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Boundary value problems for coupled systems of second order differential equations with a singularity of the first kind: explicit solutions

Applications of Mathematics, Vol. 39 (1994), No. 1, 1–13

Persistent URL: http://dml.cz/dmlcz/134239

BOUNDARY VALUE PROBLEMS FOR COUPLED SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH A SINGULARITY OF THE FIRST KIND: EXPLICIT SOLUTIONS

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(Received October 25, 1991)

Summary. In this paper we obtain existence conditions and an explicit closed form expression of the general solution of twopoint boundary value problems for coupled systems of second order differential equations with a singularity of the first kind. The approach is algebraic and is based on a matrix representation of the system as a second order Euler matrix differential equation that avoids the increase of the problem dimension derived from the standard reduction of the order method.

Keywords: Coupled differential system, boundary value problem, singularity of the first kind, Moore-Penrose pseudo-inverse

AMS classification: 34B05

1. INTRODUCTION

We consider boundary value problems of the type

$$y''(t) + \frac{A_1}{t} y'(t) + \frac{A_0}{t^2} y(t) = f(t), \quad 0 < t \leqslant 1 \tag{1.1}$$

$$E_1 y(0) + E_2 y(1) = E_3; \quad F_1 y'(0) + F_2 y'(1) = F_3, \tag{1.2}$$

where y, f are vector functions with values in $\mathbb{C}^n$, A_0, A_1, E_i, F_i are $n \times n$ matrices, elements of $\mathbb{C}^{n\times n}$ for $i = 1, 2$, and E_3, F_3 are vectors in $\mathbb{C}^n$.

Problems of type (1.1)–(1.2) appear in spherical shells theory [6,12], and they have been studied in [13], considering an equivalent extended first order system. Following the ideas developed in [3], [4], [5], the aim of this paper is to find an explicit closed form solution of problems of the type (1.1)–(1.2), avoiding the increase of the problem dimension but without the assumption of the existence of solutions for the associated algebraic matrix equation

$$Z^2 + (A_1 - I)Z + A_0 = 0. \tag{1.3}$$

The paper is organized as follows. In Section 2 we present a closed form expression for the general solution of the homogeneous equation

$$y''(t) + \frac{A_1}{t} y'(t) + \frac{A_0}{t^2} y(t) = 0, \quad 0 < t \leqslant 1 \tag{1.4}$$

as well as existence conditions and the corresponding closed form expression for the solutions of (1.4) continuously differentiable and prolongable to the point $t = 0$. In Section 3 we consider the boundary value problem (1.1), (1.2).

If S is a matrix in $\mathbb{C}^{m \times n}$, we denote by S^+ its Moore-Penrose pseudoinverse and recall that an account of uses and properties of this concept may be found in [1], [11]. The effective computation of S^+ is an easy matter using MATLAB, [9].

2. On the homogenous equation $y''(t) + (A_1/t)y'(t) + (A_0/t^2)y(t) = 0$

For the sake of clarity in the presentation we begin this section by introducing the companion matrix C associated to the matrix equation (1.3)

$$C = \begin{bmatrix} 0 & I \\ -A_0 & I - A_1 \end{bmatrix}. \tag{2.1}$$

Theorem 1. *Let $J = [\mathrm{Diag}(J_1, \ldots, J_k)]$ be the Jordan canonical form of the matrix C defined by (2.1), where J_j is a matrix in $\mathbb{C}^{m_j \times m_j}$ for $1 \leqslant j \leqslant k$. If $M = (M_{ij})$ with $M_{ij} \in \mathbb{C}^{n \times m_j}$, $1 \leqslant i \leqslant 2$, $1 \leqslant j \leqslant k$, is an invertible matrix in $\mathbb{C}^{2n \times 2n}$ such that*

$$M[\mathrm{Diag}(J_1, \ldots, J_k)] = CM \tag{2.2}$$

and $m_1 + \ldots + m_k = 2n$, then the general solution of equation (1.4) is given by

$$y(t) = \sum_{j=1}^{k} M_{1j} \exp\left(J_j \log(t)\right) d_j, \quad d_j \in \mathbb{C}^{m_j}. \tag{2.3}$$

Proof. Note that (2.2) implies that

$$M_{2j}J_j = -A_0M_{1j} + (I - A_1)M_{2j} \text{ and } M_{1j}J_j = M_{2j} \quad \text{for } 1 \leqslant j \leqslant k. \tag{2.4}$$

From (2.4) we have

$$M_{1j}J_j^2 + (A_1 - I)M_{1j}J_j + A_0M_{1j} = 0, \quad 1 \leqslant j \leqslant k. \tag{2.5}$$

Now let us consider the vector function $y_j(t) = M_{1j}\exp\big(J_j\log(t)\big)d_j$ with $d_j \in \mathbf{C}^{m_j}$. An easy computation and (2.5) yield

$$\begin{aligned} &y_j''(t) + \frac{A_1}{t}y_j'(t) + \frac{A_0}{t^2}y_j(t) \\ &= [M_{1j}J_j^2 + (A_1 - I)M_{1j}J_j + A_0M_{1j}]t^{-2}\exp\big(J_j\log(t)\big)d_j = 0 \quad \text{for } 0 < t \leqslant 1. \end{aligned}$$

Thus $y_j(t)$ is a solution of (1.4) for any vector $d_j \in \mathbf{C}^{m_j}$, $1 \leqslant j \leqslant k$ and $y(t)$ defined by (2.3) describes a set of solutions of (1.4) in $0 < t \leqslant 1$. In order to prove that (2.3) defines the general solution of (1.4), let $z(t)$ be any solution of (1.4) in $0 < t \leqslant 1$, and let $z(1) = c_0$, $z'(1) = c_1$. From the uniqueness of solutions for a Cauchy problem related to (1.4), the result is proved if we find an appropriate vector $d_j \in \mathbf{C}^{m_j}$ such that $y(t)$ defined by (2.3) satisfies the same initial conditions as $z(t)$ at $t = 1$. Note that the derivative of $y(t)$ defined by (2.3) takes the form

$$y'(t) = \sum_{j=1}^{k} M_{1j}\,\frac{J_j}{t}\,\exp\big(J_j\log(t)\big)d_j$$

and from (2.4),

$$y'(t) = \sum_{j=1}^{k} \frac{M_{2j}}{t}\exp\big(J_j\log(t)\big)d_j. \tag{2.6}$$

By virtue of (2.3) and (2.4), the initial conditions $y(1) = c_0$, $y'(1) = c_1$ are satisfied if there exist vectors $d_j \in \mathbf{C}^{m_j}$ such that

$$M\begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}. \tag{2.7}$$

Since M is invertible, the system (2.7) admits only one solution defined by

$$\begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = M^{-1}\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.$$

Thus the result is established. □

In order to study the existence of solutions of (1.4) which are continuously differentiable in the *closed* interval [0,1], it is important to recall that [10], p. 66, implies that if J_j is a Jordan block of size $p \times p$ associated to an eigenvalue λ_j, then

(2.8)

$$\exp\left(J_j \log(t)\right) = \exp\left(\lambda_j \log(t)\right) \begin{bmatrix} 1 & \log t & (\log t)^2/2! & \dots & (\log t)^{p-1}/(p-1)! \\ & 1 & \log t & \dots & \dots \\ & & 1 & & \dots \\ & & & & \log t \\ & & & & 1 \end{bmatrix}.$$

Hence it follows that if $\mathrm{Re}(\lambda_j) > 0$, then $\exp\left(J_j \log(t)\right)$ tends to the $p \times p$ zero matrix as t tends to zero. If $\mathrm{Re}(\lambda_j) < 0$, then $\exp\left(J_j \log(t)\right)$ is unbounded as t tends to zero. Finally, if $\mathrm{Re}(\lambda_j) = 0$, since $\log t$ is unbounded as t tends to zero, the matrix function $\exp\left(J_j \log(t)\right)$ is bounded if and only if the Jordan block J_j is of size 1×1. On the other hand, note that the first derivative of $y_j(t) = M_{1j} \exp(J_j \log t) d_j$ is $y_j'(t) = (M_{2j}/t) \exp(J_j \log t) d_j$, and due to the invertibility of $M = (M_{ij})$ and to (2.4), the block matrices M_{1j} are nonzero for $1 \leqslant j \leqslant k$. An analogous analysis related to $\exp(J_j \log t)$ shows that the matrix $(1/t)\exp(J_j \log t) = \exp\left((J_j - I)\log t\right)$ remains bounded as t tends to zero if $\mathrm{Re}(\lambda_j) \geqslant 1$, and when $\mathrm{Re}(\lambda_j) = 1$, the size of J_j is 1×1.

The previous comments and Theorem 1 yields the following result:

Corollary 1. *Let us consider the notation of Theorem 1. Then*

(i) *Equation* (1.4) *admits bounded solutions $y(t)$ in all the half-open interval* $]0,1]$, *if and only if there exist eigenvalues λ_j of the companion matrix C defined by* (2.1) *such that* $\mathrm{Re}(\lambda_j) \geqslant 1$ *and if* $\mathrm{Re}(\lambda_{j_0}) = 1$, *then the index of λ_{j_0} as an eigenvalue of C is one.*

(ii) *Under the hypotheses of* (i), *if $J_1, \dots, J_m$ are the Jordan blocks of the Jordan canonical form of C corresponding to those eigenvalues λ_j satisfying the conditions of* (i), *then the set of all bounded solutions $y(t)$ of* (1.4) *with bounded derivative in* $]0,1]$ *is given by*

$$(2.9) \qquad y(t) = \sum_{j=1}^{m} M_{1j} \exp(J_j \log t) d_j, \quad d_j \in \mathbb{C}^{m_j}, \ 1 \leqslant j \leqslant m \leqslant k.$$

The set of all values $y(1)$ of the bounded solutions of (1.4) *in the interval* $]0,1]$ *is the range of the matrix $\overline{M} = [M_{11}, \dots, M_{1m}]$. If d lies in the range of $\overline{M}$, then taking vectors d_j in* (2.9) *satisfying $M \,\mathrm{col}(d_1, \dots, d_m) = d$, one gets a bounded solution $y(t)$ of* (1.4) *such that $y(1) = d$.*

P r o o f. It is a consequence of the previous comments and the fact that if an eigenvalue λ of C has index one then all Jordan blocks associated to λ have size 1×1, [7]. □

Note that under the conditions of Corollary 1, all solutions $y(t)$ defined by (2.9) as well as $y'(t)$ have a well defined value at $t = 0$, if their limits as t tends to zero in (2.9) and in the expression

$$y'(t) = \sum_{j=1}^{m} M_{2j} \exp\left((J_j - I)\log t\right) d_j \tag{2.10}$$

exist. From the previous comments it is clear that $\exp\left((J_j - I)\log t\right)$ and $\exp(J_j \log t)$ tend to the zero matrix as t tends to zero, if J_j is a Jordan block associated to an eigenvalue λ_j of C with $\mathrm{Re}(\lambda_j) > 1$. If $\mathrm{Re}(\lambda_j) = 1$ and $\mathrm{Im}(\lambda_j) = b_j \neq 0$, then, having index one, it follows that

$$\begin{aligned} \exp(J_j \log t) &= \exp\left((1 + ib_j)\log t\right) = t \exp(ib_j \log t), \\ \exp\left((J_j - I)\log t\right) &= \exp(ib_j \log t) = \cos(b_j \log t) + i \sin(b_j \log t). \end{aligned} \tag{2.11}$$

Since the limits of the real and the imaginary part of (2.11) as t tends to zero do not exist, we conclude that in order to have well defined solutions $y(t)$ of (1.4) and with well defined derivative at $t = 0$, it is necessary that $b_j = 0$. Thus the following result has been proved:

Corollary 2. *Let us consider the notation of Theorem 1. Then*

(i) *Equation* (1.4) *admits continuously differentiable solutions in the closed interval* $[0, 1]$, *if and only if there exist eigenvalues* λ_j *of the companion matrix* C *such that* $\mathrm{Re}(\lambda_j) \geqslant 1$ *and the single possible eigenvalue with real part equal to one is* $\lambda_j = 1$ *and has index one.*

(ii) *Under the hypotheses of* (i), *if* $J_1, \dots, J_p$ *are the Jordan blocks of the Jordan canonical form of* C *associated to eigenvalues* $\lambda_1, \dots, \lambda_p$ *with* $\mathrm{Re}(\lambda_j) > 1$ *for* $1 \leqslant j \leqslant p$, *and* $J_{p+1} = J_{p+2} = \dots = J_k$ *coincide with the real number 1, then the set of all continuously differentiable solutions of* (1.4) *in the closed interval* $[0, 1]$ *is given by*

$$y(t) = \sum_{j=1}^{p} M_{1j} \exp(J_j \log t) d_j + t \sum_{j=p+1}^{k} M_{1j} d_j, \quad d_j \in \mathbb{C}^{m_j}, \ 1 \leqslant j \leqslant k \tag{2.12}$$

with

$$y(0) = \lim_{t \to 0} y(t) = 0, \quad y'(0) = \lim_{t \to 0} y'(t) = \sum_{j=p+1}^{k} M_{2j} d_j. \tag{2.13}$$

The set of the attainable values of these solutions at $t = 1$ is the range of the matrix $M' = [M_{11}, \ldots, M_{1k}]$.

3. The boundary value problem

We begin this section with obtaining a closed form expression for the general solution of equation (1.1) where $f(t)$ is a continuous function in $[0, 1]$. Let $M = (M_{ij})$ be the invertible matrix in $\mathbf{C}^{2n \times 2n}$ from Theorem 1 and let us write

$$V = M^{-1} = \left[\begin{array}{c|c|c|c} V_{11} & V_{12} & \cdots & V_{1k} \\ \hline V_{21} & V_{22} & \cdots & V_{2k} \end{array}\right]^T = \left[\begin{array}{c|c} V_{11} & V_{21} \\ V_{12} & V_{22} \\ \vdots & \vdots \\ V_{1k} & V_{2k} \end{array}\right] \tag{3.1}$$

where $V_{ij} \in \mathbf{C}^{m_j \times n}$ for $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant k$. Based on the expression (2.3) we are looking for a particular solution of the inhomogeneous equation (1.1) of the form

$$y_p(t) = \sum_{j=1}^{k} M_{1j} \exp(J_j \log t) d_j(t) \tag{3.2}$$

where $d_j(t)$ are $\mathbf{C}^{m_j}$ valued vector functions to be determined for $1 \leqslant j \leqslant k$. Choose the functions $d_j(t)$ so that

$$\left[\begin{array}{c|c} I & 0 \\ \hline 0 & I/t \end{array}\right] M \left[\operatorname{Diag}\big(\exp(J_1 \log t), \ldots, \exp(J_k \log t)\big)\right] \left[\begin{array}{c} d_1'(t) \\ \hline \vdots \\ \hline d_k'(t) \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline f(t) \end{array}\right]. \tag{3.3}$$

Note that (3.3) implies that $y_p(t)$ defined by (3.2) satisfies

$$\begin{aligned} y_p'(t) &= \sum_{j=1}^{k} M_{1j} J_j \exp\big((J_j - I) \log t\big) d_j(t), \\ y_p''(t) &= \sum_{j=1}^{k} (M_{1j} J_j^2 - M_{1j} J_j) \exp\big((J_2 - 2I) \log t\big) d_j(t) + f(t). \end{aligned} \tag{3.4}$$

It follows from (3.2), (3.4) and (2.5) that

$$\begin{aligned} & y_p''(t) + \frac{A_1}{t} y_p'(t) + \frac{A_0}{t^2} y_p(t) \\ &= \sum_{j=1}^{k} \big(M_{1j} J_j^2 + (A_1 - I) M_{1j} J_j + A_0 M_{1j}\big) \exp\big((J_j - 2I) \log t\big) d_j(t) + f(t) = f(t). \end{aligned}$$

Now from (3.3) and taking into account the notation (3.1), we conclude that

$$(3.5)\quad \left[\mathrm{Diag}\left(\exp(J_1 \log t), \ldots, \exp(J_k \log t)\right)\right] M^{-1} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & tI \end{array}\right] \left[\begin{array}{c} d_1'(t) \\ \hline \vdots \\ \hline d_k'(t) \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline f(t) \end{array}\right],$$

$$(3.6)\qquad d_j(t) + L_j + \int_1^t \exp(-J_j \log s) V_{2j} s f(s)\,\mathrm{d}s, \quad 1 \leqslant j \leqslant k,\ L_j \in \mathbb{C}^{m_j}.$$

Taking for L_j the zero vector in $\mathbb{C}^{m_j}$ for $1 \leqslant j \leqslant k$ and denoting

$$(3.7)\qquad u^{J_j} = \exp(J_j \log u), \quad u > 0,$$

from (3.2) and (3.6) we have

$$(3.8)\qquad y_p(t) = \sum_{j=1}^{k} M_{1j} \int_1^t \left(\frac{t}{s}\right)^{J_j} V_{2j} s f(s)\,\mathrm{d}s, \quad 0 < t \leqslant 1,$$

that is a particular solution of (1.1) such that $y_p(1) = 0$. Note also that the derivative of $y_p(t)$ takes the form

$$(3.9)\qquad \begin{aligned} y_p'(t) &= \sum_{j=1}^{k} M_{2j} t^{(J_j - I)} \int_1^t (1/s)^{J_j} V_{2j} s f(s)\,\mathrm{d}s + \sum_{j=1}^{k} M_{1j} V_{2j} t f(t) \\ &= \sum_{j=1}^{k} M_{2j} \int_1^t \left(\frac{t}{s}\right)^{(J_j - I)} V_{2j} f(s)\,\mathrm{d}s, \quad 0 < t \leqslant 1, \end{aligned}$$

and in particular at $t = 1$ we have

$$y_p'(1) = \sum_{j=1}^{k} M_{1j} V_{2j} f(1) = \left(\sum_{j=1}^{k} M_{1j} V_{2j}\right) f(1) = 0.$$

From Theorem 1 and the previous comments the following result can be established:

Theorem 2. *Let us consider the notation of Theorem 1, let $f(t)$ be a continuous function in $]0,1]$ and let $V = M^{-1} = (V_{ij})^T$ be the block partitioned matrix defined by (3.1). Then the general solution of (1.1) in $0 < t \leqslant 1$ is defined by*

$$(3.10)\qquad y(t) = \sum_{j=1}^{k} M_{1j} t^{J_j} d_j + y_p(t), \quad d_j \in \mathbb{C}^{m_j},\ 1 \leqslant j \leqslant k$$

where $y_p(t)$ is defined by (3.8) and satisfies $y_p(1) = 0$, $y_p'(1) = 0$.

Corollary 2 provides conditions obtaining solutions of (1.4) which admit a continuously differentiable extension to the point $t = 0$. Now we are interested in the study of the existence of a limit as t tends to zero of the functions $y_p(t)$ and $y_p'(t)$ defined by (3.8) and (3.9), respectively.

Let us consider the change of variable $t/s = u$ in the integral

$$H_j = \int_1^t \left(\frac{t}{s}\right)^{(J_j - I)} V_{2j} f(s)\, \mathrm{d}s, \tag{3.11}$$

which yields

$$H_j = -t \int_t^1 u^{(J_j - 3I)} V_{2j} f(t/u)\, \mathrm{d}u = t \int_t^1 \exp\big((J_j - 3I)\log u\big) V_{2j} f(t/u)\, \mathrm{d}u. \tag{3.12}$$

Now let us consider the change $\log u = v$. Then (3.12) implies that

$$H_j = -t \int_{\log t}^0 \exp(vJ_j) \mathrm{e}^{-2v} V_{2j} f(t\mathrm{e}^{-v})\, \mathrm{d}v. \tag{3.13}$$

Let us suppose that J_j is the Jordan block of size 1×1 corresponding to the eigenvalue $\lambda_j = 1$, then making the change $t\mathrm{e}^{-v} = p$ in (3.13) we obtain that

$$H_j = \int_1^t V_{2j} f(p)\, \mathrm{d}p = V_{2j} \int_1^t f(p)\, \mathrm{d}p. \tag{3.14}$$

If $\mathrm{Re}(\lambda_j) > 1$, from (2.8) the integral H_j is convergent as t tends to zero if the limit of the integrals

$$H_j(m) = t \int_{\log t}^0 \exp\big(v(\lambda_j - 2)\big) v^m V_{2j} f(t\mathrm{e}^{-v})\, \mathrm{d}v, \quad m \geqslant 0 \tag{3.15}$$

exists. To study the limit of $H_j(m)$ as t tends to zero, we distinguish the cases $\lambda_j = 2$ and $\lambda_j \neq 2$. If $\lambda_j = 2$ then since f is bounded, the integral $H_j(m)$ defined by (3.15) converges to zero as t tends to zero. If $\lambda_j \neq 2$, then from the boundedness of f and taking into account [2], p. 92, one gets for $m \geqslant 1$, $a \neq 0$,

$$\int \mathrm{e}^{av} v^m \mathrm{d}v = \mathrm{e}^{av} \Big\{ \frac{v^m}{a} + \sum_{k=1}^{m} (-1)^k \frac{m(m-1)\ldots(m-k+1)}{a^{k+1}} v^{m-k} \Big\}. \tag{3.16}$$

From (3.15) and (3.16) one concludes that if $\mathrm{Re}(\lambda_j) > 1$, $\lambda_j \neq 2$, the integral H_j defined by (3.13) converges to zero as t tends to zero. Note that from (3.14), the integral H_j corresponding to the case $\lambda_j = 1$ converges to $-V_{2j}\int_0^1 f(p)\,\mathrm{d}p$ as t tends to zero.

Note that due to the relationship

$$\int_1^t \left(\frac{t}{s}\right)^{J_j} V_{2j} s f(s)\,\mathrm{d}s = t\int_1^t \left(\frac{t}{s}\right)^{(J_j - I)} V_{2j} f(s)\,\mathrm{d}s$$

and to the previous comments related to the integral (3.11), for $\mathrm{Re}(\lambda_j) \geqslant 1$ the limit of (3.17) is the zero matrix as t tends to zero. In particular, $y_p(t)$ defined by (3.8) converges to zero as t tends to zero. Thus the following result has been established:

Theorem 3. *Le $f(t)$ be a continuous bounded function in the interval $]0,1]$ and let us consider the notation of Theorem 2.*

(i) *Equation (1.1) has solutions which admit a continuously differentiable extension to the closed interval $[0,1]$, if all eigenvalues λ_j of the companion matrix C satisfy $\mathrm{Re}(\lambda_j) \geqslant 1$ and the only possible eigenvalue with real part equal to one is $\lambda_j = 1$ and its index is one.*

(ii) *Under the hypotheses of (i), all solutions of (1.1) which admit a continuously differentiable extension to $[0,1]$ satisfy $y(0) = 0$.*

The general solution of (1.1) which admits a continuously differentiable extension to the closed interval $[0,1]$ is given by

$$y(t) = \sum_{j=1}^{k} M_{1j} t^{J_j} d_j + \sum_{j=1}^{k} M_{1j} \int_1^t \left(\frac{t}{s}\right)^{J_j} V_{2j} s f(s)\,\mathrm{d}s, \quad 0 < t \leqslant 1 \tag{3.17}$$

where d_j is an arbitrary vector in $\mathbf{C}^{m_j}$.

(iii) *If all eigenvalues λ_j of the companion matrix satisfy $\mathrm{Re}(\lambda_j) > 1$, then any solution $y(t)$ given by (3.17) satisfies $y(0) = y'(0) = 0$.*

(iv) *If the real number $\lambda = 1$ is an eigenvalue of C with index one and if the Jordan blocks $J_{p+1} = J_{p+2} = \ldots = J_k = 1$, then $y(t)$ defined by (3.17) satisfies*

$$y'(0) = \sum_{j=p+1}^{k} M_{2j} d_j - \left(\sum_{j=p+1}^{k} M_{2j} V_{2j}\right)\int_0^1 f(s)\,\mathrm{d}s \quad \textit{and} \quad y(0) = 0.$$

Now we are in a good position to study the boundary value problem (1.1), (1.2). If all eigenvalues λ_j of the companion matrix C satisfy $\mathrm{Re}(\lambda_j) > 1$, and we impose the boundary value conditions (1.2) onto the general solution of (1.1) defined by (3.17), it follows that the vectors d_j for $1 \leqslant j \leqslant k$ must verify

$$\tag{3.18} \left[\operatorname{Diag}(E_2, F_2)\right] M \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} E_3 \\ F_3 \end{bmatrix}.$$

To solve (3.18), let us consider the change defined by

$$\tag{3.19} M \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}.$$

Then system (3.18) takes the form

$$\tag{3.20} \left[\operatorname{Diag}(E_2, F_2)\right] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = \begin{bmatrix} E_3 \\ F_3 \end{bmatrix}.$$

Now from Theorem 2.3.2 of [11], p. 24, the system (3.20) is compatible if and only if

$$\tag{3.21} E_2 E_2^{+} E_3 = E_3 \quad \text{and} \quad F_2 F_2^{+} F_3 = F_3,$$

and under this condition the general solution of (3.20) is given by

$$\tag{3.22} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix} = \left[\operatorname{Diag}(E_2^{+}, F_2^{+})\right] \begin{bmatrix} E_3 \\ F_3 \end{bmatrix} + \left\{ I - \operatorname{Diag}(E_2^{+} E_2, F_2^{+} F_2) \right\} Z$$

where Z is an arbitrary vector in $\mathbb{C}^{2n}$. By virtue of (3.19), the general solution of system (3.18) is given by

$$\tag{3.23} \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = M^{-1} \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$$

where $v_1, \ldots, v_k$ are determined by (3.22).

Now we consider the boundary value problem (1.1), (1.2) in the case (iv) of Theorem 3. If we impose (1.2) onto the general solution of (1.1) defined by (3.17), it follows that vectors $d_1, \ldots, d_k$ must verify

$$\tag{3.24} \overline{V} \begin{bmatrix} d_1 \\ \vdots \\ d_k \end{bmatrix} = \begin{bmatrix} E_3 \\ F_3 + F_1 \left(\sum_{j=p+1}^{k} M_{2j} V_{2j} \right) \int_0^1 f(s)\, ds \end{bmatrix}$$

where $\overline{V}$ is the block matrix defined by

(3.25)

$$\overline{V} = \left[\begin{array}{cccc|ccc} E_2 M_{11} & E_2 M_{12} & \ldots & E_2 M_{1p} & E_2 M_{1p+1} & \ldots & E_2 M_{1k} \\ \hline F_2 M_{21} & F_2 M_{22} & \ldots & F_2 M_{2p} & (F_1 + F_2) M_{2p+1} & \ldots & (F_1 + F_2) M_{2k} \end{array}\right]$$

and $J_{p+1} = J_{p+2} = \ldots = J_k$ are the Jordan blocks that correspond to the eigenvalue $\lambda = 1$.

From Theorem 2.3.2 of [11], p. 24, the system (3.24) is compatible if and only if the matrix $\overline{V}$ satisfies

$$(I - \overline{V}\overline{V}^{+}) \begin{bmatrix} E_3 \\ F_3 + F_1 \left(\sum\limits_{j=p+1}^{k} M_{2j} V_{2j} \right) \int_0^1 F(s)\, \mathrm{d}s \end{bmatrix} = 0 \tag{3.26}$$

and under this condition the general solution of (3.24) is given by

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix} = \overline{V}^{+} \begin{bmatrix} E_3 \\ F_3 + F_1 \left(\sum\limits_{j=p+1}^{k} M_{2j} V_{2j} \right) \int_0^1 f(s)\, \mathrm{d}s \end{bmatrix} + (I - \overline{V}^{+}\overline{V}) Z, \tag{3.27}$$

where Z is an arbitrary vector in $\mathbf{C}^{2n}$.

From the previous comments and Theorem 3 the following result has been established:

Theorem 4. *Let $f(t)$ be a continuous bounded function in the interval $]0, 1]$, and let us consider the notation of Theorem 2.*

(i) *If all eigenvalues of the companion matrix C satisfy $\mathrm{Re}(\lambda_j) > 1$, then the boundary value problem (1.1), (1.2) is compatible if and only if the condition (3.21) is satisfied. Under this condition, the general solution of problem (1.1), (1.2), is given by (3.17) where $d_1, \ldots, d_k$ are given by (3.23), (3.22).*

(ii) *If all eigenvalues λ_j of C satisfy $\mathrm{Re}(\lambda_j) \geqslant 1$, but $J_{p+1} = \ldots = J_k = 1$, then the boundary value problem (1.1), (1.2) is solvable, if and only if the matrix V defined by (3.25) satisfies (3.26). Under this condition the general solution of problem (1.1), (1.2) is given by (3.17) where d_j, for $1 \leqslant j \leqslant k$, are determined by (3.27).*

Remark 1. Note that the solutions of problem (1.1), (1.2) are given in terms of the Jordan canonical form of the companion matrix C defined by (2.1). The computation of the Jordan canonical form of a matrix can be efficiently performed by using MACSYMA, [8].

In order to show that the construction of the solutions proposed by the above results is straightforward let us introduce a family of examples.

E x a m p l e 1. Let us consider the problem in $\mathbf{C}^2$ defined by

$$y''(t) + \frac{1}{t}\begin{bmatrix} -6 & -8 \\ 1 & 1 \end{bmatrix} y'(t) + \frac{1}{t^2}\begin{bmatrix} 10 & 4 \\ -2 & 0 \end{bmatrix} y(t) = f(t), \quad 0 < t \leqslant 1. \tag{3.28}$$

The companion matrix and its Jordan canonical form take the form

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -4 & 7 & 8 \\ 2 & 0 & -1 & 0 \end{bmatrix}; \qquad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

The matrix M, its inverse V and the corresponding blocks are given by

$$M = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 8 & 4 & 0 \\ 1 & 0 & -2 & 1 \end{bmatrix}, \qquad M^{-1} = V = \begin{bmatrix} 2 & 4 & -1 & -4 \\ -1/4 & -1 & 1/4 & 1 \\ 0 & 1 & 0 & -1 \\ -2 & -2 & 1 & 3 \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad M_{12} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \qquad M_{21} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad M_{22} = \begin{bmatrix} 8 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

$$V_{21} = [-1, -4], \qquad V_{22} = \begin{bmatrix} 1/4 & 1 \\ 0 & -1 \\ 1 & 3 \end{bmatrix}.$$

Note that $k = 2$, $m_1 = 1$, $m_2 = 3$, $n = 2$,

$$J_1 = (1), \qquad J_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

All eigenvalues of the companion matrix C satisfy $\mathrm{Re}(\lambda_j) \geqslant 1$, where $\lambda_1 = 1$ is with index one and $\lambda_2 = 2$. Given a bounded continuous function $f(t)$ in $]0, 1]$, then by virtue of Theorem 3-(ii), the set of all solutions of (1.1) which admit a continuously differentiable extension to the interval $[0, 1]$ is given by

$$y(t) = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} d_1 + \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} t^2 \begin{bmatrix} 1 & \log t & \log^2 t \\ 0 & 1 & \log t \\ 0 & 0 & 1 \end{bmatrix} d_2$$

$$+ t\begin{bmatrix} -1 & -4 \\ -1 & -4 \end{bmatrix} \int_1^t f(s)\,\mathrm{d}s + t^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \int_1^t s^{-J_2} \begin{bmatrix} 1/4 & 1 \\ 0 & -1 \\ 1 & 3 \end{bmatrix} s f(s)\,\mathrm{d}s$$

where d_1 is an arbitrary vector in $\mathbf{C}$ and d_2 is an arbitrary vector in $\mathbf{C}^3$.

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ACKNOWLEDGEMENT

This work has been supported by the D.G.I.C.Y.T. PS 90-0140 and the NATO grant CR900040.

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