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Mathematica Bohemica, Vol. 126 (2001), No. 3, 571–579

Persistent URL: http://dml.cz/dmlcz/134202

DENSE SUBSETS OF ORDERED SETS

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(Received June 15, 1999)

Abstract. Some modifications of the definition of density of subsets in ordered (= partially ordered) sets are given and the corresponding concepts are compared.

Keywords: ordered set, weakly dense subset, dense subset, separability

MSC 2000: 06A06

0. Introduction

In [10], some cardinal characteristics of ordered sets are studied, among others the 2-pseudodimension introduced by the first author in [8] (see also [9]). For the reader's convenience, we recall its definition. Let $\mathbb{G} = (G, \leqslant)$ be an ordered set, let $I \neq \emptyset$ be a set and let $f_i\colon G \to \{0,1\}$ be a mapping for any $i \in I$. If

$$x \leqslant y \iff f_i(x) \leqslant f_i(y) \text{ for all } i \in I \tag{1}$$

holds for any $x, y \in G$, then we say that $(f_i;\ i \in I)$ is a 2-*realizer* of $\mathbb{G}$. Further, we set

$$\text{2-pdim}\ \mathbb{G} = \min\{|I|;\ (f_i;\ i \in I) \text{ is a 2-realizer of } \mathbb{G}\}; \tag{2}$$

this cardinal is called the 2-*pseudodimension* of $\mathbb{G}$. The significance of this characteristic is given by the following fact: 2-pdim $\mathbb{G}$ is the least cardinal m such that $\mathbb{G}$ can be embedded into the cardinal power of type 2^m [8], [10].

Another characteristic of an ordered set $\mathbb{G}$, studied in [10], is its separability sep $\mathbb{G}$; by this we mean the minimum of cardinalities of dense subsets in $\mathbb{G}$. The density of subsets of $\mathbb{G}$, defined in [10], corresponds to the u-density introduced in 2.2. of this article. Here we introduce some modifications of the definition of dense subsets of ordered sets and compare the corresponding concepts. Other definitions of dense subsets of ordered sets can be found in [12], [13], [14]; for linearly ordered sets see, e.g. [3] or [4]. Some other characteristics of linearly ordered sets are studied in [6], [7], [11].

We summarize some notation used in the sequel. For basic notions concerning ordered sets see [1] or [2]. An ordered set with a carrier G and a relation $\leqslant$ will be denoted by $(G, \leqslant)$ or $\mathbb{G}$. If M is a set, then $|M|$ is its *cardinality* and $\mathbb{B}(M)$ its *power*, i.e. $\mathbb{B}(M) = \{X;\ X \subseteq M\}$. If α is an ordinal, then $|\alpha|$ denotes also the cardinality of α. For elements x, y of an ordered set $(G, \leqslant)$, $x \parallel y$ means that x, y are *incomparable* and $x \prec y$ means that y is a *cover* of x, i.e. $x < y$ and $x < z < y$ for no $z \in G$. If x is an element of an ordered set $\mathbb{G}$, then $I(x)$ denotes the *principal ideal* and $F(x)$ the *principal filter* in $\mathbb{G}$ generated by x, i.e. $I(x) = \{t \in G;\ t \leqslant x\}$, $F(x) = \{t \in G;\ t \geqslant x\}$. Further, we set $I_H(x) = I(x) \cap H$, $F_H(x) = F(x) \cap H$ whenever H is a subset of G. When the contrary is not stated, we assume $|M| \geqslant 2$ for any set M in the following chapters. The Axiom of Choice will be assumed.

1. Weakly dense subsets

1.1. Definition. Let $\mathbb{G} = (G, \leqslant)$ be an ordered set and let $H \subseteq G$. The set H will be called *weakly l-dense* in $\mathbb{G}$ if

(3) $$x, y \in G,\ x \nleqslant y \Rightarrow \text{there exists } h \in H \text{ such that } h \leqslant x,\ h \nleqslant y.$$

Further, set

(4) $$wl\text{-sep}\ \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly } l\text{-dense in } \mathbb{G}\};$$

this cardinal will be called the *weak l-separability* of $\mathbb{G}$.

The following theorem provides a complete characterization of weakly dense subsets of $\mathbb{G}$.

1.2. Theorem. *Let $\mathbb{G} = (G, \leqslant)$ be an ordered set and let $H \subseteq G$. H is weakly l-dense in $\mathbb{G}$ if and only if $x = \sup I_H(x)$ for any $x \in G$.*

Proof. 1. Let H be weakly l-dense in $\mathbb{G}$ and let $x \in G$. We have $t \leqslant x$ for all $t \in I_H(x)$. Let $y \in G$ be such an element that $t \leqslant y$ for each $t \in I_H(x)$. Suppose that $x \nleqslant y$; then there exists $t \in H$, $t \leqslant x$ such that $t \nleqslant y$, i.e. $t \in I_H(x)$, $t \nleqslant y$ contradicting our assumption. Hence $x \leqslant y$ and consequently, $x = \sup I_H(x)$.

2. Let $x = \sup I_H(x)$ for all $x \in G$ and let $x, y \in G$, $x \nleqslant y$. Suppose that there is no $t \in H$ with $t \leqslant x$, $t \nleqslant y$, i.e. that $t \in H$, $t \leqslant x$ implies $t \leqslant y$. Then

$I_H(x) \subseteq I_H(y)$ implying $x = \sup I_H(x) \leqslant \sup I_H(y) = y$, a contradiction. Thus there exists $t \in H$, $t \leqslant x$ with $t \nleqslant y$ and H is weakly l-dense in $\mathbb{G}$. $\square$

From 1.2 we easily obtain

1.3. Corollary. *Let $\mathbb{G}$ be an ordered set, let $H \subseteq G$ be weakly l-dense in $\mathbb{G}$ and let $x, y \in G$. Then $x \leqslant y$ if and only if $I_H(x) \subseteq I_H(y)$.*

Proof. If $x \leqslant y$, then, trivially, $I_H(x) \subseteq I_H(y)$. Conversely, if $I_H(x) \subseteq I_H(y)$, then $x = \sup I_H(x) \leqslant \sup I_H(y) = y$. $\square$

In other words, the mapping $x \to I_H(x)$ is an embedding of $(G, \leqslant)$ into $(\mathbb{B}(H), \subseteq)$ whenever H is weakly l-dense in $\mathbb{G}$.

As an example, let $\mathbb{G}$ be a lattice satisfying the descending chain condition. Then any element of $\mathbb{G}$ is the supremum of the join-irreducible elements of $\mathbb{G}$ lying below it. Thus the set H of join-irreducible elements is weakly l-dense in $\mathbb{G}$.

Another consequence of Theorem 1.2 is the following assertion.

1.4. Lemma. *Let $\mathbb{G}$ be an ordered set and let $H \subseteq G$ be weakly l-dense in $\mathbb{G}$. Define a mapping $f_h\colon G \to \{0, 1\}$, for any $h \in H$, in the following way: $f_h(x) = 1$ iff $h \in I_H(x)$. Then $(f_h;\ h \in H)$ is a 2-realizer of $\mathbb{G}$.*

Proof. By 1.3, $x, y \in G$, $x \leqslant y$ is equivalent to $I_H(x) \subseteq I_H(y)$ and this is equivalent to $f_h(x) \leqslant f_h(y)$ for all $h \in H$. $\square$

1.5. Corollary. *Let $\mathbb{G}$ be an ordered set. Then*

$$\text{2-pdim } \mathbb{G} \leqslant wl\text{-sep } \mathbb{G}. \tag{5}$$

If $\mathbb{G}$ is a finite antichain, $|G| = m$, then the only weakly l-dense subset in $\mathbb{G}$ is G; thus wl-sep $\mathbb{G} = m$. On the other hand, 2-pdim $\mathbb{G} = n$, where n is the least positive integer with $\binom{n}{[\frac{n}{2}]} \geqslant m$ ([8], [5], [15]). Thus 2-pdim $\mathbb{G} < wl$-sep $\mathbb{G}$ is possible.

The following definition is dual to that of 1.1.

1.6. Definition. A subset H of an ordered set $\mathbb{G} = (G, \leqslant)$ is called *weakly u-dense* in $\mathbb{G}$ if

$$x, y \in G,\ x \nleqslant y \Rightarrow \text{there exists } h \in H \text{ such that } y \leqslant h,\ x \nleqslant h. \tag{6}$$

Further,

$$wu\text{-sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly } u\text{-dense in } \mathbb{G}\}. \tag{7}$$

By considerations dual to 1.2 and 1.4 we find

1.7. Theorem. *A subset H of an ordered set $\mathbb{G}$ is weakly u-dense in $\mathbb{G}$ iff $x = \inf F_H(x)$ for each $x \in G$.*

1.8. Lemma. *Let H be a weakly u-dense subset of an ordered set $\mathbb{G}$ and let $f_h\colon G \to \{0,1\}$, $h \in H$, be such a mapping that $f_h(x) = 0$ iff $h \in F_H(x)$. Then $(f_h;\ h \in H)$ is a 2-realizer of $\mathbb{G}$.*

This yields

1.9. Corollary. *For any ordered set $\mathbb{G}$ we have*

$$\text{2-pdim}\ \mathbb{G} \leqslant \min\{wl\text{-sep}\ \mathbb{G},\ wu\text{-sep}\ \mathbb{G}\}. \tag{8}$$

Now we show that wl-sep $\mathbb{G}= wu$-sep $\mathbb{G}$ need not hold.

1.10. Example. Let M be an infinite set and let $G \subseteq \mathbb{B}(M)$ be the set of those subsets $X \subseteq M$ for which $|X| = |M|$. If $\mathbb{G} = (G, \subseteq)$, then wu-sep $\mathbb{G} = |M|$, wl-sep $\mathbb{G} > |M|$.

Proof. If $H \subseteq G$ is weakly u-dense in $\mathbb{G}$ then $M - \{x\} \in H$ for each $x \in M$. In fact, if $y \in M$, $y \neq x$, then $M - \{y\} \nsubseteq M - \{x\}$, which means that there must exist $A \in H$ such that $M - \{x\} \subseteq A$, $M - \{y\} \nsubseteq A$. This is possible only if $A = M - \{x\}$. Thus $|H| \geqslant |M|$ and wu-sep $\mathbb{G} \geqslant |M|$. On the other hand, the set $H = \{M - \{x\};\ x \in M\}$ is weakly u-dense in $\mathbb{G}$: If $A, B \in G$, $A \nsubseteq B$, then there exists $x \in A - B$ and hence $B \subseteq M - \{x\}$, $A \nsubseteq M - \{x\}$. Consequently, wu-sep $\mathbb{G}$ $= |M|$.

Assume wl-sep $\mathbb{G} \leqslant |M|$ and let $H \subseteq G$ be a weakly l-dense subset in $\mathbb{G}$ such that $|H| = |M|$. Let α be the least ordinal with $|\alpha| = |M|$ and let $(A_i;\ i < \alpha)$ be a sequence of type α composed of all elements of H. We have $|A_i| = |M|$ for each $i < \alpha$. Choose arbitrary $x_0, y_0 \in A_0$, $x_0 \neq y_0$. Let $\beta < \alpha$ be an ordinal and suppose that we have defined elements x_i, y_i for all ordinals $i < \beta$. Let us choose $x_\beta, y_\beta \in A_\beta$ so that $x_\beta \neq y_\beta$, $x_\beta \notin \{x_i;\ i < \beta\} \cup \{y_i;\ i < \beta\}$, $y_\beta \notin \{x_i;\ i < \beta\} \cup \{y_i;\ i < \beta\}$. This is possible for $|\{x_i;\ i < \beta\} \cup \{y_i;\ i < \beta\}| \leqslant 2|\beta| < |M| = |A_\beta|$. Thus by transfinite induction we have defined elements x_i, y_i for all ordinals $i < \alpha$ such that $x_i, y_i \in A_i$, $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$ and $\{x_i;\ i < \alpha\} \cap \{y_i;\ i < \alpha\} = \emptyset$. Denote $A = \{x_i;\ i < \alpha\}$, $B = \{y_i;\ i < \alpha\}$. Then $|A| = |B| = |M|$, thus $A, B \in G$ and $A \cap B = \emptyset$, especially $A \nsubseteq B$. By assumption there must exist an ordinal $i < \alpha$ such that $A_i \subseteq A$, $A_i \nsubseteq B$. But $y_i \in A_i$, thus $y_i \in A$, which is a contradiction for $y_i \in B$, $A \cap B = \emptyset$. □

If we consider the dual to the set from 1.10 we see that also wl-sep $\mathbb{G} < wu$-sep $\mathbb{G}$ is possible.

Now let $\mathbb{G} = (G, \leqslant)$ be an ordered set, let $G_1 \subseteq G$ and let $\mathbb{G}_1 = (G_1, \leqslant)$ with the induced order. One may expect that wl-sep $\mathbb{G}_1 \leqslant wl$-sep $\mathbb{G}$; the following example shows that this is not the case.

1.11. Example. Let M be an infinite set, let $G = \mathbb{B}(M)$ and let G_1 be the set of those subsets $X \subseteq M$ for which $|X| = |M|$. If $\mathbb{G} = (G, \subseteq)$, $\mathbb{G}_1 = (G_1, \subseteq)$, then wl-sep $\mathbb{G} = |M|$, wl-sep $\mathbb{G}_1 > |M|$.

Proof. We have seen that wl-sep $\mathbb{G}_1 > |M|$ in Example 1.10; but we will show that wl-sep $\mathbb{G} = |M|$. If $H \subseteq G$ is a weakly l-dense subset in $\mathbb{G}$, then $\{x\} \in H$ for all $x \in M$: choose $y \in M$, $y \neq x$ so that $\{x\} \nsubseteq \{y\}$; consequently, there exists $A \in H$ such that $A \subseteq \{x\}$, $A \nsubseteq \{y\}$. This is possible only for $A = \{x\}$; it implies wl-sep $\mathbb{G} \geqslant |M|$. On the other hand, the set $H = \{\{x\};\ x \in M\}$ is weakly l-dense in $\mathbb{G}$: if $A, B \in G$, $A \nsubseteq B$, then there exists an element $x \in A - B$ and then $\{x\} \subseteq A$, $\{x\} \nsubseteq B$. Thus wl-sep $\mathbb{G} \leqslant |M|$, which implies wl-sep $\mathbb{G} = |M|$. □

1.12. Definition. Let $\mathbb{G}$ be an ordered set and let $H \subseteq G$. We will say that H is *weakly dense* in $\mathbb{G}$ if it is both weakly l-dense and weakly u-dense in $\mathbb{G}$. Further, set

$$w\text{-sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is weakly dense in } \mathbb{G}\}. \tag{9}$$

As the union of a weakly l-dense subset of $\mathbb{G}$ and a weakly u-dense subset of $\mathbb{G}$ is a weakly dense subset of $\mathbb{G}$ we have trivially

$$\max\{wl\text{-sep } \mathbb{G},\ wu\text{-sep } \mathbb{G}\} \leqslant w\text{-sep } \mathbb{G} \leqslant wl\text{-sep } \mathbb{G} + wu\text{-sep } \mathbb{G}. \tag{10}$$

If the set G is infinite, then the cardinals wl-sep $\mathbb{G}$, wu-sep $\mathbb{G}$ are also infinite. Thus in (10) the sign = holds; especially we have

$$w\text{-sep } \mathbb{G} = \max\{wl\text{-sep } \mathbb{G}, wu\text{-sep } \mathbb{G}\} \tag{11}$$

for any infinite ordered set $\mathbb{G}$.

If G is finite, then both $\max\{wl$-sep $\mathbb{G}$, wu-sep $\mathbb{G}\} < w$-sep $\mathbb{G}$ and w-sep $\mathbb{G} <$ wl-sep $\mathbb{G} +$ wu-sep $\mathbb{G}$ is possible. For the first relation take $\mathbb{G} = (\mathbb{B}(M), \subseteq)$, where M is finite, $|M| \geqslant 3$; then $H_1 = \{\{x\};\ x \in M\}$ is the least weakly l-dense subset of $\mathbb{G}$, $H_2 = \{M - \{x\};\ x \in M\}$ is the least weakly u-dense subset of $\mathbb{G}$ and $H_1 \cup H_2$ is the least weakly dense subset of $\mathbb{G}$. Consequently, wl-sep $\mathbb{G} = wu$-sep $\mathbb{G} = |M|$, w-sep $\mathbb{G} = 2|M|$. For the other relation, note that if $\mathbb{G}$ is a chain, $x, y \in G$ and $x \prec y$, then every weakly l-dense subset of $\mathbb{G}$ contains y, every weakly u-dense subset of $\mathbb{G}$ contains x and every weakly dense subset of $\mathbb{G}$ contains x, y. Hence we have: if $\mathbb{G}$ is a finite chain, then $wl\text{-sep}\ \mathbb{G} = wu\text{-sep}\ \mathbb{G} = |G| - 1$, $w\text{-sep}\ \mathbb{G} = |G|$.

2. Dense subsets

2.1. Definition. Let $\mathbb{G} = (G, \leqslant)$ be an ordered set and let $H \subseteq G$. We will call H *l-dense* in $\mathbb{G}$ if the following conditions are satisfied:

$$x, y \in G,\ x < y \Rightarrow \text{there exist } h_1, h_2 \in H \text{ such that } x \leqslant h_1 < h_2 \leqslant y, \tag{12}$$

$$x, y \in G,\ x \parallel y \text{ and } I(x) - \{x\} \subseteq I(y) \Rightarrow x \in H. \tag{13}$$

The condition (12) was formulated already in Hausdorff [3], p. 89, for chains. (13) is a slight modification of a condition which appeared in Novotný [12].

Further, we define the *l-separability* of $\mathbb{G}$:

$$l\text{-sep}\ \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is } l\text{-dense in } \mathbb{G}\}. \tag{14}$$

The u-density is defined dually:

2.2. Definition. A subset H of an ordered set $\mathbb{G}$ is called *u-dense* in $\mathbb{G}$ if it satisfies (12) and the condition

$$x, y \in G,\ x \parallel y \text{ and } F(x) - \{x\} \subseteq F(y) \Rightarrow x \in H. \tag{15}$$

Further,

$$u\text{-sep}\ \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is } u\text{-dense in } \mathbb{G}\}. \tag{16}$$

Note that if H is l-dense (u-dense) in $\mathbb{G}$ and $x \in G$ is a minimal (maximal) element which is not the least (the greatest), then $x \in H$. Also, if $x, y \in G$ and $x \prec y$, then $x, y \in H$.

2.3. Theorem. *Let $\mathbb{G}$ be an ordered set and let $H \subseteq G$ be an l-dense subset of $\mathbb{G}$. Then H is weakly l-dense in $\mathbb{G}$.*

P r o o f. Let H be l-dense in $\mathbb{G}$ and let $x, y \in G$, $x \nleqslant y$. If $y < x$, then there exist elements $h_1, h_2 \in H$ such that $y \leqslant h_1 < h_2 \leqslant x$, which means that $h_2 \leqslant x$, $h_2 \nleqslant y$. Let $x \parallel y$ and suppose that there is no $h \in H$ such that $h \leqslant x$, $h \nleqslant y$. Consequently, $h \in H$, $h \leqslant x$ implies $h \leqslant y$. Let $z \in I(x) - \{x\}$ be an arbitrary element. Then $z < x$ and thus there exist $h_1, h_2 \in H$ such that $z \leqslant h_1 < h_2 \leqslant x$. By our assumption

$h_2 \leqslant y$, which means that $z < y$, i.e. $z \in I(y)$. Thus $I(x) - \{x\} \subseteq I(y)$ and by (13) $x \in H$. As $x \leqslant x$ we have $x \leqslant y$ contradicting the assumption $x \parallel y$. Hence there must exist an element $h \in H$ such that $h \leqslant x$, $h \nleqslant y$ and H is weakly l-dense in $\mathbb{G}$. □

The dual assertion to 2.3 is also valid.

2.4. Corollary. *Let $\mathbb{G}$ be an ordered set. Then*

$$wl\text{-sep } \mathbb{G} \leqslant l\text{-sep } \mathbb{G}, \quad wu\text{-sep } \mathbb{G} \leqslant u\text{-sep } \mathbb{G}. \tag{17}$$

Let M be a finite set, $|M| \geqslant 3$ and let $\mathbb{G} = (\mathbb{B}(M), \subseteq)$. We have stated above that wl-sep $\mathbb{G} = |M|$. Let $A \in \mathbb{B}(M)$ be arbitrary. If $A \neq \emptyset$ and $x \in A$, then $A - \{x\} \prec A$; if $A = \emptyset$ and $x \in M$, then $A \prec \{x\}$. Thus $\mathbb{B}(M)$ is the only l-dense subset of $\mathbb{G}$ and l-sep $\mathbb{G} = 2^{|M|}$. Hence wl-sep $\mathbb{G} < l$-sep $\mathbb{G}$ is possible; analogously for wu-sep $\mathbb{G}$, u-sep $\mathbb{G}$.

2.5. Definition. Let $\mathbb{G}$ be an ordered set and let $H \subseteq G$. The set H will be called *dense* in $\mathbb{G}$ if it is both l-dense and u-dense in $\mathbb{G}$. Further, set

$$\text{sep } \mathbb{G} = \min\{|H|;\ H \subseteq G \text{ is dense in } \mathbb{G}\}. \tag{18}$$

Trivially, we have

$$\max\{l\text{-sep } \mathbb{G},\ u\text{-sep } \mathbb{G}\} \leqslant \text{sep } \mathbb{G} \leqslant l\text{-sep } \mathbb{G} + u\text{-sep } \mathbb{G} \tag{19}$$

and if $\mathbb{G}$ is infinite, then the sign $=$ holds. But we will show that, on the contrary to weak density, $=$ always holds in the left inequality of (19). This is a consequence of the following trivial assertion.

2.6. Lemma. *Let $\mathbb{G}$ be a finite ordered set. If $H \subseteq G$ is l-dense in $\mathbb{G}$, then $H = G$.*

P r o o f. Let $x \in G$. If x is not an isolated element of $\mathbb{G}$ then there exists an element $y \in G$ such that either $x \prec y$ or $y \prec x$. Consequently, $x \in H$. If x is isolated, then it is a minimal and not the least element, thus $x \in H$ again. □

The same holds for u-density; thus l-sep $\mathbb{G} = u$-sep $\mathbb{G} = |G|$ for a finite ordered set $\mathbb{G}$.

2.7. Corollary. *Let $\mathbb{G}$ be an ordered set. Then*

$$\text{sep } \mathbb{G} = \max\{l\text{-sep } \mathbb{G},\ u\text{-sep } \mathbb{G}\}. \tag{20}$$

Note that (5), (17) and (20) imply 2-pdim $\mathbb{G} \leqslant$ sep $\mathbb{G}$; this fact (with another formulation) is the main result in [12].

Trivially, w-sep $\mathbb{G} \leqslant$ sep $\mathbb{G}$ for any ordered set $\mathbb{G}$. If M is an infinite set and $\mathbb{G} = (\mathbb{B}(M), \subseteq)$, then it is easy to show wl-sep $\mathbb{G} = |M|$, l-sep $\mathbb{G} = 2^{|M|}$ and similarly wu-sep $\mathbb{G} = |M|$, u-sep $\mathbb{G} = 2^{|M|}$. Then (11) implies w-sep $\mathbb{G} = |M|$ and from (20) we have sep $\mathbb{G} = 2^{|M|}$. Thus w-sep $\mathbb{G} <$ sep $\mathbb{G}$ is possible.

We prove a simple assertion.

2.8. Lemma. *Let $\mathbb{G}$ be a chain and let $H \subseteq G$. Then H is dense in $\mathbb{G}$ if and only if it is weakly dense in $\mathbb{G}$.*

Proof. If H is dense in $\mathbb{G}$, then it is weakly dense by 2.3 and the dual assertion. Let H be weakly dense in $\mathbb{G}$ and let $x, y \in G$, $x < y$. As $y \nleqslant x$, there exists $h_2 \in H$ such that $h_2 \leqslant y$, $h_2 \nleqslant x$, i.e. $x < h_2 \leqslant y$. As $h_2 \nleqslant x$, there exists $h_1 \in H$ such that $x \leqslant h_1$, $h_2 \nleqslant h_1$, i.e. $h_1 < h_2$. Then $x \leqslant h_1 < h_2 \leqslant y$, which means that (12) holds and H is dense in $\mathbb{G}$. □

As a corollary, we have w-sep $\mathbb{G} =$ sep $\mathbb{G}$ if $\mathbb{G}$ is a chain.

At the end, we summarize the relations obtained:

$$\begin{aligned} \text{2-pdim } \mathbb{G} &\leqslant \min\{wl\text{-sep } \mathbb{G}, wu\text{-sep } \mathbb{G}\} \\ \max\{wl\text{-sep } \mathbb{G}, wu\text{-sep } \mathbb{G}\} &\leqslant w\text{-sep } \mathbb{G} \leqslant wl\text{-sep } \mathbb{G} + wu\text{-sep } \mathbb{G} \\ wl\text{-sep } \mathbb{G} &\leqslant l\text{-sep } \mathbb{G} \\ wu\text{-sep } \mathbb{G} &\leqslant u\text{-sep } \mathbb{G} \\ w\text{-sep } \mathbb{G} &\leqslant \text{sep } \mathbb{G} \\ \text{sep } \mathbb{G} &= \max\{l\text{-sep } \mathbb{G},\ u\text{-sep } \mathbb{G}\}. \end{aligned}$$

The following two problems remain open:

1. Let $\mathbb{G} = (G, \leqslant)$ be an ordered set, let $G_1 \subseteq G$ and let $\mathbb{G}_1 = (G_1, \leqslant)$ be an ordered subset of $\mathbb{G}$. Does then l-sep $\mathbb{G}_1 \leqslant l$-sep $\mathbb{G}$ (u-sep $\mathbb{G}_1 \leqslant u$-sep $\mathbb{G}$) hold?
2. Can there exist an ordered set $\mathbb{G}$ such that l-sep $\mathbb{G} \neq u$-sep $\mathbb{G}$?

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