i = \sin\alpha_i$, $c_i = \cos u_i$, $s_i = \sin u_i$.

$$(4) \qquad y_6 = \begin{pmatrix} -\kappa L_5 - \sigma(c_4 M_5 - s_4 F_5) \\ -\sigma L_5 + \kappa(c_4 M_5 - s_4 F_5) \\ s_4 M_5 + c_4 F_5 \end{pmatrix},$$

$$z_6 = \begin{pmatrix} \kappa[B_5 - \frac{1}{2}a_3(c_4 M_5 - s_4 F_5)] - \sigma(c_4 A_5 - s_4 P_5 - \frac{1}{2}a_3 L_5 \\ \sigma[B_5 - \frac{1}{2}a_3(c_4 M_5 - s_4 F_5)] + \kappa(c_4 A_5 - s_4 P_5 - \frac{1}{2}a_3 L_5 \\ s_4 A_5 + c_4 P_5 \end{pmatrix},$$

where

$$(5) \qquad \begin{aligned} M_5 &= C_4 S_5 c_5 + S_4 C_5, \quad L_5 = S_4 S_5 c_5 - C_4 C_5, \quad F_5 = s_5 S_5, \\ B_5 &= -a_4 M_5 - a_5(S_4 C_5 c_5 + C_4 S_5) + d_5 S_4 F_5, \\ A_5 &= -a_4 L_5 + a_5(C_4 C_5 c_5 - S_4 S_5) - F_5(d_4 + C_4 d_5), \\ P_5 &= a_5 C_5 s_5 + d_5 S_5 c_5 + d_4 M_5. \end{aligned}$$

We used the Denavitt-Hartenberg parameters:
$\alpha_i \rightarrow$ the angle from X_i to X_{i+1},
$a_i \rightarrow$ the distance from X_i to X_{i+1},
$d_i \rightarrow$ the offset between X_{i-1}, X_i and X_i, X_{i+1},
$u_i \rightarrow$ the angle of revolution around the axis X_i.

The formulas for Y_1, Y_2, Y_3 are obtained from formulas for Y_6, Y_5, Y_4 by the following substitution:

$$\alpha \rightarrow -\alpha_{6-i},\; a_i \rightarrow -a_{6-i},\; u_i \rightarrow u_{7-i},\; d_i \rightarrow d_{7-i}. \tag{6}$$

Now we are going to solve the equation $Y_2 = Y_6$. We obtain 6 equations for six Plücker coordinates, from which only 4 are independent (y_2 is a unit vector, y_2 and z_2 are perpendicular). The angles of revolution u_3, u_4, u_5 are the unknowns, at least one of them must be different from a constant. This follows that in general we obtain two equations as solvability conditions. The equations $Y_2 = Y_6$ can be written as follows:

$$\begin{gathered}
\kappa(C_2 + L_5) + \sigma(-S_2c_3 + M_5c_4 - F_5s_4) = 0,\\
\sigma(-C_2 + L_5) + \kappa(-S_2c_3 - M_5c_4 - F_5s_4) = 0,\\
S_2s_3 - F_5c_4 - M_5s_4 = 0,\\
\kappa[-G_2 + B_5 - \tfrac{1}{2}a_3(M_5c_4 - F_5s_4)] + \sigma(-H_5 - \tfrac{1}{2}a_3L_5 + A_5c_4 - P_5s_4) = 0,\\
\sigma[G_2 + B_5 - \tfrac{1}{2}a_3(M_5c_4 - F_5s_4)] + \kappa(-H_2 + \tfrac{1}{2}a_3L_5 - A_5c_4 + P_5s_4) = 0,\\
R_2 - P_5 - A_5s_4 = 0,
\end{gathered} \tag{7}$$

where G_2, H_2, R_2 are defined analogically to G_4, H_4, R_4 using (6).

After making suitable linear combinations in (7) we obtain for $S_3 \neq 0$:

$$\begin{gathered}
r_1 \equiv -C_3M_5c_4 + F_5C_3s_4 + L_5S_3 - S_2c_3 = 0\\
r_2 \equiv S_3M_5c_4 - F_5S_3s_4 + C_2 + C_3L_5 = 0\\
r_3 \equiv -F_5c_4 - M_5s_4 + S_2s_3 = 0\\
r_4 \equiv -A_5C_3c_4 + C_3P_5s_4 - B_5S_3 - C_2a_2c_3 + S_2d_3s_3 = 0\\
r_5 \equiv A_5S_3c_4 - S_3P_5s_4 - B_5C_3 - S_2a_2 - S_2a_3c_3 = 0\\
r_6 \equiv -P_5c_4 - A_5s_4 + S_2c_3d_3 + C_2a_2s_3 = 0
\end{gathered} \tag{8}$$

Let $S_2 \neq 0$. From r_1 and r_3 we obtain

$$c_3 = \frac{1}{S_2}(-C_3M_5c_4 + F_5C_3s_4 + L_5S_3),\; s_3 = \frac{1}{S_2}(F_5c_4 + M_5s_4). \tag{9}$$

Substitution and combination with r_2 yields:

$$(10) \qquad \begin{aligned} r_4 &\equiv (F_5 d_3 - A_5 C_3) S_2 S_3 c_4 + (C_3 P_5 + M_5 d_3) S_2 S_3 s_4 - B_5 S_2 S_3^2 \\ &\quad - C_2^2 C_3 a_2 - C_2 L_5 a_2 - C_2 S_2 S_3 a_3 = 0, \\ r_5 &\equiv A_5 S_3^2 c_4 - P_5 S_3^2 s_4 - B_5 C_3 S_3 - S_2 S_3 a_2 - C_2 C_3 a_3 - L_5 a_3 = 0, \\ r_6 &\equiv (-P_5 S_2 + C_2 F_5 a_2) S_3 c_4 + (-A_5 S_2 + C_2 M_5 a_2) S_3 s_4 + C_2 C_3 S_2 d_3 \\ &\quad + L_5 S_2 d_3 = 0. \end{aligned}$$

We consider equations r_2, r_5 and r_6 as linear equations in c_4 and s_4. They can have common solution only if their determinant is equal to zero;we shall write

$$(11) \qquad \det |r_2, r_5, r_6| = 0.$$

Similarly we have

$$(12) \qquad \det |r_2, r_4, r_5| = 0.$$

Remark. Let

$$(13) \qquad a_1 \cos\varphi + b_1 \sin\varphi + c_1 = 0, \; a_2 \cos\varphi + b_2 \sin\varphi + c_2 = 0,$$

be two equations for unknown angle φ. Equations (13) have a common solution iff

$$(14) \qquad \det \begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix}^2 + \det \begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}^2 - \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}^2 = 0.$$

Equations (11), (12) and (14) for r_2 and r_5 are algebraic in $\cos u_5$ and $\sin u_5$. (We have to take into account the identity $\cos^2 u_5 + \sin^2 u_5 = 1$, we can change (11), (12) and (14) in such a way that there are linear in $\cos u_5$.)If u_5 is constant, we can leave out the axis Y_5 and we obtain a solution of the equation $Y_2 = Y_5$. It is easy to see from the expression for Y_5 in (2) that the equation $Y_2 = Y_5$ has no nontrivial solutions. Therefore we can suppose that (11), (12) and (14) must be satisfied on some interval and so they must be identically zero.

The expanded form of (11) and (14) is very long and complicated, (11) has 197 terms, (14) has 304 terms and therefore we shall not write this equations here in full. We shall solve these equations step by step by choosing suitable coefficients at various powers of $\cos u_5$ and $\sin u_5$.

During the computations we shall use also the following simple fact:

To each solution of the equation $Y_2 = Y_6$ we obtain a new solution of this equation by the substitution $Y_2 \to Y_6$, $Y_3 \to Y_5$, $Y_5 \to Y_3$, $Y_6 \to Y_2$, Y_4 remains.

Now we are ready to solve equations (11) and (14). The coefficients at the highest power in (14) yield: At $c_5 s_5^3$:

$$(15) \qquad -2S_3^4 S_4^4 S_5^4 a_4 d_4 = 0,$$

at s_5^4 :

$$(16) \qquad S_3^2 S_4^2 S_5^4 (S_4^2 a_3^2 - S_3^2 a_4^2 + S_3^2 S_4^2 d_4^2) = 0.$$

A) Let $S_5 = 0$. At s_5^2 in (14) we obtain $S_4 = 0$, this yields $M_5 = F_5 = 0$. From r_2 and r_5 we compute c_4 and c_5 and substitute into (9). We obtain $s_3 = 0$ and therefore there is no solution in this case. If r_2 and r_5 are linearly dependent, we obtain $c_3 = 0$ from r_1.

B) Let $S_5 \neq 0$, $S_4 = 0$. The coefficient at s_5^2 in (14) and (11) leads to $C_3 = C_5 = 0$. The substitution into (9) yields $c_3 = 0$ and we have again no solution.

C) Let $S_5 S_4 \neq 0$. Then from (15) and (16) we obtain $d_4 = 0$, $a_4 = a_3 m S_4 S_3^{-1}$, $m^2 = 1$.

C_1) Let $a_3 = 0$. Then (14) at $c_5 s_5$ yields $d_5(-C_2 S_5 a_2 + C_5 S_2 a_5) = 0$.

$C_1\alpha$). Let $d_5 = 0$.At $c_5 s_5$ in (11) we obtain $a_5 d_3 = 0$.

αo) Let $a_5 \neq 0$. Then $d_3 = 0$. From s_5^2 in (13) we obtain

$$a_2^2 = -2C_2 C_5 S_2^{-1} S_5^{-1} a_2 a_5 - a_5^2 + a_5^2 S_2^{-2} + a_5^2 S_5^{-2}.$$

Substitution into the coefficient at s_5^2 in (11) yields $C_5 S_2 a_2 = C_2 S_5 a_5$. The coefficient at $c_5 s_5$ in (12) yields $C_2 S_5 a_2 = C_5 S_2 a_5$. This yields $S_2^2 = S_5^2$, $a_2^2 = a_5^2$. Similarly we obtain $C_3 S_2 a_2 = C_4 S_5 a_5$ which yields a solution.

αoo) Let $a_5 = 0$. From the coefficient at s_5^2 in (14) we obtain $a_2 = 0$, from the equation r_2 we obtain $d_3 = 0$, we obtain a trivial solution with all axes passing through one point.

$C_1\beta$) From (12) at $c_5 s_5$ we obtain $C_5 d_5 + C_2 d_3 = 0$.

a) Let $C_5 \neq 0$. Then from (11) at s_5^2 we obtain $S_2^2 = S_5^2$ and (12) yields $s_4 = 0$ and we have no solution in this case.

b) Let $C_5 = 0$. From the coefficient at s_5^2 in (11) we obtain $C_2 = 0$, remaining coefficients in (11) lead to $s_4 = 0$ and we have no solution.

C_2) Let $a_3 \neq 0$. The coefficient at s_5^3 in (11) yields $S_3 a_4 d_3 + S_4 a_3 d_5 = 0$, so $d_5 = -d_3 m$. The coefficient at s_5^3 in (14) yields $a_3 d_3 (C_2 - C_5 m) = 0$.

a) Let $C_2 - C_5 m \neq 0$. Then $d_3 = 0$.

From coefficients at $c_5 s_5^2$ in (11) and (14) we obtain

$$a_5 = a_3 S_5 S_3^{-1} (C_5 - C_2 m)^{-1} (C_2 C_4 C_5 + C_3 S_2^2 - C_4 m),$$
$$a_2 = -a_3 S_2 S_3^{-1} (C_5 - C_2 m)^{-1} (C_2 C_3 C_5 + C_4 S_5^2 - C_3 m).$$

$a\alpha$). Let $C_5 = -C_2m$. Then from coefficients at s_5^2 in (11) and (14) we obtain $S_3^2 = S_4^2$, $S_2^2 = S_3^2$, the coefficient at c_5 in (11) yields $C_4 = -C_3m$ and we have a solution.

$a\beta$) Let $S_2^2 \neq S_5^2$. We compute the coefficient at s_5^2 in (11) and we obtain an equation of the type

$AC_3C_4 + B = 0$, where $A = -(S_2^2 + S_5^2)(C_5 - C_2m)$, $B = (C_5 - C_2m)(S_2^2 - 2S_3^2 - 2S_4^2 + S_5^2) - (S_3^2 + S_4^2)(C_2S_5^2 - C_5S_2^2m)$.

From it we obtain the equation $A^2(1 - S_3^2)(1 - S_4^2) - B^2 = 0$, which is of the type $KC_2C_5 + L = 0$, where K and L are polynomials in S_2^2, S_3^2, S_4^2, S_5^2.

The coefficient at s_5^2 in (14) is an equation of the type $PC_3C_4+Q = 0$. Substitution from previous equations leads to the equation

$$(S_2^2 - S_5^2)(S_2^2S_3^2 - S_4^2S_5^2) = 0,$$

which yields $S_5^2 = S_2^2S_3^2S_4^{-2}$. Substitution into the equation $AC_3C_4 + B = 0$ yields $(S_4^2 - S_2^2)(S_4^2 - S_3^2)^4 = 0$. Because $S_4^2 = S_3^2$ leads to $S_5^2 = S_2^2$, we must have $S_4^2 = S_2^2$, $S_5^2 = S_3^2$.

The coefficient at s_5^2 in (13) now yields

$$(S_2^2 - 1)(S_3^2 - 1)(S_3^2 - S_2^2)(C_4C_5 - C_2C_3) = 0.$$

$S_2^2 = S_3^2$ leads to S_2^2, which is impossible. The only possibility is $C_4C_5 = C_2C_3$ as the other possibilities are special cases of this one. We obtain a solution.

b) Let $C_5 = C_2m$. The coefficient at $c_5s_5^2$ in (14) yields $a_2S_2 = S_5a_5m$, the coefficient at $c_5s_5^2$ in (11) yields $C_4 = C_3m$. Now we solve equations r_2 and r_5 for c_4 and s_4 and we obtain $s_4 = 0$. In the case that equations r_2 and r_5 are linearly dependent, we obtain the trivial case with all axes passing through one point. This shows that in this case we also have no solution.

D) Let $S_2 = 0$, $S_3 \neq 0$. We obtain equations

$$F_5c_4 + M_5s_4 = 0, \ -C_3M_5c_4 + F_5C_3s_4 + L_5S_3 = 0. \tag{17}$$

Let at first $S_5 = 0$. Then $M_5 = S_4C_5 = 0$, and therefore $S_4 = 0$. We must have $L_5 = C_4C_5 = 0$, which is impossible. Therefore we must have $S_5 \neq 0$. If $S_4 \neq 0$, we have one of the previous cases for the inverse motion. Therefore we can suppose $S_4 = 0$. (17) implies $C_2C_3 = C_4C_5$. We compute c_3 and s_3 from equations r_4 and r_6 in (8) and consider the equation $c_3^2 + s_3^2 = 1$. This equation yields $a_4 = 0$ which is impossible and there is no solution in this case.

E) Let $S_2 = S_3 = 0$. Then $C_2 + C_3L_5 = 0$, so $C_2 = C_3C_4C_5$ and $S_4 = S_5 = 0$, a solution.All axes are parallel and we have a trivial solution.

F) Let $S_2 \neq 0$, $S_3 = 0$. If $S_4 \neq 0$, we obtain one of the previous cases by taking the inverse motion. So we can suppose $S_4 = 0$. We obtain $C_2 + L_5 = 0$, which yields $C_2 = C_4C_5$. Therefore $S_5 \neq 0$. From equations $-\frac{1}{2}a_3(M_5c_4 - F_5s_4) + B_5 - G_2 = 0$ and $M_5c_4 - F_5s_4 + S_2c_3 = 0$ we obtain $a_4C_4S_5 = 0$, which is impossible and we have no solution in this case.

References

[1] *Karger A.*: Geometry of the motion of robot-manipulators. Manuscr. Math. *62* (1988), 115–126.

[2] *Karger A.*: Classification of singular robot-manipulators. Submitted to Mech. Mach. Theory.

[3] *Karger A.*: Robot-manipulators as submanifolds. Submitted to Mathematica Pannonica.

Author's address: *Adolf Karger*, MÚ UK, Sokolovská 83, 186 00 Praha 8.