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ONE-STEP METHODS FOR TWO-POINT BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS WITH PARAMETERS

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Summary. A general theory of one-step methods for two-point boundary value problems with parameters is developed. On nonuniform nets h_n, one-step schemes are considered. Sufficient conditions for convergence and error estimates are given. Linear or quadratic convergence is obtained by Theorem 1 or 2, respectively.

Keywords: One-step methods, two-point boundary value problems.

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1. INTRODUCTION.

We study the first order nonlinear system of ordinary differential equations

$$y'(t) = f(t, y(t), \lambda), \quad t \in I = [a, b], \quad a < b, \tag{1}$$

with the boundary conditions

$$y(a) = y_a \in \mathbf{R}^q, \tag{2}$$

$$B_1\lambda + B_2 y(b) = b_0 \in \mathbf{R}^p, \tag{3}$$

where $f\colon I \times \mathbf{R}^q \times \mathbf{R}^p \to \mathbf{R}^q$ is continuous and $\lambda \in \mathbf{R}^p$ is a parameter. Here B_1 is a matrix of dimension $p \times p$ and B_2 is a matrix of dimension $p \times q$. By a solution (φ, λ) of the BVP(1–3) we mean a function $\varphi \in C^1(I, \mathbf{R}^q)$ and a parameter $\lambda \in \mathbf{R}^p$ that satisfy the BVP(1–3) ($C^1(I, \mathbf{R}^q)$ denotes the space of all continuous functions from I into $\mathbf{R}^q$ with a continuous first derivative). Conditions under which (1–3) has a solution were determined in many papers (for example, see [4, 9, 10, 11]).

Indeed, $y(t) = y(t;\lambda)$. It is well known that if f has continuous first order partial derivatives f_y and f_λ with respect to the second and third variables, then

$$\frac{\partial y(t;\lambda)}{\partial \lambda} \equiv Y(t;\lambda),$$

where the $q \times p$ matrix Y is the solution of

$$\text{(4)} \qquad \begin{cases} Y'(t;\lambda) = f_y(t, y(t;\lambda), \lambda)Y(t;\lambda) + f_\lambda(t, y(t;\lambda), \lambda), & t \in I, \\ Y(a;\lambda) = 0_{q\times p}. \end{cases}$$

Let $y(t) = y(t;\lambda)$ be a solution of (1-2). It is also a solution of the BVP (1–3) provided (3) is satisfied, that is if λ is a root of the equation

$$\text{(5)} \qquad \Phi(\lambda) \equiv B_1\lambda + B_2 y(b;\lambda) = b_0.$$

Since

$$\text{(6)} \qquad \Phi'(\lambda) = B_1 + B_2 Y(b;\lambda),$$

Newton's method can be used for finding the root of (5).

In the present paper we discuss the numerical solution of the BVP (1–3) using a variable step size $h_n > 0$. On the interval I we place a net of points $\{t_n\}$ with

$$\text{(7)} \qquad t_0 = a, \quad t_{n+1} = t_n + h_n, \quad n = 0, 1, \ldots, N-1 \quad \text{and} \quad t_N = b.$$

Our analysis refers to a family of such nets in which $N \to \infty$ while $h \to 0$ where $h = \max_{n=0,1,\ldots,N-1} h_n$. Now the numerical solution (y_h, λ_{hj}) of (1–3) at each point t_n may be defined by

$$\text{(8)} \qquad \begin{cases} y_h(t_0;\lambda_{hj}) = y_a, \\ y_h(t_{n+1};\lambda_{hj}) = y_h(t_n;\lambda_{hj}) + h_n F(t_n, h_n, y_h(t_n;\lambda_{hj}), \lambda_{hj}), \end{cases}$$

$$\text{(9)} \qquad \begin{cases} Y_h(t_0;\lambda_{hj}) = 0_{q\times p'} \\ Y_h(t_{n+1};\lambda_{hj}) = [I + h_n F_y(t_n, h_n, y_h(t_n;\lambda_{hj}), \lambda_{hj})]Y_h(t_n;\lambda_{hj}) \\ \qquad\qquad + h_n F_\lambda(t_n, h_n, y_h(t_n;\lambda_{hj})\lambda_{hj}), \end{cases}$$

and

$$(10)\qquad \begin{cases} \lambda_{h0} = \lambda_0 \in \mathbf{R}^p, \\ \lambda_{h,j+1} = \lambda_{hj} - [B_1 + B_2 Y_h(b;\lambda_{hj})]^{-1}[B_1\lambda_{hj} + B_2 y_h(b;\lambda_{hj}) - b_0] \end{cases}$$

for $n = 0, 1, \ldots, N-1$ and $j = 0, 1, \ldots$. Here the increment function F has first order partial derivatives F_y and F_λ with respect to the third and fourth variables, respectively. Taking $F = f$ we have the Euler scheme. Sometimes it is useful to write (9) in the following way:

$$(9')\qquad Y_h(t_n;\lambda_{hj}) = \sum_{i=0}^{n-1}\left(\prod_{r=i+1}^{n-1} A_{n+i-r,j}\right) B_{ij},$$

where

$$A_{nj} = I + h_n F_y(t_n, h_n, y_h(t_n;\lambda_{hj}), \lambda_{hj}),$$
$$B_{nj} = h_n F_\lambda(t_n, h_n, y_h(t_n;\lambda_{hj}), \lambda_{hj}).$$

Assume for a moment that $p = q$ and the matrix $B_1 + B_2$ is nonsingular. In such a situation we can determine another sequence $\{\lambda^*_{hj}\}$ by

$$(11)\qquad \lambda^*_{h,j+1} = \lambda^*_{hj} - (B_1 + B_2)^{-1}[B_1\lambda^*_{hj} + B_2 y_h(b;\lambda^*_{hj}) - b_0], \qquad j = 0, 1, \ldots$$

It means that in this case we do not need the approximate solution Y_h of (4). Now the method (8,11) is convergent to the solution (φ, λ) of the BVP(1–3) if we suppose among other that the condition

$$(12)\qquad \|(B_1 + B_2)^{-1}B_2\| \left[1 + \frac{M_2}{M_1}(\exp(M_1(b-a)) - 1)\right] < 1$$

holds where $M_1, M_2 > 0$ are Lipschitz constants of F with respect to the last two variables. This was obtained in [5] for the constant step size h. The condition (12) does not differ too much from the corresponding Keller result [7] (see also [2, 12]).

The condition (12) is superfluous for the convergence of the method (8–10). Assuming that the derivatives F_y and F_λ satisfy the Lipschitz condition we can prove the convergence of (8–10) if λ_0 is not too far from λ. The location of λ_0 is one of the problems in computations. The estimates of errors are given, too. The result of this paper extends the corresponding Keller result [8] to boundary value problems with parameters.

2. DEFINITIONS

We introduce the usual definitions.

Definition 1. We say that the method (8–10) is convergent to the solution (φ, λ) of the BVP(1–3) if

$$\lim_{\substack{N\to\infty \\ j\to\infty}} \max_{n=0,1,\ldots,N} \|y_h(t_n;\lambda_{hj}) - \varphi(t_n)\| = 0$$
$$\lim_{\substack{h\to 0 \\ j\to\infty}} \|\lambda_{hj} - \lambda\| = 0.$$

Definition 2. We say that the method (8–10) is consistent with the problem (1–3) on the solution (φ, λ) if there exist functions $\gamma_1, \gamma_2 \colon I \times H \to \mathbf{R}_+ = [0,\infty)$, $H = [0, h^*]$, $h^* > 0$ such that

(i) $$\left\| h_n F(t_n, h_n, \varphi(t_n), \lambda) + \varphi(t_n) - \varphi(t_{n+1}) \right\| \leqslant \gamma_1(t_n, h_n),$$

(ii) $$\left\| (I + h_n F_y(t_n, h_n, \varphi(t_n), \lambda))\, Y(t_n;\lambda) + h_n F_\lambda(t_n, h_n, \varphi(t_n), \lambda) - Y(t_{n+1};\lambda) \right\| \leqslant \gamma_2(t_n, h_n)$$

for $n = 0, 1, \ldots, N-1$ and

(iii) $$\lim_{h\to 0} \bar{\gamma}_s(h) = 0, \quad \bar{\gamma}_s(h) = \sum_{i=0}^{N-1} \gamma_s(t_i, h_i), \quad s = 1, 2, \quad h = \max_i h_i,$$

where Y is the bounded solution of the IVP(4).

The method (8–10) is said to be H-consistent with (1–3) on (φ, λ) if only the conditions (i) and (iii) (for $s = 1$) are satisfied.

Remark 1. Because (φ, λ) and Y are solutions of (1–3) and (4), respectively, the conditions (i) and (ii) can also be written in the following way:

$$\left\| h_n F(t_n, h_n, \varphi(t_n), \lambda) - \int_{t_n}^{t_{n+1}} f(\tau, \varphi(\tau), \lambda)\, \mathrm{d}\tau \right\| \leqslant \gamma_1(t_n, h_n),$$
$$\Big\| h_n \left[F_y(t_n, h_n, \varphi(t_n), \lambda) Y(t_n;\lambda) + F_\lambda(t_n, h_n, \varphi(t_n), \lambda \right] - \int_{t_n}^{t_{n+1}} \left[f_y(\tau, \varphi(\tau), \lambda) Y(\tau;\lambda) + f_\lambda(\tau, \varphi(\tau), \lambda) \right] \mathrm{d}\tau \Big\| \leqslant \gamma_2(t_n, h_n).$$

It is known that our method is consistent with (1–3) on (φ, λ) if

$$\lim_{h\to 0} F(t,h,y,\lambda) = f(t,y,\lambda),$$
$$\lim_{h\to 0} F_y(t,h,y,\lambda) = f_y(t,y,\lambda),$$
$$\lim_{h\to 0} F_\lambda(t,h,y,\lambda) = f_\lambda(t,y,\lambda)$$

for all $(t,y,\lambda) \in I \times \mathbf{R}^q \times \mathbf{R}^p$.

3. Convergence

We are now in a position to establish the main convergence theorems and the associated error estimates.

Let

$$0 \leqslant z_{n+1} \leqslant D[Az_n^2 + Bz_n + C], \qquad A,B,C,D > 0, \qquad n = 0,1,\ldots.$$

We will need the following lemma.

Lemma 1 (see [6]). *Assume that there exists d such that*

$$DB < d < 1, \quad 4\bar{p}^2AC < 1, \quad \text{where } \bar{p} = \frac{D}{d - DB}.$$

If $z_0 \leqslant \varepsilon = DC/(1-d) \leqslant 1/(\bar{p}A)$ *then*

$$z_n \leqslant d^n\varepsilon + DC\,\frac{1-d^n}{1-d'} \quad n = 0,1,\ldots.$$

Remark 2. It is easy to see that $z_n \leqslant \varepsilon, \quad n = 0,1,\ldots.$

Proof of Lemma 1 [6]. We can write

$$Q(z) = D[Az^2 + Bz + C] = Dq(z) + dz, \quad \text{where} \quad q(z) = Az^2 - z/\bar{p} + C.$$

The quadratic function q has two distinct positive zeros z_- and z_+, where $z_+ > z_- > 0$. The function Q is increasing for $z > 0$ so if $z_0 \leqslant \varepsilon$ then $q(z) \leqslant C$ for $0 \leqslant z \leqslant \varepsilon$ and by induction $z_n \leqslant \varepsilon$ for $n = 0,1,\ldots.$ Now

$$z_{n+1} \leqslant DC + dz_n, \quad n = 0,1,\ldots,$$

and hence we have our estimate for z_n. □

Now we can formulate the theorem.

Theorem 1. *Let the following assumptions be satisfied:*

1° *there exists a unique solution* (φ, λ) *of the BVP* (1–3),

2° *the function* $F\colon I \times H \times \mathbf{R}^q \times \mathbf{R}^p \to \mathbf{R}^q$ *is continuous and has first order partial derivatives* F_y *and* F_λ *with respect to the third and fourth variables, respectively,*

3° *there exist constants* $L_1, L_2, K_1, K_2, K_3 \geqslant 0$ *and functions* $\varepsilon_1, \varepsilon_2\colon I \times H \to \mathbf{R}_+$ *such that for* $(t, h, x, \bar{x}, \mu, \bar{\mu}) \in I \times H \times \mathbf{R}^q \times \mathbf{R}^q \times \mathbf{R}^p \times \mathbf{R}^p$ *we have*

$$\text{(i)} \qquad \|F_y(t,h,x,\mu\| \leqslant L_1, \qquad \|F_\lambda(t,h,x,\mu\| \leqslant L_2;$$

$$\text{(ii)} \qquad \|F_y(t,h,x,\mu) - F_y(t,h,\bar{x},\mu)\| \leqslant K_1\|x-\bar{x}\| + \varepsilon_1(t,h);$$

$$\text{(iii)} \qquad \|F_\lambda(t,h,x,\mu) - F_\lambda(t,h,\bar{x},\bar{\mu})\| \leqslant K_2\|x-\bar{x}\| + K_3\|\mu-\bar{\mu}\| + \varepsilon_2(t,h),$$

and

$$\lim_{h\to 0} \delta_s(h) = 0, \qquad \delta_s(h) = \sum_{i=0}^{N-1} h_i\varepsilon_s(t_i, h_i), \quad s = 1,2, \quad h = \max_i h_i,$$

where the matrix norm is consistent with the vector norm (see [12]);

4° *the method* (8–10) *is* H*-consistent with the BVP*(1–3) *on the solution* (φ, λ)*;*

5° *the matrix* $B_1 + B_2Y_h(b;\lambda_{hj})$ *is nonsingular for* $j = 0, 1, \ldots$ *and there exists a constant* $D > 0$ *such that*

$$\|(B_1 + B_2Y_h(b;\lambda_{hj}))^{-1}B_2\| \leqslant D, \quad j = 0, 1, \ldots.$$

Then for sufficiently small $\bar{h}$ *there exists a positive constant* $d < 1$ *such that the method* (8–10) *is convergent to the solution* (φ, λ) *of the BVP* (1–3) *provided*

$$\text{(13)} \qquad \|\lambda_0 - \lambda\| \leqslant u_0(h) = \sup_{x \leqslant \bar{h}} \frac{DC(x)}{1-d}, \quad h \leqslant \bar{h}.$$

Moreover, the estimates

$$\text{(14)} \qquad \|\lambda_{hj} - \lambda\| \leqslant u_j(h), \quad j = 0, 1, \ldots$$

$$\text{(15)} \qquad \max_{n=0,\ldots,N} \|y_h(t_n;\lambda_{hj}) - \varphi(t_n)\| \leqslant c[L_2(b-a)u_j(h) + \bar{\gamma}_1(h)], \quad j = 0, 1, \ldots$$

hold for $h = \max_i h_i \leqslant \bar{h}$ *with*

$$u_j(h) = d^j \|\lambda_0 - \lambda\| + DC(h)\frac{1-d^j}{1-d}, \quad j = 1, 2, \ldots$$

and

$$C(h) = c\bar{\gamma}_1(h)\left[\frac{K_1}{2}(b-a)c^2\bar{\gamma}_1(h) + c\delta_1(h) + 1\right], \quad c = \exp(L_1(b-a)).$$

Proof. Put

$$\begin{aligned}
&v_n^j = y_h(t_n; \lambda_{hj}) - \varphi(t_n), \quad V_n^j = \|v_n^j\|, \\
&z_h^j = \lambda_{hj} - \lambda, \quad Z_h^j = \|z_h^j\|, \\
&w_n^j = Y_h(t_n; \lambda_{hj}) z_h^j - v_n^j, \quad W_n^j = \|w_n^j\|, \\
&C_n = h_n F(t_n, h_n, \varphi(t_n), \lambda) + \varphi(t_n) - \varphi(t_{n+1}).
\end{aligned}$$

The mean value theorem yields the relation

$$\begin{aligned}
(16) \qquad v_{n+1}^j &= v_n^j + h_n[F(t_n, h_n, y_h(t_n; \lambda_{hj})\lambda_{hj}) \\
&\quad - F(t_n, h_n, \varphi(t_n), \lambda_{hj}) \\
&\quad + F(t_n, h_n, \varphi(t_n), \lambda_{hj}) - F(t_n, h_n, \varphi(t_n), \lambda)] + C_n \\
&= \left[I + h_n \int_0^1 F_y(t_n, h_n, \varphi(t_n) + \tau v_n^j, \lambda_{hj})\, \mathrm{d}\tau\right] v_n^j \\
&\quad + h_n \int_0^1 F_\lambda(t_n, h_n, \varphi(t_n), \lambda + \tau z_h^j)\, \mathrm{d}\tau\, z_h^j + C_n, \\
&\qquad\qquad n = 0, 1, \ldots, N-1,
\end{aligned}$$

or

$$V_{n+1}^j \leqslant (1 + h_n L_1) V_n^j + h_n L_2 Z_h^j + \gamma_1(t_n, h_n), \quad n = 0, 1, \ldots, N-1.$$

Hence we get

$$V_n^j \leqslant \sum_{i=0}^{n-1} \left(\prod_{r=i+1}^{n-1} (1 + h_r L_1)\right) (h_i L_2 Z_h^j + \gamma_1(t_i, h_i))$$

for $n = 0, 1, \ldots, N$, $\quad j = 0, 1, \ldots$ (here $\sum_r^s = 0$, $\prod_r^s = 1$, if $r > s$, or

$$(17) \qquad V_n^j \leqslant c[(b-a)L_2 Z_h^j + \bar{\gamma}_1(h)], \quad n = 0, 1, \ldots, N.$$

Now we need some relation for z_h^j. By the definition (10) we have

$$z_h^{j+1} = (B_1 + B_2 Y_h(b; \lambda_{hj}))^{-1} B_2 w_N^j, \quad j = 0, 1, \ldots \tag{18}$$

By (9) it is easy to see

$$w_{n+1}^j = A_{nj} w_n^j + A_{nj} v_n^j - v_{n+1}^j + B_{nj} z_h^j, \quad n = 0, 1, \ldots, N-1,$$

where A_{nj} and B_{nj} are defined in (9′). According to 3° and (16), the last relation implies

$$W_{n+1}^j \leqslant (1 + h_n L_1) W_n^j + b_n^j$$

with

$$b_n^j = h_n \left[\frac{K_1}{2} (V_n^j)^2 + K_2 V_n^j Z_h^j + \frac{K_3}{2} (Z_h^j)^2 \right] + \gamma_1(t_n, h_n) + h_n \left[\varepsilon_1(t_n, h_n) V_n^j + \varepsilon_2(t_n, h_n) Z_h^j \right]$$

for $n = 0, 1, \ldots, N-1$ and $W_0^j = 0$.

Using now (17) we have

$$W_n^j \leqslant \sum_{i=0}^{n-1} \left(\prod_{r=i+1}^{n-1} (1 + h_r L_1) \right) b_i^j, \quad n = 0, 1, \ldots, N-1, \quad j = 0, 1, \ldots,$$

and hence

$$W_N^j \leqslant A (Z_h^j)^2 + B(h) Z_h^j + C(h), \quad j = 0, 1, \ldots, \tag{19}$$

where

$$A = c(b-a) \left\{ \frac{K_1}{2} (c(b-a) L_2)^2 + K_2 c(b-a) L_2 + \frac{K_3}{2} \right\}.$$

$$B(h) = c \left\{ (b-a) c \left[K_1 c(b-a) L_2 + K_2 \right] \bar{\gamma}_1(h) + c(b-a) L_2 \delta_1(h) + \delta_2(h) \right\}.$$

Combining this with (18) we see that

$$Z_h^{j+1} \leqslant D[A (Z_h^j)^2 + B(h) Z_h^j + C(h)], \quad j = 0, 1, \ldots \tag{20}$$

Now for a sufficiently small $\bar{h}$ there exists a positive constant $d < 1$ such that

$$\begin{cases} DB(h) < d < 1, \\ 4\bar{p}^2(h) A C(h) < 1, \qquad \bar{p}(h) = D/(d - DB(h)), \\ DC(h) A \bar{p}(h) + d \leqslant 1 \end{cases} \tag{21}$$

hold for $h = \max_i h_i \leqslant \bar{h}$. Hence by Lemma 1 we can get (14) and (15) for $h \leqslant \bar{h}$.

The proof is completed. □

Remark 3. Let $p = q = 1$ and

$$F_y(t,h,x,\mu) = h^\alpha(|\sin(x)|)^{1/2} + \xi(t,h,\mu),$$

where $\alpha > 0$ and $\xi: I \times H \times \mathbf{R} \to \mathbf{R}$. The function F_y does not satisfy the Lipschitz condition with respect to the third variable but it satisfies (ii) with $K_1 = 0$ and $\varepsilon_1(t,h) = 2h^\alpha$. Hence $\delta_1(h) \leqslant 2h^\alpha(b-a)$ and $\delta_1(h) \to 0$ as $h \to 0$.

Now we try to formulate some conditions which guarantee that 5° of Theorem 1 holds. We have

Lemma 2. *Let the assumptions 1° – 3° of Theorem 1 hold with* (ii) *replaced by* $\|F_y(t,h,x,\mu) - F_y(t,h,\bar{x},\bar{\mu}\| \leqslant K_1\|x-\bar{x}\| + K_0\|\mu-\bar{\mu}\| + \varepsilon_1(t,h), \quad K_1, K_0 \geqslant 0$. *Let the method (8–10) be consistent with the BVP(1–3) on the solution* (φ, λ). *Moreover, let the matrix* $B_1 + B_2Y(b;\lambda)$ *be nonsingular and*

$$\|(B_1 + B_2Y(b;\lambda))^{-1}\| \leqslant \beta_1, \quad \|B_2\| \leqslant \beta_2.$$

Then for sufficiently small $h \leqslant \bar{h}$ *the condition* 5° *of Theorem 1 holds if* λ_0 *is not too far from* λ.

Proof. Put

$$Q_n(u) = B_1 + B_2Y_h(b;u) \quad Q(u) = B_1 + B_2Y(b;u).$$

Note that for $j = 0, 1, \ldots$

$$Q_h(\lambda_{hj}) = Q(\lambda)\left\{I + Q^{-1}(\lambda)\left[Q_h(\lambda_{hj}) - Q(\lambda)\right]\right\} \tag{22}$$

and

$$Q_h(\lambda_{hj}) - Q(\lambda) = B_2q^j_N, \tag{23}$$

where

$$q^j_n = Y_h(t_n;\lambda_{hj}) - Y(t_n;\lambda), \quad n = 0,1,\ldots,N, \quad j = 0,1,\ldots$$

Now we need an estimate for q^j_N. By the definition of Y_n we have

$$\begin{aligned} q^j_{n+1} = {} & [I + h_nF_y(t_n,h_n,y_h(t_n;\lambda_{hj}),\lambda_{hj})]\,[Y_h(t_n;\lambda_{hj}) - Y(t_n;\lambda)] + Y(t_n;\lambda) \\ & + h_n\,[F_y(t_n,h_n,y_h(t_n;\lambda_{hj}),\lambda_{hj}) - F_y(t_n,h_n,\varphi(t_n),\lambda)]\,Y(t_n;\lambda) \\ & + h_nF_y(t_n,h_n,\varphi(t_n),\lambda)Y(t_n;\lambda) + h_nF_\lambda(t_n,h_n,\varphi(t_n),\lambda) - Y(t_{n+1};\lambda) \\ & + h_n\,[F_\lambda(t_n,h_n,y_h(t_n;\lambda_{hj}),\lambda_{hj}) - F_\lambda(t_n,h_n,\varphi(t_n),\lambda)]\,. \end{aligned}$$

Our assumptions yield

$$Q^j_{n+1} \leqslant (1+h_nL_1)Q^j_n + h_n\left[K_1V^j_n + K_0Z^j_h + \varepsilon_1(t_n,h_n)\right]Y_b + \gamma_2(t_n,h_n)$$
$$+ h_n\left[K_2V^j_n + K_3Z^j_h + \varepsilon_2(t_n,h_n)\right], \qquad Q^j_n = \|q^j_n\|,$$

where Y is bounded by Y_b, V^j_n and Z^j_n are defined in the proof of Theorem 1. Now using the estimate (17) we get

$$Q^j_{n+1} \leqslant (1+h_nL_1)Q^j_n + h_n\left[P_1Z^j_h + P_2\bar{\gamma}_1(h) + Y_b\varepsilon_1(t_n,h_n) + \varepsilon_2(t_n,h_n)\right] + \gamma_2(t_n,h_n)$$

for $n = 0,1,\dots,N-1, j = 0,1,\dots$, where P_1 and P_2 are some nonnegative constants. Hence

$$Q^j_N \leqslant c(b-a)P_1Z^j_h + \eta(h),$$

and for $\beta = \beta_1\beta_2$ we have

$$\|Q^{-1}(\lambda)\,[Q_h(\lambda_{hj}) - Q(\lambda)]\,\| \leqslant \beta Q^j_N \leqslant c\beta(b-a)P_1Z^j_h(h) + \beta\eta(h), \tag{24}$$

where

$$\eta(h) = c\left[(b-a)P_2\bar{\gamma}_1(h) + Y_b\delta_1(h) + \beta_2(h) + \bar{\gamma}_2(h)\right].$$

Let

$$\|\lambda_0 - \lambda\| \leqslant \varrho = \sup_{h\leqslant\bar{h}} DC(h)/(1-d) \quad \text{and} \quad c\beta(b-a)P_1\varrho \leqslant \alpha_1 < 1,$$

where $\bar{h}$ is sufficiently small that (21) holds. It means that there exists α such that for sufficiently small $h < \bar{h}$ we get

$$c\beta(b-a)P_1\varrho + \beta\eta(h) \leqslant \alpha < 1.$$

By Lemma 4.4.14([12]), p. 180) we conclude that $I + Q^{-1}(\lambda)[Q_h(\lambda_0) - Q(\lambda)]$ is nonsingular. Now by (22), $Q_h(\lambda_0)$ is also nonsingular and

$$\|Q_h^{-1}(\lambda_0)\| \leqslant \frac{\beta_1}{1-\alpha}. \tag{25}$$

Hence the condition 5° of Theorem 1 is true for $j = 0$ with $D = \beta/(1-\alpha)$.

Put $u_0(h) = \varrho$. By (20) and Remark 2 we have $Z^1_h \leqslant \varrho$. Moreover, (24) yields

$$\|Q^{-1}(\lambda)\,[Q_h(\lambda_{h1}) - Q(\lambda)]\,\|\alpha < 1.$$

It means that $I + Q^{-1}(\lambda)[Q_h(\lambda_{h1}) - Q(\lambda)]$ is nonsingular and

$$\|Q_h^{-1}(\lambda_{h1})\| \leqslant \frac{\beta_1}{1-\alpha},$$

and hence the condition 5° of Theorem 1 is true for $j = 1$. Now by induction with respect to n we can prove that 5° holds.

This completes the proof. □

Theorem 1 says that under some assumptions the method (8–10) converges to (φ, λ) provided that λ_0 is not far from λ. This convergence is linear. Under a little stronger assumptions we can get quadratic convergence of (8–10). To this end λ_0 must be nearer to λ than it was in Theorem 1. We have

Theorem 2. *Assume that the assumptions of Lemma 2 are satisfied with* $\varepsilon_1(t,h) = \varepsilon_2(t,h) = 0$, $t \in I$, $h \in H$. *Then*

$$\|\lambda_{h,j+1} - \lambda_{hj}\| \leqslant T\|Q_{hj}^{-1}\|\,\|\lambda_{hj} - \lambda_{h,j-1}\|^2, \quad j = 1, 2, \dots \tag{26}$$

where

$$T_0 = c(b-a)[K_2(b-a)L_2c + K_3]/2 + c(b-a)^2L_2[K_1(b-a)L_2c + K_0]/2,$$
$$T = \|B_2\|T_0, \quad Q_{hj} = B_1 + B_2Y_h(b;\lambda_{hj}).$$

Moreover, for a sufficiently small $\bar{h}$ *and* $\|\lambda_{h1} - \lambda_{h0}\| \leqslant e < 1/(TD)$ *the method (8–10) is convergent to* (φ, λ) *and the estimates (14-15) hold for* $h = \max_i h_i \leqslant \bar{h}$ *with*

$$u_j(h) = \frac{1}{TD}(TDe)^{2^{j-1}} + m(h), \quad j = 1, 2, \dots,$$
$$u_0(h) = m(h),$$

where $\|Q_{hj}^{-1}\| \leqslant D$ *and*

$$m(h) = 2\frac{C(h)}{x_h + (x_h^2 - 4AC(h))^{1/2}}, \quad x_h = \frac{1 - DB(h)}{D}.$$

Proof. Let

$$k_{nj} = y_h(t_n;\lambda_{hj}) - y_h(t_n;\lambda_{h,j-1}),$$
$$\bar{A}_{nj} = I + h_n\int_0^1 F_y(t_n, h_n, y_n(t_n;\lambda_{h,j-1}) + \tau k_{nj}, \lambda_{h,j-1} + \tau(\lambda_{hj} - \lambda_{h,j-1}))\,\mathrm{d}\tau,$$
$$\bar{B}_{nj} = h_n\int_0^1 F_\lambda\big(t_n, h_n, y_n(t_n;\lambda_{h,j-1}) + \tau k_{nj}, \lambda_{h,j-1} + \tau(\lambda_{hj} - \lambda_{h,j-1})\big)\,\mathrm{d}\tau.$$

for $n = 0, 1, \dots, N$, $j = 1, 2, \dots$. Then we have

$$\left\|\prod_{r=i+1}^{n-1}\bar{A}_{n+i-r,j}\right\| \leqslant \prod_{r=i+1}^{n-1}(1 + h_{n+1-r}L_1) \leqslant c, \quad i = 0, 1, \dots, n-1, \ n = 1, 2, \dots, N.$$

Moreover, for $n = 0, 1, \ldots, N$ we have

$$k_{n+1,j} = k_{nj} + h_n \left[F(t_n, h_n, y_h(t_n; \lambda_{hj}), \lambda_{hj}) - F(t_n, h_n y_h(t_n; \lambda_{h,j-1}), \lambda_{h,j-1})\right],$$

and by the mean value theorem this yields

$$k_{n+1,j} = \overline{A}_{nj} k_{nj} + \overline{B}_{nj}(\lambda_{hj} - \lambda_{h,j-1}), \quad n = 0, 1, \ldots, N-1, \quad j = 1, 2, \ldots.$$

Hence

$$k_{nj} = \sum_{i=0}^{n-1} \left(\prod_{r=i+1}^{n-1} \overline{A}_{n+i-r,j} \right) \overline{B}_{ij}(\lambda_{hj} - \lambda_{h,j-1}), \quad n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots,$$

or

$$\|k_{nj}\| \leqslant c(b-a)L_2 \|\lambda_{hj} - \lambda_{h,j-1}\|, \quad n = 0, 1, \ldots, N, \quad j = 1, 2, \ldots.$$

We can also get an estimate for $\overline{B}_{ij} - B_{ij}$, where B_{ij} is defined in (9′). We have now

$$\begin{aligned}\|\overline{B}_{ij} - B_{ij}\| &\leqslant h_i \int_0^1 [K_2(1-\tau)\|k_{ij}\| + K_3(1-\tau)\|\lambda_{hj} - \lambda_{h,j-1}\|]\, \mathrm{d}\tau \\ &\leqslant \frac{h_i}{2}[K_2(b-a)L_2 c + K_3]\|\lambda_{hj} - \lambda_{h,j-1}\|, \\ &\qquad\qquad i = 0, 1, \ldots, N, \quad j = 1, 2, \ldots\end{aligned}$$

and

$$(27)\qquad \begin{aligned}&\left\| \sum_{i=0}^{N-1} \left(\prod_{r=i+1}^{N-1} \overline{A}_{N+i-r,j} \right) [\overline{B}_{ij} - B_{ij}] \right\| \\ &\leqslant \frac{c}{2}(b-a)[K_2(b-a)L_2 c + K_3]\|\lambda_{hj} - \lambda_{h,j-1}\|, \quad j = 1, 2, \ldots.\end{aligned}$$

Put

$$\begin{aligned}\xi_{ij} &= \prod_{r=i+1}^{N-1} \overline{A}_{N+i-r,j} - \prod_{r=i+1}^{N-1} A_{N+i-r,j'} \quad i = 0, 1, \ldots, N-2, \quad j = 1, 2, \ldots, \\ \xi_{N-1,j} &= 0_{q\times q}.\end{aligned}$$

We will prove that

$$(28)\qquad \begin{aligned}&\|\xi_{N-s,j}\| \leqslant K \|\lambda_{hj} - \lambda_{h,j-1}\| \sum_{i=N-s+1}^{N-1} \prod_{\substack{r=N-s+1 \\ r\neq i}}^{N-1} (1 + h_r L_1) h_i, \\ &s = 1, 2, \ldots, N, \quad j = 1, 2, \ldots,\end{aligned}$$

where

$$K = \tfrac{1}{2}[K_1(b-a)L_2c + K_0].$$

Indeed, it is true for $s = 1$. For $s = 2$ we have

$$\begin{aligned}
\|\xi_{N-2,j}\| &= \|\bar{A}_{N-1,j} - A_{N-1,j}\| \\
&\leqslant h_{N-1}\int_0^1 [K_1(1-\tau)\|k_{N-1,j}\| + K_0(1-\tau)\|\lambda_{hj} - \lambda_{h,j-1}\|]\,\mathrm{d}\tau \\
&\leqslant h_{N-1}K\|\lambda_{hj} - \lambda_{h,j-1}\|.
\end{aligned}$$

so (28) is true for $s = 2$.

Now we assume that (28) is satisfied for some $s < N$. Then we see that

$$\begin{aligned}
\|\xi_{N-s-1,j}\| &= \|\bar{A}_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j}\bar{A}_{N-s,j} - A_{N-1,j} \times \ldots \times A_{N-s+1,j}A_{N-s,j} \\
&\quad - \bar{A}_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j}A_{N-s,j} \\
&\quad + A_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j}A_{N-s,j}\| \\
&\leqslant \|\bar{A}_{N-1,j} \times \ldots \times \bar{A}_{N-s+1,j}\|\,\|\bar{A}_{N-s,j} - A_{N-s,j}\| + \|\xi_{N-s,j}\|\,\|A_{N-s,j}\| \\
&\leqslant \prod_{r=N-s+1}^{N-1}(1+h_rL_1)Kh_{N-3}\|\lambda_{hj} - \lambda_{h,j-1}\| \\
&\quad + (1+h_{N-s}L_1)K\|\lambda_{hj} - \lambda_{h,j-1}\| \sum_{i=N-s+1}^{N-1} \prod_{\substack{r=N-s+1 \\ r\neq i}}^{N-1}(1+h_rL_1)h_i \\
&= K\|\lambda_{hj} - \lambda_{h,j-1}\| \sum_{i=N-s}^{N-1} \prod_{\substack{r=N-s \\ r\neq i}}^{N-1}(1+h_rL_1)h_i.
\end{aligned}$$

Hence (28) is true for any value of $s = 1, 2, \ldots, N$, $j = 1, 2, \ldots$. Moreover, from (28) we may get the estimate

$$\begin{aligned}
\|\xi_{N-s,j}\| &\leqslant K\|\lambda_{hj} - \lambda_{h,j-1}\| \sum_{i=N-s+1}^{N-1} \prod_{r=N-s+1}^{N-1}(1+h_rL_1)h_i \\
&\leqslant cK(b-a)\|\lambda_{hj} - \lambda_{h,j-1}\|, \quad s = 1, 2, \ldots, N, \quad N = 1, 2, \ldots
\end{aligned}$$

and hence

$$\begin{aligned}
\left\|\sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{N+i-r,j} - \prod_{r=i+1}^{N-1}A_{N+i-r,j}\right)B_{ij}\right\| &\leqslant \sum_{i=0}^{N-1}\|\xi_{ij}\|\,\|B_{ij}\| \\
&\leqslant cK(b-a)^2L_2\|\lambda_{hj} - \lambda_{h,j-1}\|, \quad j = 1, 2, \ldots.
\end{aligned} \tag{29}$$

By the definition of $\lambda_{h,j+1}$ and by (9′) we have

$$(30)\quad \begin{aligned}\|\lambda_{h,j+1}-\lambda_{hj}\| &= \|Q_{hj}^{-1}\|\,\|B_1(\lambda_{hj}-\lambda_{h,j-1}) \\ &\quad + B_2k_{Nj} - Q_{h,j-1}(\lambda_{hj}-\lambda_{h,j-1})\| \\ &= \|Q_{hj}^{-1}\|\,\|\lambda_{hj}-\lambda_{h,j-1}\| \\ &\quad \times \left\| B_1 + B_2 \sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{N+i-r,j}\right)\bar{B}_{ij} - Q_{h,j-1}\right\| \\ &= \|Q_{hj}^{-1}\|\,\|\lambda_{hj}-\lambda_{h,j-1}\|\,\|B_2\| \left\| \sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{n+i-r,j}\right)\bar{B}_{ij} \right. \\ &\quad \left. - \sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}A_{N+i-r,j}\right)B_{ij}\right\|.\end{aligned}$$

Using (27) and (29) we find

$$(31)\quad \begin{aligned}&\left\|\sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{N+i-r,j}\right)\bar{B}_{ij} - \sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}A_{N+i-r,j}\right)B_{ij}\right\| \\ &\leqslant \left\|\sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{N+i-r,j}\right)(\bar{B}_{ij}-B_{ij})\right\| \\ &\quad + \left\|\sum_{i=0}^{N-1}\left(\prod_{r=i+1}^{N-1}\bar{A}_{N+i-r,j} - \prod_{r=i+1}^{N-1}A_{N+i-r,j}\right)B_{ij}\right\| \\ &\leqslant T_0\|\lambda_{hj}-\lambda_{h,j-1}\|, \quad j=1,2,\ldots\end{aligned}$$

Combining (27), (30) and (31) we have (26).

By Lemma 2 we know that for sufficiently small h the matrix Q_{hj} is nonsingular and $\|Q_{hj}^{-1}\| \leqslant D$. It means that

$$\|\lambda_{h,j+1}-\lambda_{hj}\| \leqslant TD\|\lambda_{hj}-\lambda_{h,j-1}\|^2, \quad j=1,2,\ldots.$$

and

$$\|\lambda_{h,j+1}-\lambda_{hj}\| \leqslant \frac{1}{TD}\left(TD\|\lambda_{h1}-\lambda_{h0}\|\right)^{2^j}, \quad j=0,1,\ldots.$$

We see that all assumptions of Theorem 1 are satisfied, so (20) yields

$$Z_h^{j+1} \leqslant D[A(Z_h^j)^2 + B(h)Z_h^j + C(h)] = Dp_h(Z_h^j) + Z_h^j,$$

where

$$p_h(z) = Az^2 - x_h z + C(h).$$

The quadratic function p_h has two distinct zeros z_-^h and z_+^h where $z_+^h > z_-^h > 0$. If $\|\lambda_{h0} - \lambda\| \leqslant \min\left[z_-^h, \max_{h \leqslant \bar{h}} DC(h)/(1-d)\right]$ then $\|\lambda_{hj} - \lambda\| \leqslant z_-^h, j = 1, 2, \ldots$. Hence

$$\begin{aligned}\|\lambda_{h,j+1} - \lambda\| &\leqslant \|\lambda_{h,j+1} - \lambda_{hj}\| + \|\lambda_{hj} - \lambda\| \\ &\leqslant \frac{1}{TD}(TD\|\lambda_{h1} - \lambda_{h0}\|)^{2^j} + z_-^h, \quad j = 0, 1, \ldots\end{aligned}$$

so we have (14). The rest follows from Theorem 1.

This completes the proof. □

References

[1] *E.A. Coddington and N. Levinson*: Theory of ordinary differentical equations. McGraw-Hill, New York, 1955.

[2] *J.W. Daniel and R.E. Moore*: Computation and theory in ordinary differential equations. W.H. Freeman, San Francisco, 1970.

[3] *P. Henrici*: Discrete variable methods in ordinary differential equations. John Wiley, New York, 1962.

[4] *T. Jankowski*: Boundary value problems with a parameter of differentical equations with deviated arguments. Math. Nachr. *125* (1986), 7-28.

[5] *T. Jankowski*: One-step methods for ordinary differential equations with parameters. Apl. Mat. *35* (1990), 67-83.

[6] *T. Jankowski*: On the convergence of multistep methods for nonlinear two-point boundary value problems. APM (1991), 185-200.

[7] *H.B. Keller*: Numerical methods for two point boundary value problems. Waltham, Blaisdell, 1968.

[8] *H.B. Keller*: Numerical solution of two-point boundary value problems. Society for Industrial and Applied Mathematics, Philadelphia 24, 1976.

[9] *A. Pasquali*: Un procedimento di calcolo connesso ad un noto problema ai limiti per l'equazione $\dot{x} = f(t, x, \lambda)$. Mathematiche *23* (1968), 319-328.

[10] *T. Pomentale*: A constructive theorem of existence and uniqueness for the problem $y' = f(x, y, \lambda), y(a) = \alpha, y(b) = \beta$. ZAMM *56* (1976), 387-388.

[11] *Z.B. Seidov*: A multipoint boundary value problem with a parameter for systems of differential equations in Banach space. Sibirski Math. Z. *9* (1968), 223-228. (In Russian.)

[12] *J. Stoer and R. Bulirsch*: Introduction to numerical analysis. Springer-Verlag, New York, 1980.

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